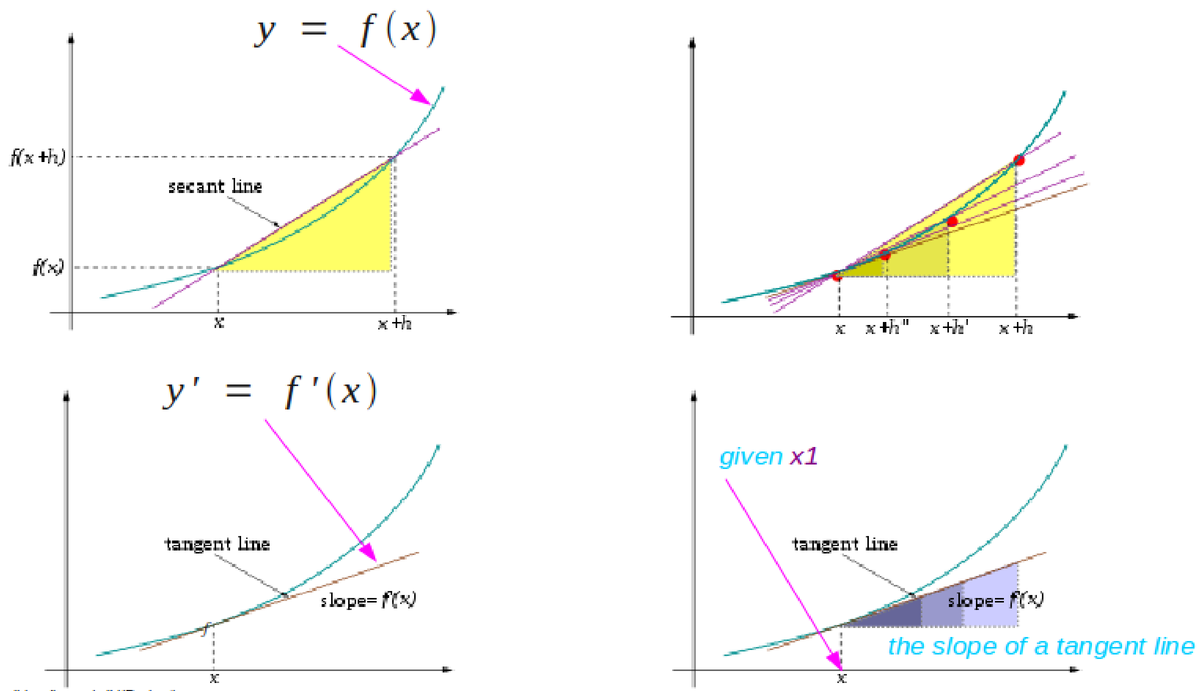
First Order ODEs (H.1)

.

20151228

Another kind of triangles and their slopes

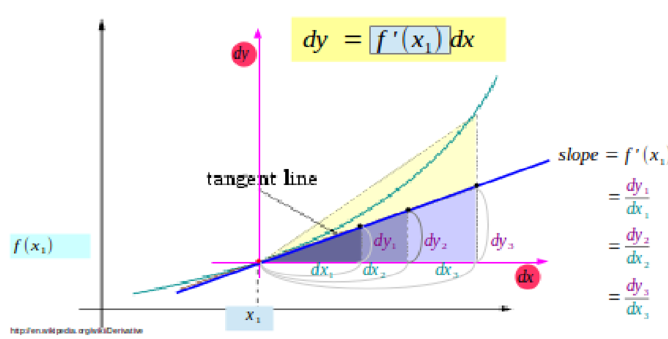


http://en.wikipedia.org/wiki/Derivative

Differentials and Derivatives (1)

$$dy = \boxed{f'(x)}\, dx$$

$$dy = \boxed{\frac{df}{dx}}\, dx$$



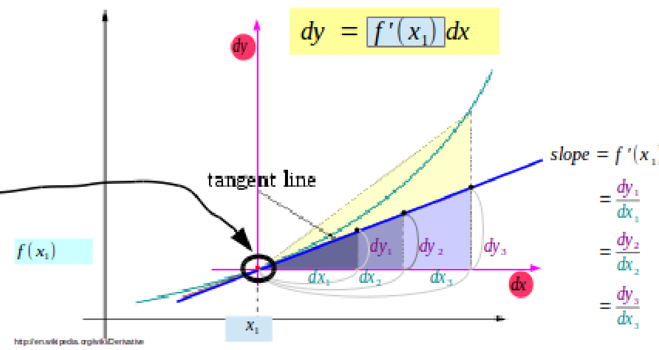
differentials **derivative**

$$\frac{dy}{dx} = f'(x)$$

ratio not a ratio

Differentials and Derivatives (2)

$$f(x_1 + dx) \approx f(x_1) + dy$$
$$= f(x_1) + f'(x_1)\,dx$$



for small enough dx

$\lim_{dx \to 0}$

$$f(x_1 + dx) = f(x_1) + dy$$
$$= f(x_1) + f'(x_1)\,dx$$

$$\lim_{dx \to 0} \frac{f(x_1 + dx) - f(x_1)}{dx} = f'(x_1)$$

Differentials and Derivatives (3)

$$dy = f'(x)\,dx \quad \Rightarrow \quad \int dy = \int f'(x)\,dx$$

$$dy = \frac{df}{dx}\,dx \quad \Rightarrow \quad \int dy = \int \frac{df}{dx}\,dx$$

$$y = f(x)$$

$$dy = \dot{f}\,dx$$

$$dy = D_x f\,dx$$

$$\int dy = \int 1\,dy = y$$

Integration Constant C

place a constant

place another constant

$$\int dy = \int f'(x)\, dx$$

$$\int dy = \int \frac{df}{dx}\, dx$$

$$y + C_1 = f(x) + C_2$$

$$\boxed{y = f(x) + C}$$

differs by a constant

place only one constant from the beginning

$$\int dy = \int f'(x)\, dx + C$$

$$\int dy = \int \frac{df}{dx}\, dx + C$$

$$\boxed{y = f(x) + C}$$

Types of First Order ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = \boxed{g(x,y)}$$

$$y' = \boxed{g(x,y)}$$

Separable Equations

$$\frac{dy}{dx} = g_1(x)g_2(y)$$

$$y' = g_1(x)g_2(y)$$

$$y = f(x)$$

Linear Equations

$$a_1(x)\boxed{\frac{dy}{dx}} + a_0(x)\boxed{y} = g(x)$$

$$a_1(x)y' + a_0(x)y = g(x)$$

$$y = f(x)$$

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0$$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$$

$$z = f(x,y)$$

1차 미분 방정식 . ch 2.

- ⓐ Separable eq. 변수분리 2.2 $y = f(x)$
- ⓑ Linear eq 선형방정식 2.3
- ⓒ Exact eq 완전방정식 2.4 $y = g(x)$

$f(x, y)$

$= f(x, g(x))$

1st order differential eq

Solution : $y(x)$: y는 x의 함수

y' $\cancel{y''}$ $\cancel{y'''}$

y, x

1차 미분 방정식

ODE Ordinary Differential Equation 상미분 방정식

PDE Partial Differential Equation 편미분 방정식

$\frac{df}{dx}$ $\qquad$ $f(x)$ ..... 2차원 그림

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ $\qquad$ $f(x, y)$ ... 3차원 그림

partial derivative.

