Laplace Equations (H.1)

20160622

Classical BVP's

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0$$ one-dim heat eq

$$a\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$ one-dim wave eq

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ two-dim Laplace's eq

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Initial Conditions

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0 \qquad \text{one-dim heat eq}$$

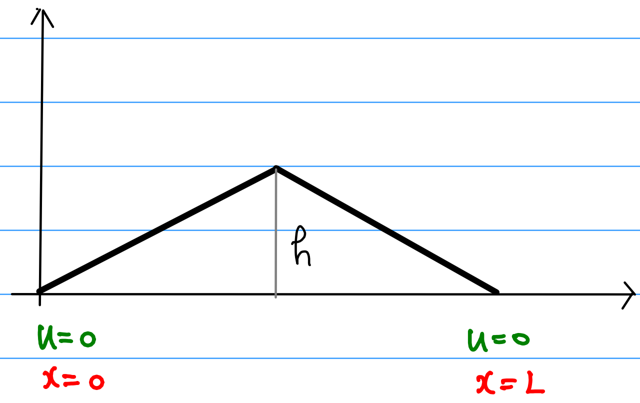
$$a\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \text{one-dim wave eq}$$

$u(x,t) \longrightarrow u(x,0)$ IC (Initial Conditions)

$$\begin{cases} u(x,0) = f(x) & 0 < x < L \\ \frac{\partial}{\partial t}u(x,0) = g(x) \end{cases}$$

$$\begin{cases} u(x,t)\Big|_{t=0} = f(x) & 0 < x < L \\ \frac{\partial}{\partial t}u(x,t)\Big|_{t=0} = g(x) \end{cases}$$

Boundary Conditions



plucked string

$u(0, t) = 0$ $\quad$ $u(L, t) = 0$ $\quad$ $t > 0$

Dirichelet Condition

Dirichlet problem for a disk

Solve Laplace's equation

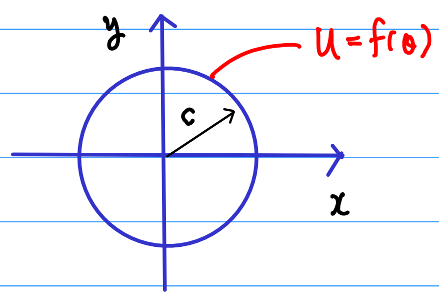
for the steady-state temperature $u(r, \theta)$

in a circular disk or plate of radius c

when the temperature of the circumference

is $u(c, \theta) = f(\theta)$, $0 < \theta < 2\pi$

two faces of the plate are insulated



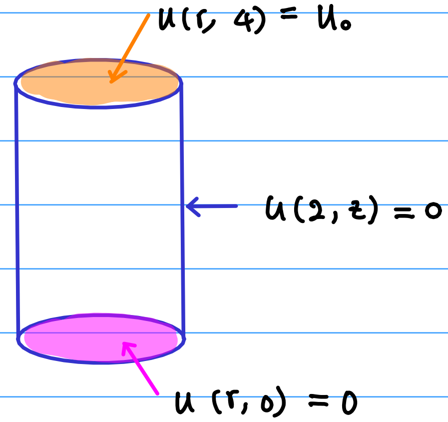
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Steady Temperature in a Circular Cylinder

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad 0<r<2, \qquad 0<z<4$$

$u(2, z) = 0 \qquad 0<z<4$

$u(r, 0) = 0 \qquad u(r, 4) = u_0 \qquad 0<r<2$



Laplace Equations

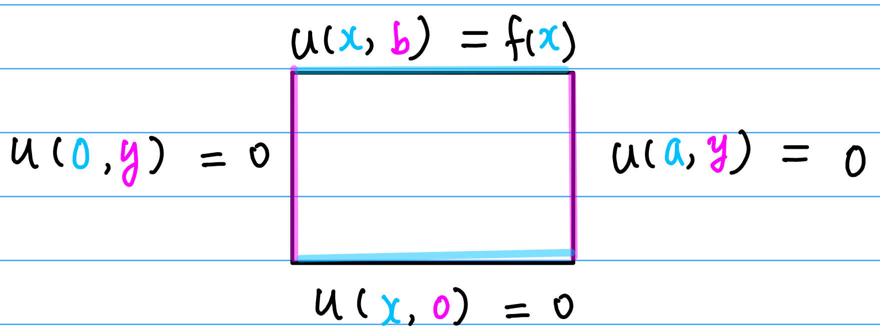
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0<x<a \qquad 0<y<b$$