Separable Equation

1차 미분 방정식 $\underbrace{y'}_{\text{x에 대해서 y를 미분}} = \underbrace{g(x, y)}_{\text{x, y의 수식}}$ $y(x) = ?$

$$y' = \underbrace{\boxed{g_1(x)}}_{(x\text{식})} \times \underbrace{\boxed{g_2(y)}}_{(y\text{식})}$$ separable

$$y'\left(\frac{1}{g_2(y)}\right) = \left(g_1(x)\right)$$

$$\underline{y'}\ (y\text{식}) = (x\text{식})$$

$$\underline{y'}\ (y\text{식})\ dx = (x\text{식})\ dx$$

접선의 기울기

$$(y\text{식})\ \boxed{\frac{dy}{dx}\ dx} = (x\text{식})\ dx$$

$$(y\text{식})\ \boxed{dy} = (x\text{식})\ dx$$

$$\int (y\text{식})\ dy = \int (x\text{식})\ dx$$

(y식)을 y에 대하여 적분

(x식)을 x에 대하여 적분

$$\frac{dy}{dx} = 6y^2 x = (6x)(y^2)$$

① $$y' = \frac{3x^2+4x-4}{2y-4} = (3x^2+4x-4)\left(\frac{1}{2y-4}\right)$$

② $$y' = \frac{xy^3}{\sqrt{1+x^2}} = \left(\frac{x}{\sqrt{1+x^2}}\right)(y^3)$$

③ $$y' = e^{-y}(2x-4) = (2x-4)(e^{-y})$$

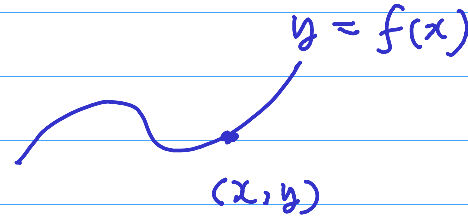
④ $$\frac{dr}{d\theta} = \frac{r^2}{\theta}$$ r(θ) 함수를 θ에 대하여 미분

$$\underline{\frac{d}{d\theta} r(\theta) = \frac{r^2(\theta)}{\theta}} \Rightarrow \underline{r(\theta)\,?}$$

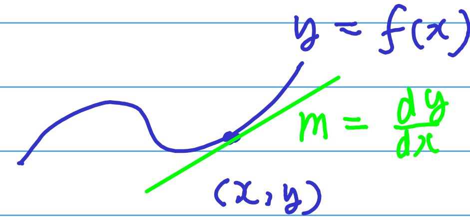
$$\frac{dr}{d\theta} = \frac{r^2}{\theta} = \left(\frac{1}{\theta}\right)(r^2)$$

⑤ $$\frac{dy}{dt} = e^{y-t}\sec(y)(1+t^2) = \left(e^{-t}(1+t^2)\right)\left(e^{y}\sec(y)\right)$$

$$\frac{dy}{dx} = 6y^2 x = (6x)(y^2)$$



$$\frac{dy}{dx} \frac{1}{y^2} = 6x$$



$$\frac{1}{y^2} \frac{dy}{dx} \, dx = 6x \, dx$$

$$\int \frac{1}{y^2} \boxed{\frac{dy}{dx} \, dx} = \int 6x \, dx$$

$$\int \frac{1}{y^2} \, dy = \int 6x \, dx$$

$$-\frac{1}{3} y^{-3} = 3x^2 + C$$

$$y^{-3} = -9x^2 - 3C$$

$$y^3 = \frac{1}{-9x^2 + C'}$$

$$y = \frac{1}{\sqrt[3]{-9x^2 + C'}}$$

Linear Equation

1차 미분 방정식 $\underbrace{y'}_{x\text{에 대해서 } y\text{를 미분}} = \underbrace{g(x, y)}_{x,\, y\text{의 수식}}$

y(x) ?

Linear Eq

$$a_1(x)\boxed{y'} + a_2(x)\boxed{y} = g(x)$$

x의 수식 or 상수 / x의 수식 or 상수 / x의 수식

$\frac{1}{a_1(x)}$

$$1 \cdot \boxed{y'} + p(x)\boxed{y} = g(x)$$

x의 수식 or 상수 / x의 수식

1. $\boxed{y'} + p(x)\boxed{y} = g(x)$

① $\boxed{\frac{dv}{dt}} = 9.8 - 0.196\boxed{v}$ $\quad v(t) = ?$

$1\boxed{\frac{dv}{dt}} + 0.196\boxed{v} = 9.8$

② $\cos(x)\boxed{y'} + \sin(x)\boxed{y} = 2\cos^3(x)\sin(x) - 1$

$1\boxed{y'} + \frac{\sin(x)}{\cos(x)}\boxed{y} = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$

③ $t\boxed{y'} + 2\boxed{y} = t^2 - t + 1$

$\boxed{y'} + \frac{2}{t}\boxed{y} = t - 1 + \frac{1}{t}$

④ $t\boxed{y'} - 2\boxed{y} = t^5\sin(2t) - t^3 + 4t^4$

$1\boxed{y'} - \frac{2}{t}\boxed{y} = t^4\sin(2t) - t^2 + 4t^3$

⑤ $2\boxed{y'} - \boxed{y} = 4\sin(3t)$ $\quad y(t)?$

$1\boxed{y'} - \frac{1}{2}\boxed{y} = 2\sin(3t)$

homogeneous eq

$$y' + p(x)\, y = 0 \quad \text{sol} \quad y_h$$

$$\Leftrightarrow \quad y_h' + p(x)\, y_h = 0$$

non-homogeneous eq

$$y' + p(x)\, y = Q(x) \quad \text{sol} \quad y_p$$

$$\Leftrightarrow \quad y_p' + p(x)\, y_p = Q(x)$$

$$y_h' + p(x)\, y_h = 0$$

$$y_p' + p(x)\, y_p = Q(x)$$

$$(y_p + y_h)' + p(x)(y_p + y_h) = Q(x)$$

$$\Leftrightarrow \quad \boxed{1.\ y' + p(x)\, y = Q(x)} \quad \text{sol} \quad (y_p + y_h)$$

homogeneous eq

$y' + p(x)\, y = 0$ sol $y_h = c \cdot y_1$

$= c \cdot e^{-\int p(x)dx}$

non-homogeneous eq

$y' + p(x)\, y = Q(x)$ sol $y_p = u \cdot y_1$

$= u(x)\, e^{-\int p(x)dx}$

① $\dfrac{dy}{dx} + y = 1$

$y = 1 + ce^{-x}$

② $\dfrac{dy}{dx} + y = x$

$y = (x-1) + ce^{-x}$

③ $\dfrac{dy}{dx} + y = x^2$

$y = x^2 - 2x + 2 + ce^{-x}$

particular solutions

$y=1$

$\frac{d}{dx}(1) + (1) = 0+1=1$

$y = x-1$

$\frac{d}{dx}(x-1) + (x-1) = 1 + x - 1 = x$

$y = x^2 - 2x + 2$

$(x^2-2x+2)' + (x^2-2x+2)$

$= 2x - 2 + x^2 - 2x + 2$

$= x^2$

y_p

homogeneous solutions

$y = ce^{-x}$

$(ce^{-x})' + (ce^{-x})$

$= -ce^{-x} + ce^{-x} = 0$

$y = ce^{-x}$

$(ce^{-x})' + (ce^{-x})$

$= -ce^{-x} + ce^{-x} = 0$

$y = ce^{-x}$

$(ce^{-x})' + (ce^{-x})$

$= -ce^{-x} + ce^{-x} = 0$

y_h

general solutions

$y = 1 + ce^{-x}$

$(1+ce^{-x})' + (1+ce^{-x})$

$= \cancel{-ce^{-x}} + 1 + \cancel{ce^{-x}}$

$= 1$

$y = x-1 + ce^{-x}$

$(x-1+ce^{-x})' + (x-1+ce^{-x})$

$= \cancel{1} - \cancel{ce^{-x}} + x \cancel{-1} + \cancel{ce^{-x}}$

$= x$

$y = x^2 - 2x + 2 + ce^{-x}$

$= x^2$

$y_p + y_h$

$$a_1(x)\boxed{y'} + a_2(x)\boxed{y} = g(x)$$

general Solution

1개의 y_p

infinitely many Solutions

...