Neuman Condition (BC) $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \qquad 0<y<b$

Dirichlet Condition (BC) $u(x,0) = 0, \quad u(x,b) = f(x) \qquad 0<x<a$

Dirichlet Condition (BC) $u(0,y) = 0, \quad u(a,y) = 0 \qquad 0<y<b$

Dirichlet Condition (BC) $u(x,0) = 0, \quad u(x,b) = f(x) \qquad 0<x<a$



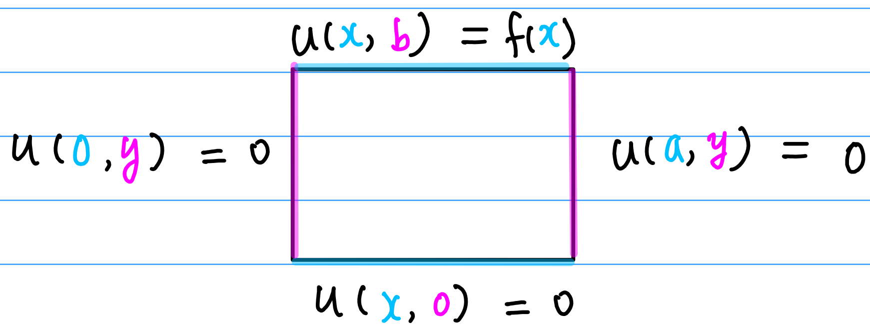
Dirichlet Problem

elliptic partial differential equation

$$\nabla^2 u = 0 \qquad \text{Laplace Eq}$$

within a region R (in the plane or 3-space)

u takes a prescribed values
on the entire boundary of the region



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < a \qquad 0 < y < b$$

$\boxed{\text{Dirichlet Condition}}$ (BC) $u(0, y) = 0,\quad u(a, y) = 0 \qquad 0 < y < b$

$\boxed{\text{Dirichlet Condition}}$ (BC) $u(x, 0) = 0,\quad u(x, b) = f(x) \qquad 0 < x < a$

Dirichlet Problem

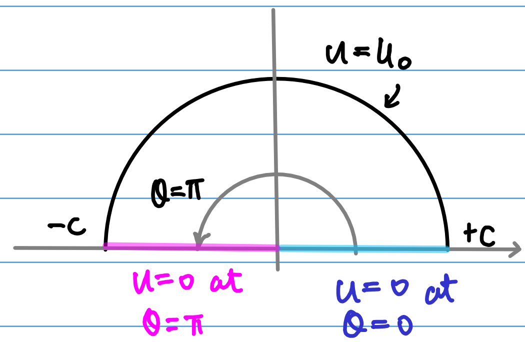
Steady-state temperature in a semi-circle plate

$u(r, \theta)$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad 0<\theta<\pi, \quad 0<r<c$$

$u(c, \theta) = u_0 \qquad 0<\theta<\pi$

$u(r, \theta) = 0 \qquad u(r, \pi) = 0 \qquad 0<r<c$

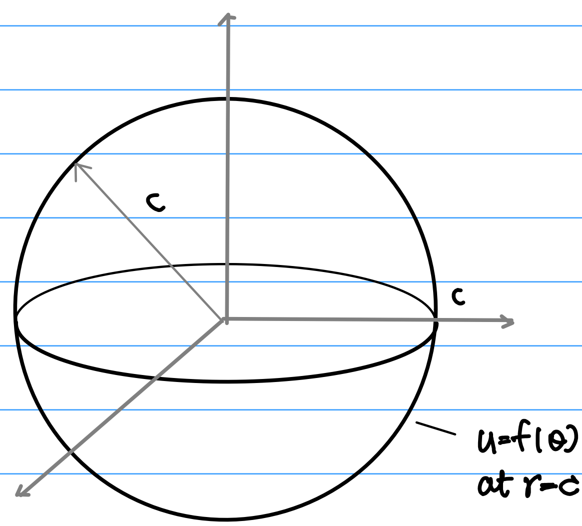


Dirichlet Problem

Steady-state temperature in a sphere

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad 0<r<c, \qquad 0<\theta<\pi$$