$y_p + y_h$

y_p

$$y_p + \underbrace{C \cdot \boxed{x\text{식.}}}_{y_h}$$

$g(x)$

$$a_1(x)\boxed{y'} + a_2(x)\boxed{y} = 0$$

$$1\boxed{y'} + p(x)\boxed{y}$$

$$C \cdot e^{-\int p(x)\,dx}$$

Solving the Homogeneous DE : finding y_h

$$\frac{dy}{dx} + P(x)y = 0$$

$$y' + P(x)y = 0$$

homogeneous solution

$$y_h = f_h(x)$$

$$\frac{dy}{dx} = -P(x)y \qquad y' = -P(x)y$$

$$\frac{1}{y}\frac{dy}{dx} = -P(x) \qquad \frac{1}{y}y' = -P(x)$$

not a ratio

$$\int \frac{1}{y}\boxed{\frac{dy}{dx}dx} = -\int P(x)dx \qquad \int \frac{1}{y}y'dx = -\int P(x)dx \qquad dy = \frac{df}{dx}dx$$

$$\int \frac{1}{y}\boxed{dy} = -\int P(x)dx \qquad \int \frac{1}{y}dy = -\int P(x)dx$$

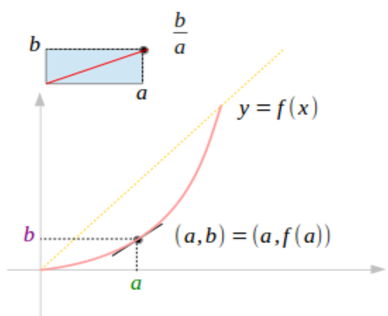
$$\ln|y| = -\int P(x)dx + C \qquad \ln|y| = -\int P(x)dx + C$$

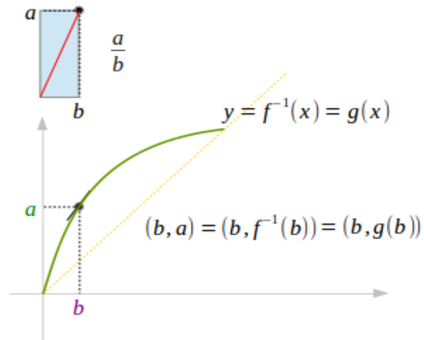
$$|y| = e^{-\int P(x)dx + C} \qquad |y| = e^{-\int P(x)dx + C}$$

$$y = \pm e^{-\int P(x)dx} \cdot e^{C} = \pm e^{C} \cdot e^{-\int P(x)dx}$$

$$y = c\,e^{-\int P(x)dx} \qquad y = c\,e^{-\int P(x)dx}$$

Derivatives of Inverse Functions





$m_1 = f'(a)$

$m_2 = g'(b)$

$$\boxed{m_1 m_2 = f'(a)\,g'(b) = 1} \Rightarrow \boxed{g'(b) = \frac{1}{f'(a)}} \Rightarrow \boxed{g'(b) = \frac{1}{f'(g(b))}}$$

$g(b) = a$

Derivatives of Inverse Functions

$$\boxed{m_1 m_2 = f'(a)\,g'(b) = 1} \Rightarrow \boxed{g'(b) = \frac{1}{f'(a)}} \Rightarrow \boxed{g'(b) = \frac{1}{f'(g(b))}}$$

$g(b) = a$

$$\Downarrow$$

$$\boxed{g'(x) = \frac{1}{f'(g(x))}}$$

To find g'(x)
(1) find f'(x)
(2) find 1 / f'(x)
(3) substitute x with g(x)

	$\frac{d}{dx}\ln x \Rightarrow \frac{1}{x}$	$\frac{d}{dx}e^x \Rightarrow e^x$
$f'(x)$	$(e^x)' \Rightarrow e^x$	$(\ln x)' = \frac{1}{x}$
$\frac{1}{f'(x)}$	➡ $\frac{1}{e^x}$	➡ $\frac{1}{1/x} = x$
$g'(x)$	➡ $\frac{1}{e^{\ln x}} = \frac{1}{x}$	➡ e^x

Derivative of ln |x|

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad x>0$$

$$\frac{d}{dx}\ln(-x) = \frac{1}{(-x)}\cdot(-1) = \frac{1}{x} \qquad x<0$$

$\ln x \qquad x>0 \qquad \xrightarrow{\frac{d}{dx}} \qquad \frac{1}{x}$

$\ln(-x) \qquad x<0 \qquad \xrightarrow{\frac{d}{dx}} \qquad \frac{1}{x}$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad x\neq 0$$

Method of Solving First Order ODEs

$$\frac{dy}{dx} + P(x)y = Q(x)$$ ← y_p?

$$y' + P(x)y = Q(x)$$

$$y = f(x)$$

$$\left[e^{+\int P(x)dx}\right]\left[\frac{dy}{dx} + P(x)y\right] = \left[e^{+\int P(x)dx}\right]Q(x)$$

$$y_1 = e^{-\int P(x)dx} \qquad \frac{1}{y_1} = e^{+\int P(x)dx}$$

$$\left[e^{+\int P(x)dx}\right]\frac{dy}{dx} + \left[\left[e^{+\int P(x)dx}\right]P(x)\right]y = \left[e^{+\int P(x)dx}\right]Q(x)$$

$$\left[e^{+\int P(x)dx}\right]\cdot P(x) = \frac{d}{dx}\left[e^{+\int P(x)dx}\right] \qquad f'(g(x))g'(x)$$

$$\left[e^{+\int P(x)dx}\right]\frac{dy}{dx} + \frac{d}{dx}\left[e^{+\int P(x)dx}\right]y = \left[e^{+\int P(x)dx}\right]Q(x)$$

$$\left[e^{+\int P(x)dx}\right]\frac{dy}{dx} + \frac{d}{dx}\left[e^{+\int P(x)dx}\right]y = \frac{d}{dx}\left[e^{+\int P(x)dx}\cdot y\right]$$

$$f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left[\left[e^{+\int P(x)dx}\right]\cdot y\right] = \left[e^{+\int P(x)dx}\right]Q(x)$$

$$\int \frac{d}{dx}\left[\left[e^{+\int P(x)dx}\right]\cdot y\right]dx = \int \left[e^{+\int P(x)dx}\right]Q(x)\,dx + c$$

$$\left[\left[e^{+\int P(x)dx}\right]\cdot y\right] = \int \left[e^{+\int P(x)dx}\right]Q(x)\,dx + c \Rightarrow y(x) = ce^{-\int P(x)dx} + e^{-\int P(x)dx}\cdot\left[\int Q(x)\cdot e^{+\int P(x)dx}\,dx\right]$$

$$\left(e^{+\int p(x)dx}\cdot y\right)' = e^{+\int p(x)dx}\cdot y' + e^{+\int p(x)dx}\cdot y$$

$$= e^{+\int p(x)dx}\cdot y' + e^{+\int p(x)dx}\, p(x)\cdot y$$

$$= e^{+\int p(x)dx}\cdot\left(y' + p(x)\cdot y\right)$$

think the reverse order

the given problem $\quad y' + p(x)\,y = q(x)$

$$e^{+\int p(x)dx}\cdot\left(y' + p(x)\,y\right) = q(x)\cdot e^{+\int p(x)dx}$$

$$\left(e^{+\int p(x)dx}\cdot y\right)' = q(x)\cdot e^{+\int p(x)dx}$$

$$\left(e^{+\int p(x)dx}\cdot y\right) = \int q(x)\cdot e^{+\int p(x)dx}\,dx$$

the given problem $\quad y' + p(x)\,y = q(x)$

$$\Rightarrow \quad \boxed{e^{+\int p(x)\,dx}} \cdot \left(y' + p(x)\,y \right) = q(x) \cdot \boxed{e^{+\int p(x)\,dx}}$$