$$u(c,\theta) = f(\theta) \qquad 0<\theta<\pi$$



Dirichlet Problem: Superposition

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < a, \quad 0 < y < b$$

$u(0, y) = F(y)$ $\qquad$ $u(a, y) = G(y)$ $\qquad$ $0 < y < b$

$u(x, 0) = f(x)$ $\qquad$ $u(x, b) = g(x)$ $\qquad$ $0 < x < a$

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \qquad 0 < x < a, \quad 0 < y < b$$

$u_1(0, y) = 0$ $\qquad$ $u_1(a, y) = 0$ $\qquad$ $0 < y < b$

$u_1(x, 0) = f(x)$ $\qquad$ $u_1(x, b) = g(x)$ $\qquad$ $0 < x < a$

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \qquad 0 < x < a, \quad 0 < y < b$$

$u_2(0, y) = F(y)$ $\qquad$ $u_2(a, y) = G(y)$ $\qquad$ $0 < y < b$

$u_2(x, 0) = 0$ $\qquad$ $u_2(x, b) = 0$ $\qquad$ $0 < x < a$

Dirichlet Problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0<x<2, \quad 0<y<2$$

$$u(0,y) = 0 \qquad u(2,y) = y(2-y) \qquad 0<y<b$$

$$u(x,0) = 0 \qquad u(x,2) = \begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \end{cases}$$

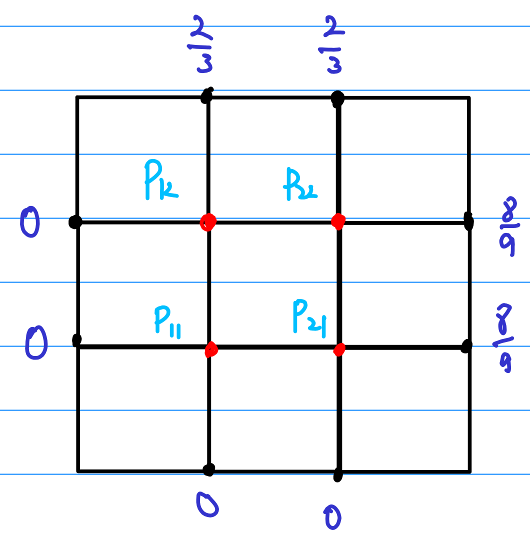
the Dirichlet Problem for Laplace Eq $\nabla^2 u = 0$

$u(x,y)$ - given on the boundary C of a region R

the approximate solution by using numerical method

interior mesh points

u_{ij}



Harmonic Functions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

the real (u) & imaginary (v) parts of an analytic function
cannot be chosen arbitrarily

both u and v must satisfy Laplace eq

the real-valued function $\phi(x, y)$
— has continuous
2nd order partial derivatives
in a domain D

— satisfies Laplace eq

$\Leftrightarrow$ $\phi(x, y)$: harmonic in D

In mathematics, mathematical physics and the theory of stochastic processes, a **harmonic function** is a twice continuously differentiable function $f : U \to \mathbf{R}$ (where U is an open subset of $\mathbf{R}^n$) which satisfies Laplace's equation, i.e. real-valued fn.