$$\Rightarrow \quad \left(\boxed{e^{+\int p(x)\,dx}} \cdot \boxed{y} \right)' = q(x) \cdot \boxed{e^{+\int p(x)\,dx}}$$

$$\left(\boxed{e^{+\int p(x)\,dx}} \cdot \boxed{y} \right) = \int q(x) \cdot \boxed{e^{+\int p(x)\,dx}}\, dx + C$$

$$\left(\boxed{e^{+\int p(x)\,dx}} \cdot \boxed{y} \right) = \int q(x) \cdot \boxed{e^{+\int p(x)\,dx}}\, dx + C$$

$$\boxed{y} = \frac{1}{\boxed{e^{+\int p(x)\,dx}}} \int q(x) \cdot \boxed{e^{+\int p(x)\,dx}}\, dx + \frac{C}{\boxed{e^{+\int p(x)\,dx}}}$$

$$\boxed{y} = \underbrace{\boxed{e^{-\int p(x)\,dx}} \int q(x) \cdot \boxed{e^{+\int p(x)\,dx}}\, dx}_{\Downarrow\; y_p.} + \underbrace{C\,\boxed{e^{-\int p(x)\,dx}}}_{\Downarrow\; y_h}$$

1st Order Linear Equations

y_p 구하기

$$\boxed{1\cdot y' + \boxed{P(x)}\, y = Q(x)}$$

$$y_h = c\cdot e^{-\int P(x)\,dx}$$

$$y_p = u(x)\, e^{-\int P(x)\,dx}$$

$$\boxed{e^{+\int P(x)\,dx}}$$ Integrating Factor

- const. c X
- sign X

$$\boxed{e^{+\int P(x)\,dx}}\ \boxed{\left(1\cdot y' + P(x)\, y\right) = \left(Q(x)\right)}\ \boxed{e^{+\int P(x)\,dx}}$$

$$\left(\boxed{e^{+\int P(x)\,dx}\cdot}\ \boxed{y_p}\right)' = Q(x)\, e^{+\int P(x)\,dx}$$

$$e^{+\int P(x)\,dx}\cdot y_p = \int Q(x)\, e^{+\int P(x)\,dx}\,dx$$

$$y_p = e^{-\int P(x)\,dx}\cdot \int Q(x)\, e^{+\int P(x)\,dx}\,dx$$

$$y = y_h + y_p$$

$$y = \underbrace{c\cdot e^{-\int P(x)\,dx}}_{y_h} + \underbrace{e^{-\int P(x)\,dx}\cdot \int Q(x)\, e^{+\int P(x)\,dx}\,dx}_{y_p}$$

Calculus 1

Review : Exponential

Review : Logarithmic

Derivatives : Trig Derivatives

Differential Equation

First Order : Linear Equation

First Order Differential Equations (1P.pdf) .....
Linear Equation (2A.pdf)

cf) Partial Derivatives (9.4 Zill & Wright)

Exact Equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$\boxed{y(x) = ?}$ … 1 variable

devise another function $f(x,y)$ … 2 variables

$\boxed{\text{Exact}} \longrightarrow f(x,y)$ exists

total differential

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

$$= M(x,y)\,dx + N(x,y)\,dy = 0$$

Check $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$

$$\frac{\partial}{\partial y}(M) = \frac{\partial}{\partial x}(N)$$

$$df = 0 \qquad \boxed{f(x,y) = c}$$

$y(x) \leftarrow$ Curve $\leftarrow$ Surface $\cap$ 평면

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Check $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) \overset{?}{=} \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$

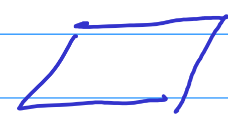
if yes, there exists $f(x,y)$ and

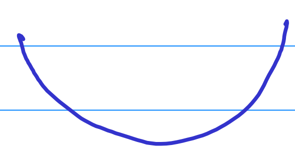
$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} \Rightarrow \boxed{df = 0}$$

$$\Rightarrow f = c$$

$$\underset{\displaystyle z}{\underset{\shortparallel}{f(x,y)}} = \underset{\displaystyle z}{\underset{\shortparallel}{\text{const.}}}$$

a surface in $\mathbb{R}^3$ — a plane parallel to x-y plane





intersection "$\cap$"

$\Downarrow$

curve . .

$\Downarrow$

$y = f(x)$ . . . Solution.

$$(2xy - 9x^2) + (2y + x^2 + 1)\frac{dy}{dx} = 0$$

$$(2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0$$

$$\boxed{\begin{matrix} \| \\ M \\ \| \\ \dfrac{\partial f}{\partial x} \end{matrix}}\ \begin{matrix} \\ dx + \\ \\ dx + \end{matrix}\ \boxed{\begin{matrix} \| \\ N \\ \| \\ \dfrac{\partial f}{\partial y} \end{matrix}}\ \begin{matrix} \\ dy = 0 \\ \\ dy = 0 \end{matrix}$$

* Exact?

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$2x = 2x \qquad \therefore \text{ exact}$$

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx + c$$

variable y may be included

$$= \int \frac{\partial f}{\partial x}\,dx + g(y)$$

$$= \int (2xy - 9x^2)\,dx + g(y)$$

$$f(x,y) = x^2y - 3x^3 + g(y) \quad ?$$

$$\frac{\partial f}{\partial y} = \cancel{x^2} + g'(y) = (2y + \cancel{x^2} + 1)$$

$$g'(y) = 2y + 1 \qquad g(y) = y^2 + y$$

$$f(x,y) = x^2y - 3x^3 + y^2 + y$$

$$df = 0$$

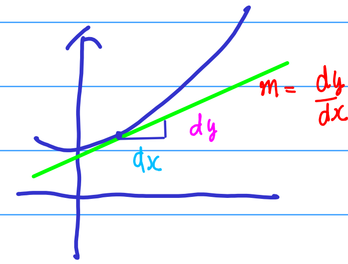
$$f(x,y) = x^2y - 3x^3 + y^2 + y = C$$

$$a(x,y) + b(x,y)\frac{dy}{dx} = 0 \qquad \underline{y(x)} = ?$$

$$\Downarrow$$

$$a(x,y)\,dx + b(x,y)\boxed{\frac{dy}{dx}\,dx} = 0$$

$$a(x,y)\,dx + b(x,y)\,dy = 0$$



$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} \Longrightarrow \boxed{df = 0}$$

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial N}{\partial x}\,dy = 0$$

$$\Downarrow$$

$$\boxed{f(x,y) = c}$$

$f(x,y)$

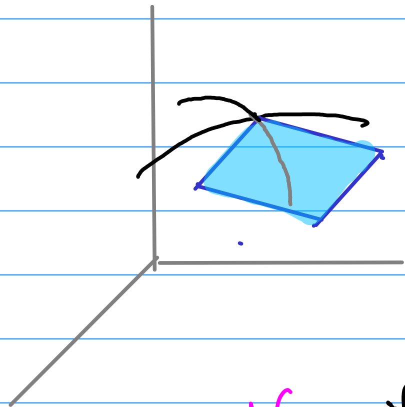
when

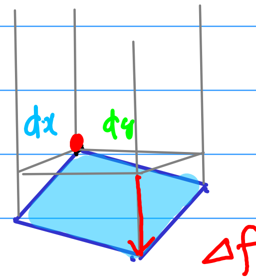
$x : dx$ — $x \to x+dx$

$y : dy$ — $y \to y+dy$

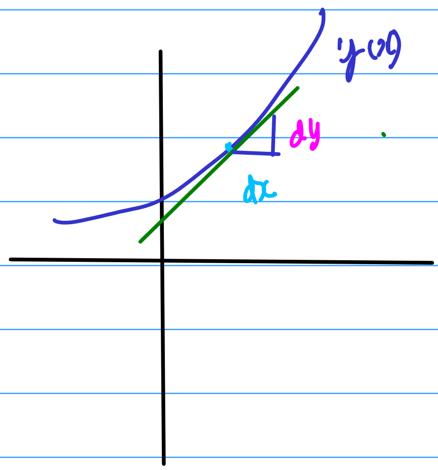
$f : df$ — change in the value of $f(x,y)$