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

everywhere on U. This is usually written as

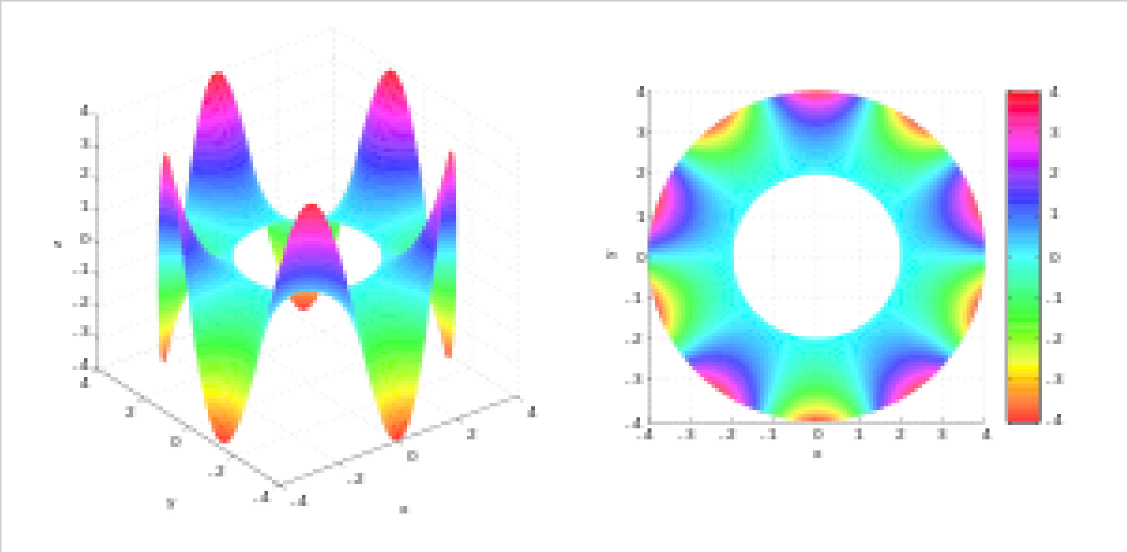
$$\nabla^2 f = 0$$

or

$$\Delta f = 0$$

Etymology of the term "harmonic" [edit]

The descriptor "harmonic" in the name harmonic function originates from a point on a taut string which is undergoing harmonic motion. This solution to the differential equation for this type of motion can be written in terms of sines and cosines, functions which are thus referred to as *harmonics*. Fourier analysis involves expanding periodic functions on the unit circle in terms of a series of these harmonics. Considering higher dimensional analogues of the harmonics on the unit n-sphere, one arrives at the spherical harmonics. These functions satisfy Laplace's equation and over time "harmonic" was used to refer to all functions satisfying Laplace's equation.[1]

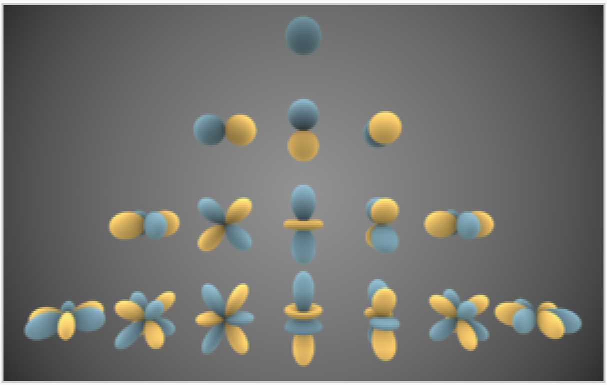


A harmonic function defined on an annulus.

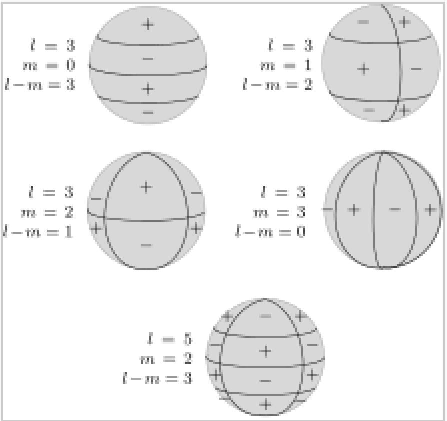
Spherical Harmonic *

In mathematics and physical science, **spherical harmonics** are special functions defined on the surface of a sphere. They are often employed in solving partial differential equations that commonly occur in science. The spherical harmonics are a complete set of orthogonal functions on the sphere, and thus may be used to represent functions defined on the surface of a sphere, just as circular functions (sines and cosines) are used to represent functions on a circle via Fourier series. Like the sines and cosines in Fourier series, the spherical harmonics may be organized by (spatial) angular frequency, as seen in the rows of functions in the illustration on the right. Further, spherical harmonics are basis functions for SO(3), the group of rotations in three dimensions, and thus play a central role in the group theoretic discussion of SO(3).