$df = 0 \Rightarrow$ no change $\Rightarrow$ constant





total differential $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ ...... $\mathbb{R}^3$, $z = f(x,y)$



$$dy = \frac{dy}{dx} dx \quad \cdots\cdots \mathbb{R}^2, \; y = f(x)$$

$$\left(df = \frac{df}{dx} dx \right)$$

∗ exactness condition

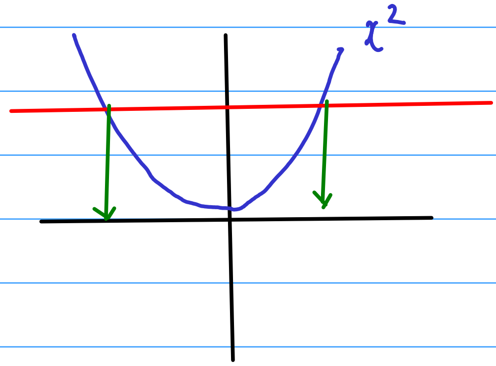
$$\boxed{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}$$

$$\frac{\partial}{\partial y}(M) = \frac{\partial}{\partial x}(N)$$

if this condition is not met,

there is no such $f(x,y)$

$x^2 = 2$



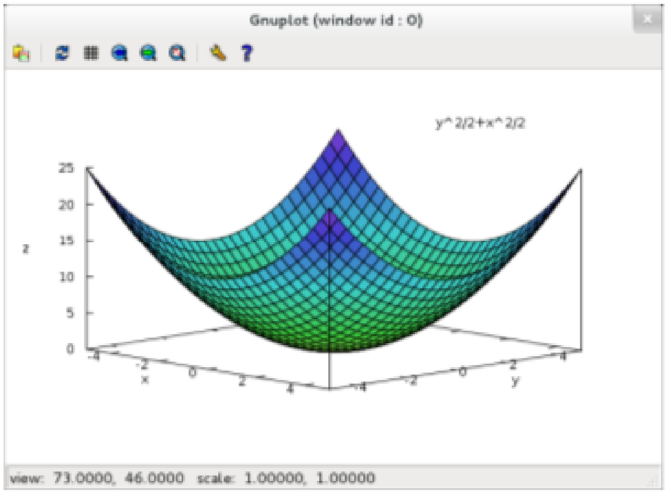
Exact Equation Method (2)

A differential form

$$P(x,y)dx + Q(x,y)dy$$

this differential form is **exact** in a region R if there is a function f(x,y) such that

$$P(x,y)dx + Q(x,y)dy$$
$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$



differential form

$$x\,dx + y\,dy$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dy = df$$

exact differential form

Since there exists such

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1$$

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + C$$

$$df = 0 \iff f = c$$

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} \qquad = c'$$

Exact Equation Method (4)

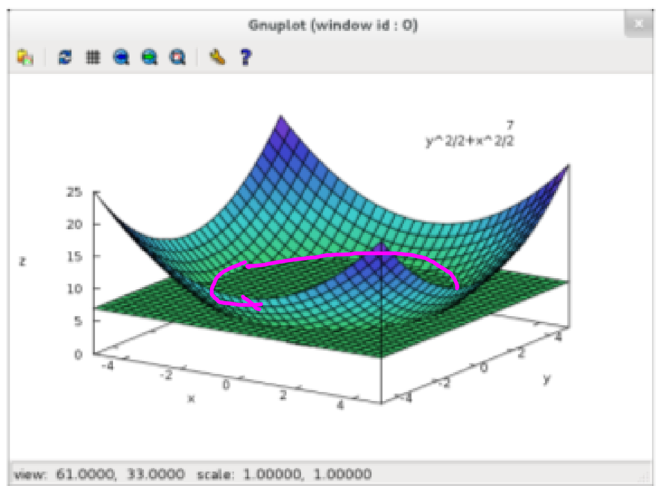
differential equation

$$x\,dx + y\,dy = 0$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dy = \boxed{df = 0}$$

The implicit solution is

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1 = c$$

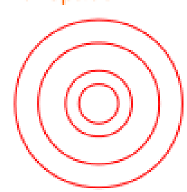


$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} = 7$$

The solution is all the points (x, y) that satisfies f(x,y) = 7

different c
In 3-d space

different c
In 2-d space



$f(x, y)$ $\qquad df = \begin{matrix} x : dx \rightarrow \\ y : dy \uparrow \end{matrix}$

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} \Longrightarrow \boxed{df = 0}$$

$\boxed{y(x) = ?}$ … 1 variable $\qquad \underline{f(x,y) = c}$

$f(x,y)$ … 2 variables $\qquad \boxed{f(x, y(x))}$

$$\underbrace{M(x,y)}_{= \frac{\partial f}{\partial x}}\,dx + \underbrace{N(x,y)}_{= \frac{\partial f}{\partial y}}\,dy = 0$$

① Verify "exactness"

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

② $\underbrace{M(x,y)}_{= \frac{\partial f}{\partial x}}$ $\qquad f(x,y) = \int M(x,y)\,dx + g(y)$

③ $\underbrace{N(x,y)}_{= \frac{\partial f}{\partial y}}$ $\qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + g'(y)$

$$= N(x,y)$$

④ $\underbrace{f(x,y)}_{= z} = \underbrace{c}_{= z}$ $\qquad$ f의 surface $\cap$ 평면 $\Rightarrow$ $y(x)$

Non-Exact Case

Zill (Ex 4)

$$xy\,dx + (2x^2+3y^2-20)\,dy = 0$$

$$\frac{\partial}{\partial y}(xy) = x \neq \frac{\partial}{\partial x}(2x^2+3y^2-20) = 4x$$

non-exact eq

$\boxed{y^3}$ ← Integrating Factor $\mu(x)$ $\mu(y)$

$$\boxed{xy \cdot y^3}\,dx + \boxed{(2x^2+3y^2-20)\,y^3}\,dy = 0$$

$$xy^4\,dx + (2x^2y^3+3y^5-20y^3)\,dy = 0$$

$$\frac{\partial}{\partial y}(xy^4) = 4xy^3 = \frac{\partial}{\partial x}(2x^2y^3+3y^5-20y^3) = 4xy^3$$

Now exact!