Special Orthogonal Group



Visual representations of the first few real spherical harmonics. Blue portions represent regions where the function is positive, and yellow portions represent where it is negative. The distance of the surface from the origin indicates the value of $Y_\ell^m(\theta,\phi)$ in angular direction (θ,ϕ).



Schematic representation of $Y_{\ell m}$ on the unit sphere and its nodal lines. $\mathrm{Re}[Y_{\ell m}]$ is equal to 0 along m great circles passing through the poles, and along $\ell - m$ circles of equal latitude. The function changes sign each time it crosses one of these lines.

Criterion for Analyticity

$\boxed{1}$ Suppose the real-valued functions

$u(x,y)$, $v(x,y)$: continuous

the 1st order partial derivatives

$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$: continuous

in a domain D

$\boxed{2}$ And if $u(x,y)$ & $v(x,y)$ meets the Cauchy – Rieman eq

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}}$$

Then the complex function $f(z) = u(x,y) + i\, v(x,y)$

is analytic in D

$$\boxed{\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \end{aligned}} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \end{aligned}}$$

$$\boxed{\frac{1}{i} = -i}$$

$$\boxed{\Delta z = \Delta x + i\Delta y}$$

① $\Delta x \to 0, \ \Delta y = 0$

$$f'(z) = \frac{f(z+\Delta x) - f(z)}{\Delta x}$$

$$= \frac{u(x+\Delta x, y) + i\,v(x+\Delta x, y) - u(x, y) - i\,v(x, y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

② $\Delta y \to 0 \quad \Delta x = 0$

$$f'(z) = \frac{f(z+\Delta y) - f(z)}{i\Delta y}$$

$$= \frac{u(x, y+\Delta y) + i\,v(x, y+\Delta y) - u(x, y, \Delta y) - i\,v(x, y+\Delta y)}{i\Delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Harmonic Function Definition

A real-valued function

$\phi(x, y)$ is harmonic in D

$\Longleftrightarrow$ ① the 2^{nd} order partial derivatives :

continous in a domain D

② satisfy Laplace's equation

real $\phi(x, y)$ is harmonic in D

$\Longleftrightarrow$ $\dfrac{\partial^2 \phi}{\partial x^2}$, $\dfrac{\partial^2 \phi}{\partial y^2}$, $\dfrac{\partial^2 \phi}{\partial x \partial y}$, $\dfrac{\partial^2 \phi}{\partial y \partial x}$: continous

$\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$ Laplace's equation the source of Harmonic Functions

$f(z) = u(x, y) + i\, v(x, y)$

$\boxed{\text{analytic}}$ in domain D

the function $u(x, y)$ & $v(x, y)$ are $\boxed{\text{harmonic}}$ functions

u & v : continuous 2nd order partial derivatives

f : analytic → Cauchy - Rieman eq

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

① $$\frac{\partial^2 u}{\partial x^2} = \boxed{\frac{\partial^2 v}{\partial x \partial y}} \qquad \frac{\partial^2 u}{\partial y^2} = -\boxed{\frac{\partial^2 v}{\partial x \partial y}}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$u(x, y)$: harmonic

② $$\boxed{\frac{\partial^2 u}{\partial y \partial x}} = \frac{\partial^2 v}{\partial y^2} \qquad \boxed{\frac{\partial^2 u}{\partial x \partial y}} = -\frac{\partial^2 v}{\partial x^2}$$