Non-Exact Case : Finding Integrating Factor

Exact $\longrightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\quad M_y = N_x$

$$M_y - N_x = 0$$

Non-Exact $\longrightarrow$ $M_y - N_x \neq 0$

$\frac{M_y - N_x}{N}$: x의 수식 ✓ $\quad I.F. = e^{\int \frac{M_y - N_x}{N} dx}$

$-\frac{M_y - N_x}{M}$: y의 수식 ✓ $\quad I.F. = e^{-\int \frac{M_y - N_x}{M} dy}$

Integrating Factor

- sign x
- const c x

$$\boxed{\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} \Leftarrow \frac{M_y - N_x}{N} \cdots\cdots x\text{의 수식}}$$

$$\boxed{\mu(y) = e^{\int \frac{-M_y + N_x}{M} dy} \Leftarrow -\frac{M_y - N_x}{M} \cdots y\text{의 수식}}$$

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = M_y \qquad \frac{\partial N}{\partial x} = N_x$$

difference

x 또는

y 수식

$$xy\,dx + (2x^2+3y^2-20)\,dy = 0$$

$$\frac{\partial}{\partial y}(xy) = x \neq \frac{\partial}{\partial x}(2x^2+3y^2-20) = 4x$$

non-exact eq

$$\frac{\partial M}{\partial y} = M_y = x \qquad \frac{\partial N}{\partial x} = N_x = 4x$$

x only expression

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$$

$$\mu(x) = e^{\int \frac{-3x}{(2x^2+3y^2-20)}\,dx}$$

$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$

$\frac{\partial}{\partial y}(xy) = x \neq \frac{\partial}{\partial x}(2x^2 + 3y^2 - 20) = 4x$

non-exact eq

$\frac{\partial M}{\partial y} = M_y = x \qquad \frac{\partial N}{\partial x} = N_x = 4x$

x only expression

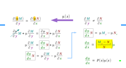
$\mu(y) = e^{\int \frac{-M_y + N_x}{M} dy} \Leftarrow -M + N \cdots$ y에 대해

$\mu(y) = e^{\int \frac{3\cancel{x}}{\cancel{x}y} dy} = e^{3\int \frac{1}{y} dy} = e^{\ln|y|^3}$

$= |y|^3$

$\mu(y) = y^3, \; -y^{-3} \Rightarrow y^3$

$xy \cdot y^3\,dx + (2x^2 + 3y^2 - 20)\,y^3\,dy = 0$



integrating factor 공식 유도 과정

$(f\cdot g)' = f'g + fg'$

② $\dfrac{\partial(\mu M)}{\partial y} = \dfrac{\partial(\mu N)}{\partial x}$ $\xleftarrow{\mu(x)}$ ① $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

y가 없는 수식

$$\cancel{\frac{\partial \mu}{\partial y}}^{\,0} M + \mu \frac{\partial M}{\partial y} = \boxed{\frac{\partial \mu}{\partial x}} N + \mu \frac{\partial N}{\partial x}$$

$$\mu \frac{\partial M}{\partial y} = \boxed{\frac{d \mu}{d x}} N + \mu \frac{\partial N}{\partial x}$$

$$\boxed{\frac{d \mu}{d x}} N = \mu \frac{\partial M}{\partial y} - \mu \frac{\partial N}{\partial x}$$

$$\boxed{\frac{d \mu}{d x}} N = \mu M_y - \mu N_x$$

$$\frac{d \mu}{d x} = \left(\frac{M_y - N_x}{N}\right)\mu$$

$$\frac{d \mu}{d x} = P(x)\mu(x)$$

만일 □에 y가 없는 수식이면

$$\frac{d \mu}{d x} - P(x)\mu = 0$$

$$\mu(x) = c e^{\int P(x)\,dx}$$

$$\mu(x) = \cancel{c}\, e^{\int \left(\frac{M_y - N_x}{N}\right) dx}$$

Substitution Method

$$\frac{dy}{dx} = g(x, y) \rightsquigarrow \left(\frac{y}{x}\right)$$ 항상 이런 형태가 있다.

$\left(\frac{y}{x}\right) = u \qquad y = ux$

Substitute   ex) $y = ux$

$$\boxed{y = h(x,u)} \Leftarrow \boxed{u = \Phi(x)}$$

$$\Downarrow$$

$$\boxed{\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}}$$

$$\frac{d}{dx}y = \frac{d}{dx}h(x,u)$$

$\frac{\partial h}{\partial x} = u \quad \frac{\partial h}{\partial u} = x \quad \frac{du}{dx} = u' \quad \Rightarrow \quad u + xu'$

$$g(x, h(x,u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$g(x, h(x,u)) = h_x(x,y) + h_u(x,u)\frac{du}{dx}$$

$$\frac{dy}{dx} = g(x, y)$$

* substitute y

and y'

both y & y'

$\left(\frac{y}{x}\right) \to u$

$y \to ux$

$y' \to u'x + u$

Paul's online math note ex1)

$$xyy' + 4x^2 + y^2 = 0$$

divide by x^2

$$\left(\frac{y}{x}\right)y' + 4 + \left(\frac{y}{x}\right)^2 = 0$$

u u

$$u = \frac{y}{x} = \frac{y(x)}{x}$$

$$\frac{x\text{의 수식}}{x\text{의 식}}$$

substitute both y and y'

$$y = xu$$

$$y' = u + xu'$$

$$\left(\frac{y}{x}\right)y' + 4 + \left(\frac{y}{x}\right)^2 = 0$$

$$\left(\frac{xu}{x}\right)\cdot\left(u + xu'\right) + 4 + \left(\frac{xu}{x}\right)^2 = 0$$

$$u(u + xu') + 4 + u^2 = 0$$

$$2u^2 + xuu' + 4 = 0$$

$$2u^2 + xuu' + 4 = 0$$

$$\frac{xu'u}{u} = \frac{-2u^2 - 4}{u}$$

$$xu' = -\frac{4+2u^2}{u}$$

$$\left(\frac{u}{4+2u^2}\right) u' = -\frac{1}{x} \qquad u' = \frac{du}{dx}$$

$$\frac{u}{4+2u^2} \cdot \frac{du}{dx} dx = -\frac{1}{x} dx$$

$$\frac{u}{4+2u^2} du = -\frac{1}{x} dx$$

$$\frac{1}{4}\int \frac{4u}{4+2u^2} du = \int -\frac{1}{x} dx$$

$$\ln\left(4+2v^2\right)^{\frac{1}{4}} = \ln\left(x\right)^{-1} + c$$

sides and do a little rewriting

$$\left(4+2v^2\right)^{\frac{1}{4}} = \mathbf{e}^{\ln(x)^{-1}+c} = \mathbf{e}^{c}\mathbf{e}^{\ln(x)^{-1}} = \frac{c}{x}$$

$$\frac{1}{4}\ln|4+2u^2| = -\ln|x| + C$$

$$\ln|4+2u^2|^{\frac{1}{4}} = \ln|x|^{-1} + C$$

$$|4+2u^2|^{\frac{1}{4}} = e^{\ln|x|^{-1}+C} = e^{\ln|x|^{-1}} e^{C}$$

$$(4+2u^2)^{\frac{1}{4}} = \frac{1}{x} C$$

$$(4+2u^2)^{\frac{1}{4}} = \frac{1}{x} C$$

$$(4+2u^2) = \left(\frac{c}{x}\right)^4$$

$$u = \frac{y}{x}$$

$$4+2\left(\frac{y}{x}\right)^2 = \left(\frac{c}{x}\right)^4$$

$$2\left(\frac{y}{x}\right)^2 = \left(\frac{C_1}{x^4} - 4\right)$$

$$\frac{y^2}{x^2} = \frac{1}{2}\left(\frac{c_1 - 4x^4}{x^4}\right)$$

$$y^2 = \frac{1}{2}x^2\left(\frac{C_1 - 4x^4}{x^4}\right)$$

$$= \frac{C_1 - 4x^4}{2x^2}$$

Paul's online math note ex2)

$$xy' = y(\ln x - \ln y) \qquad \boxed{x > 0}$$

$$xy' = y \ln\left(\frac{x}{y}\right)$$

$$y' = \left(\frac{y}{x}\right) \ln\left(\frac{x}{y}\right) \qquad \frac{y}{x} = u \qquad y = xu$$

$$y' = u + xu'$$

$$(u + xu') = u \ln\left(\frac{1}{u}\right)$$

$$xu' = u \ln\left(\frac{1}{u}\right) - u$$

$$\int \frac{1}{u\left(\ln\left(\frac{1}{u}\right) - 1\right)} \frac{du}{dx} dx = \int \frac{1}{x} dx$$

u

$$\boxed{\begin{aligned} t &= \ln\left(\tfrac{1}{u}\right) - 1 \\ dt &= \left(\frac{1}{\frac{1}{u}}\right)\left(\tfrac{1}{u}\right)' du \\ dt &= u \cdot (-u^{-2})\, du \\ dt &= -\tfrac{1}{u}\, du \end{aligned}}$$