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

$v(x, y)$: harmonic

$f(z) = u(x, y) + i\, v(x, y)$ $\boxed{\text{analytic}}$ in domain D

$$f(z) = \underbrace{u(x, y)}_{\text{harmonic}} + i\, \underbrace{v(x, y)}_{\text{harmonic}}$$

continous

$$\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^2 u}{\partial x\, \partial y}, \quad \frac{\partial^2 u}{\partial y\, \partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

continous

$$\frac{\partial^2 v}{\partial x^2}, \quad \frac{\partial^2 v}{\partial y^2}, \quad \frac{\partial^2 v}{\partial x\, \partial y}, \quad \frac{\partial^2 v}{\partial y\, \partial x}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

harmonic conjugate

In several ways, the harmonic functions are real analogues to holomorphic functions. All harmonic functions are analytic, i.e. they can be locally expressed as power series. This is a general fact about elliptic operators, of which the Laplacian is a major example.

$$f(z) = u(x, y) + i\, v(x, y) \quad \boxed{\text{analytic}} \text{ in domain } D \quad \rightleftarrows$$

the function $u(x, y)$ & $v(x, y)$ are $\boxed{\text{harmonic}}$ functions

Harmonic Conjugate Functions

$f(z) = u(x,y) + i v(x,y)$ analytic in D

$\Rightarrow$ u, v : harmonic in D

given $u(x,y)$: harmonic in D

find $v(x,y)$: harmonic in D

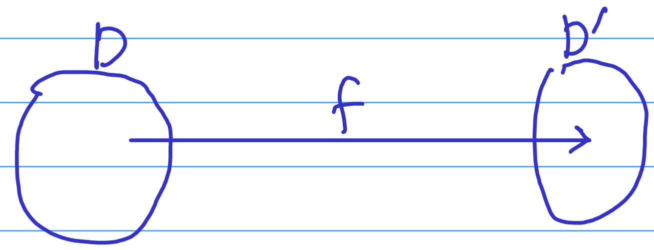
such that $u(x,y) + i v(x,y)$: analytic in D

u, v : harmonic conjugate function

Harmonic Functions & the Dirichlet Problem

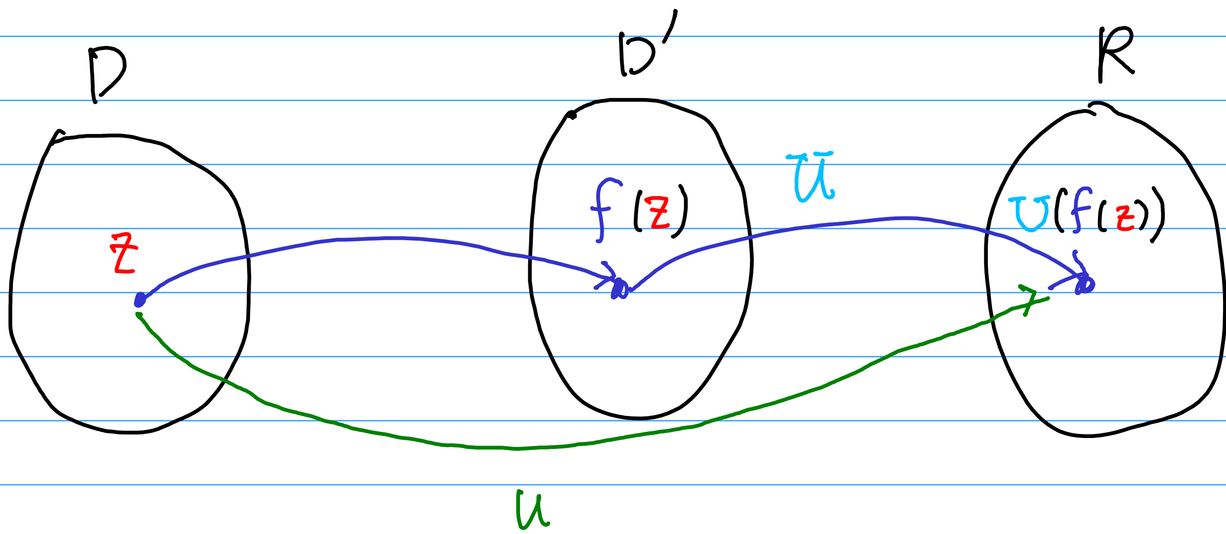
Transformation Theorem for Harmonic Functions

f : analytic function
that maps a domain D onto a domain D'



real valued function $u(x,y) = U(f(z))$

u is harmonic in D $\Leftarrow$ U is harmonic in D'