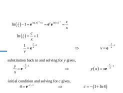
$$\int \frac{1}{u\left(\ln\left(\frac{1}{u}\right) - 1\right)} du = \int \frac{1}{x} dx$$

$$\int \frac{-1}{\left(\ln\left(\frac{1}{u}\right) - 1\right)} \left(\frac{-1}{u} du\right)$$

$$-\int \frac{1}{\left(\ln\left(\frac{1}{u}\right) - 1\right)} dt = \int \frac{1}{x} dx \qquad dt = u\, du$$

$$\ln\left(\ln\left(\tfrac{1}{u}\right) - 1\right) = -(\ln x + c)$$

$$\ln\left(\ln\left(\tfrac{1}{u}\right) - 1\right) = c - \ln x$$



$$\ln\left(\ln\left(\frac{1}{u}\right) - 1\right) = C - \ln x$$

$$\ln\left(\frac{1}{u}\right) - 1 = e^{C - \ln x}$$

$$= c_1 e^{\ln \frac{1}{x}}$$

$$= \frac{c_1}{x}$$

$$\ln\left(\frac{1}{u}\right) - 1 = \frac{c_1}{x}$$

$$\ln\left(\frac{1}{u}\right) = \frac{c_1}{x} + 1$$

$$\frac{1}{u} = e^{\frac{c_1}{x} + 1}$$

$$u = e^{-\frac{c_1}{x} - 1}$$

$$\frac{y}{x} = e^{-\frac{c_1}{x} - 1}$$

$$\boxed{y = x \cdot e^{-\frac{c_1}{x} - 1}}$$

* $\int \tan x \, dx$ → substitution $u = \cos x$

$\int t^2 \sin(2t) \, dt$ →

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

$u = \cos x$

$du = -\sin x \, dx$

$= \int \frac{-1}{u} du = -\ln|u| = \ln|u|^{-1}$

$= \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|$

integrating factor $\mu(x)$

→ multiplied to the original differential eq

$y' + p(x) y = Q(x)$

$\mu(x)\left(y' + p(x) y\right) = \left(Q(x)\right) \mu(x)$

40p Ex 6)

$\frac{2\pi}{39}$

$$\int f \, g' \, dt = f \, g - \int f' \, g \, dt$$

$$\int t^2 \sin(2t) \, dt$$

$$= -\tfrac{1}{2} t^2 \cos(2t) - \int (t^2)' \left[\tfrac{-1}{2} \cos(2t)\right] dt$$

$$= -\tfrac{1}{2} t^2 \cos(2t) + \boxed{\int t \cos(2t) \, dt}$$

$$\boxed{t \left[\tfrac{1}{2} \sin(2t)\right] - \int t' \left[\tfrac{1}{2} \sin(2t)\right] dt}$$

$$\tfrac{1}{2} t \sin(2t) - \underline{\tfrac{1}{2} \int \sin(2t) \, dt}$$

$$-\tfrac{1}{2} \left[-\tfrac{1}{2} \cos(2t) \right]$$

$$-\tfrac{1}{2} t^2 \cos(2t) + \tfrac{1}{2} t \sin(2t) + \tfrac{1}{4} \cos(2t)$$

$$-\tfrac{1}{2} \left(t^2 - \tfrac{1}{2}\right) \cos(2t) + \tfrac{1}{2} t \sin(2t)$$

(%i2) integrate(%, t);

(%o2) $$\frac{4\,t \sin(2\,t) + \left(2 - 4\,t^2\right) \cos(2\,t)}{8}$$

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + c_1 = (x-1)e^x + c_1$$

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx = x^2 e^x - 2x e^x + 2e^x + c_2 = (x^2 - 2x + 2)e^x + c_2$$

$$\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c_3 = (x^3 - 3x^2 + 6x - 6)e^x$$

$$\int x e^x\,dx = (x-1)e^x + c_1$$

$$\int x^2 e^x\,dx = (x^2 - 2x + 2)e^x + c_2$$

$$\int x^3 e^x\,dx = (x^3 - 3x^2 + 6x - 6)e^x + c_3$$

$$\int e^x\,dx = \int \left\{\frac{d}{dx}e^x\right\} dx = e^x + c$$

$$\int x e^{x^2/2}\,dx = \int \left\{\frac{d}{dx}e^{x^2/2}\right\} dx = e^{x^2/2} + c$$

$$\int x^2 e^{x^3/3}\,dx = \int \left\{\frac{d}{dx}e^{x^3/3}\right\} dx = e^{x^3/3} + c$$

Chain Rule

$$\frac{df}{dg} \cdot \frac{dg}{dx}$$

$$f(g(x)) \xrightarrow{\frac{d}{dx}} f'(g(x))\, g'(x)$$

$$\xleftarrow[\int \cdot\, dx]{}$$

Substitution rule

Product Rule

$$(fg)' = f'g + fg'$$

$$fg' = (f.g)' - f'g$$

$$\int fg'\,dx = \int (f.g)'\,dx - \int f'g\,dx$$

$$\int fg'\,dx = fg - \int f'g\,dx$$

$$(x^n)' = n x^{n-1}$$

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}$$

$$\int \frac{-1}{t^2} dt = \int t^{-2} dt = \frac{1}{-2+1} t^{-2+1} = -1 \cdot t^{-1} = \frac{-1}{t}$$

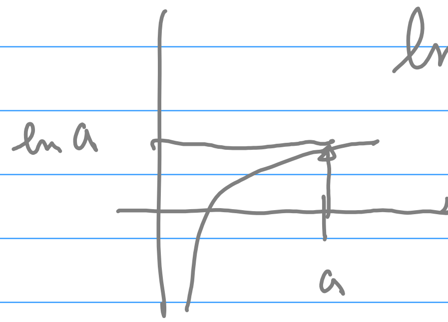
$$\left(-\frac{1}{t}\right)' = (-t^{-1})' = -(-1) t^{-1-1} = t^{-2} = \frac{1}{t^2}$$

$$(e^x)' = e^x$$

$$(e^{2x+3})' = e^{2x+3}\cdot(2x+3)' = e^{2x+3}\cdot 2$$

$$(e^{x^2+1})' = e^{x^2+1}\cdot(x^2+1)' = e^{x^2+1}\cdot 2x$$

$$(e^{x\ln a})' = e^{x\ln a}\cdot(x\ln a)' = e^{x\ln a}\cdot \ln a$$



$\ln a = \log_e a$ ⋯ constant

Δx

$$a = e^{\log_e a} = 10^{\log_{10} a} = 99^{\log_{99} a}$$

$$= e^{\ln a}$$

$$e^{x\ln a} = (e^x)^{\ln a} = (e^{\ln a})^x$$

$$(2^2)^3 = 4^3 = 64$$

$$(2^3)^2 = 8^2 = 64$$

$$\Downarrow$$

$$a^x$$

$$(e^{x\ln a})' = e^{x\ln a}\cdot \ln a$$

$$(a^x)' = (a^x)\cdot \ln a$$

$$\frac{d}{dx}\left(\log_a x\right) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$$

$$= \frac{1}{\ln a}\frac{d}{dx}\left(\ln x\right)$$

$$= \frac{1}{x \ln a}$$

that a was a constant and so $\ln a$ is also

Putting all this together gives,

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}$$

Here is a summary of the derivatives in this section.