⑥ prove

$$\boxed{\begin{array}{ll} U(z) \text{ harmonic in } D' & \Rightarrow \\ U(f(z)) \text{ harmonic in } D & \boxed{f: D \to D'} \end{array}}$$

a special case where D is simply connected

U, V harmonic conjugate in D'

$\Rightarrow H = U + iV$ analytic in D'

$\Rightarrow H(f(z)) = U(f(z)) + i\,V(f(z))$ is analytic in D

$f(z) = u(x, y) + i\,v(x, y)$ analytic in domain D

the function $u(x, y)$ & $v(x, y)$ are harmonic functions

$\Rightarrow u(x, y) = U(f(z))$ harmonic in D

⑥ prove

$U(z)$ has a harmonic conjugate $V(z)$

let $g(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$

To show $g(z)$: analytic

$Re\{g(z)\}$ and $Im\{g(z)\}$

① are continuous

② has continuous 1st order partial derivatives

③ satisfy the Cauchy-Rieman equations

$\frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial y}$ continuous

$\frac{\partial^2 U}{\partial x^2}, \quad \frac{\partial^2 U}{\partial x \partial y},$

$\frac{\partial^2 U}{\partial y \partial x}, \quad \frac{\partial^2 U}{\partial y^2}$ continuous

$$\begin{cases} \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial U}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right) = -\frac{\partial}{\partial x}\left(-\frac{\partial U}{\partial y}\right) \end{cases}$$

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial U}{\partial y}\right)} \qquad \boxed{\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right) = -\frac{\partial}{\partial x}\left(-\frac{\partial U}{\partial y}\right)}$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y}$$

U : harmonic in D′ — equal 2nd order mixed partial derivatives

⑥ g(z) satisfies the Cauchy - Rieman eq

$$\boxed{\frac{\partial U}{\partial x}}, \ -\boxed{\frac{\partial U}{\partial y}}$$

① are continuous
② has continuous 1st order partial derivatives
③ satisfy the Cauchy – Rieman equations

→ $g(z) = \boxed{\frac{\partial U}{\partial x}} - i\boxed{\frac{\partial U}{\partial y}}$: analytic in D′

$$\boxed{\begin{array}{l}\text{Fundamental Theorem for Contour Integrals} \\ \\ f \quad \text{continuous in a domain } D \\ F \quad \text{antiderivative of } f \text{ in } D \\ C \quad \text{any contour in } D \\ \qquad \text{initial pt } z_0, \quad \text{final pt } z_1 \\ \\ \qquad \displaystyle\int_C f(z)\,dz = F(z_1) - F(z_0)\end{array}}$$

$$g(z) = \boxed{\frac{\partial U}{\partial x}} - i\boxed{\frac{\partial U}{\partial y}} \quad : \text{analytic in } D'$$

$\Rightarrow$ $g(z)$ has anti-derivative $G(z) = U_1 + i V_1$

$$G(z) = U_1 + i V_1$$

Cauchy - Rieman eq

$$\boxed{\frac{\partial m}{\partial x} = \frac{\partial n}{\partial y}}$$

$$\boxed{\frac{\partial n}{\partial y} = -\frac{\partial n}{\partial x}}$$

$$G'(z) = \frac{\partial U_1}{\partial x} + i\frac{\partial V_1}{\partial x}$$

$$= \frac{\partial U_1}{\partial x} - i\frac{\partial U_1}{\partial y}$$

$$g(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$$\frac{\partial U_1}{\partial x} = \frac{\partial U}{\partial x}$$

$$\frac{\partial U_1}{\partial y} = \frac{\partial U}{\partial y}$$

equal 1st order partial derivatives

$$H(z) = U + i V_1 \quad \text{analytic in } D'$$

$$g(z) = \boxed{\frac{\partial U}{\partial x}} - i \boxed{\frac{\partial U}{\partial y}}$$: analytic in D'

$\Rightarrow$ $g(z)$ has anti-derivative $G(z) = U_1 + i V_1$

$\Rightarrow$ $H(z) = U + i V_1$ analytic in D'

U has harmonic conjugate V_1 in D'