$\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x$	$\frac{d}{dx}\left(a^x\right) = a^x \ln a$
$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$	$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}$

Pauls online math note
Calculus 1

Integral :
    Computing Indefinite Integrals
    Substitution Rule
    More about substitution rule

a : c : e :

a ~~b~~
c ~~d~~
e ~~f~~
g ~~h~~

Differential Equation
1st Order
    Exact Equation
    Substitution

다루고

$f(x,y) = c$

range   Graph X

상수 주어지기

initial   condi

IVP

2.2 Separable

Sep Ex1) $(1+x)\,dy - y\,dx = 0$

Ex 2) $\dfrac{dy}{dx} = -\dfrac{x}{y}$ $y(4) = -3$

$\begin{cases} x = 4 \\ y = -3 \end{cases}$ $(4, -3)$

Ex3) $\dfrac{dy}{dx} = y^2 - 4$

2.3 Linear Eq

60p Ex1) $\frac{dy}{dx} - 3y = 6$

Ex2) $x\frac{dy}{dx} - 4y = x^6 e^x$

Ex3) $(x^2-9)\frac{dy}{dx} + xy = 0$

Exact Equation

P90 Ex1) $2xy\,dx + (x^2-1)\,dy = 0$

Ex2) $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$

Ex3) $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$, $y(0)=2$

$\int x e^x \, dx =$

$\int x^2 e^x \, dx =$

$\int x^3 e^x \, dx =$

$\int x \cos x \, dx =$ $\qquad$ $\int x \cos \frac{x}{k} \, dx$

$\int x^2 \cos x \, dx =$ $\qquad$ $\int x^2 \cos \frac{x}{k} \, dx$

$\int x \sin x \, dx =$ $\qquad$ $\int x \sin \frac{x}{k} \, dx$

$\int x^2 \sin x \, dx =$ $\qquad$ $\int x^2 \sin \frac{x}{k} \, dx$

(%i1) `integrate( x*%e^x, x );`

(%o1) $(x-1)\,\%e^{x}$

(%i2) `integrate( x^2*%e^x, x );`

(%o2) $\left(x^2-2\,x+2\right)\,\%e^{x}$

(%i3) `integrate( x^3*%e^x, x );`

(%o3) $\left(x^3-3\,x^2+6\,x-6\right)\,\%e^{x}$

(%i4) `integrate( x*cos(x), x );`

(%o4) $x\,\sin(x)+\cos(x)$

(%i5) `integrate( x^2*cos(x), x );`

(%o5) $\left(x^2-2\right)\,\sin(x)+2\,x\,\cos(x)$

(%i6) `integrate( x*sin(x), x );`

(%o6) $\sin(x)-x\,\cos(x)$

(%i7) `integrate( x^2*sin(x), x );`

(%o7) $2\,x\,\sin(x)+\left(2-x^2\right)\,\cos(x)$

Substitution Rule

$$\int f(\underset{\substack{\| \\ u}}{g(x)})\, g'(x)\, dx = \int f(u)\, du$$

Chain rule

$$\frac{d}{dx} f(g(x)) = \frac{d}{dg} f(g)\, \frac{d}{dx} g(x)$$

① $\int \cos x \boxed{\sin x \quad dx} = -\int (\cos x)(\cos x)' \, dx$

$= -\int u \, du$

$= -\frac{1}{2}u^2 + C$

$\cos x \sin x = \frac{1}{2}$ $\qquad = -\frac{1}{2}(\cos x)^2 + C$

$\parallel$

② $\int \frac{1}{2}\sin 2x \, dx = -\frac{1}{4}\cos 2x + C$

$= -\frac{1}{4}\left((\cos x)^2 - (\sin x)^2\right) + C$

$= -\frac{1}{4}\left((\cos x)^2 - (1 - (\cos x)^2)\right) + C$

$= -\frac{1}{2}(\cos x)^2 + \frac{1}{4} + C$

$= -\frac{1}{2}(\cos x)^2 + C'$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Ubuntu . CAS (Computer Algebra System)

wx Maxima ← Macsyma

x Maxima

60 MIT

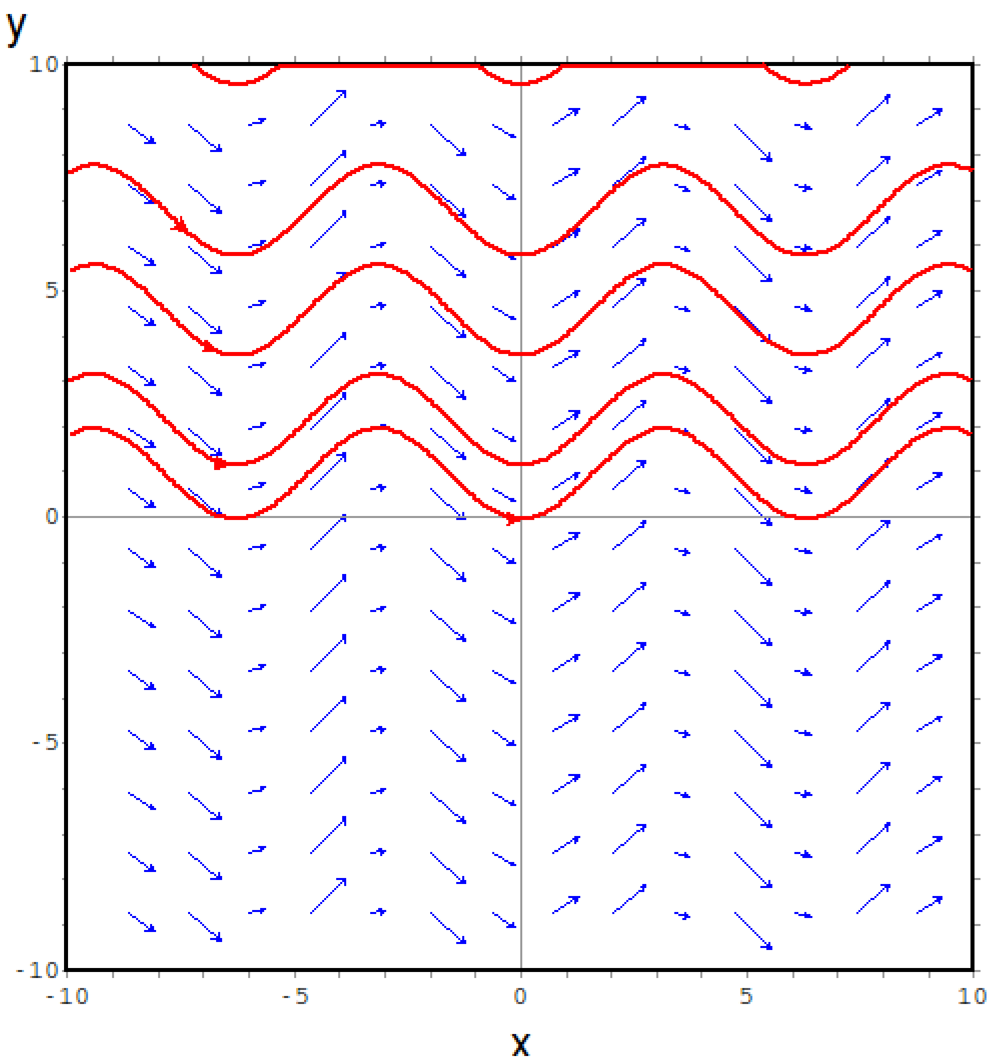
direction field → plot df expert

Octave

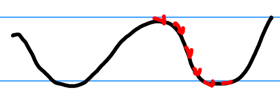
Matlab ⟷ octave

Mapple

Mathematica ⟷ wxMaxima



y
10
5
0
-5
-10
-10
-5
0
5
10
x



$$\frac{dy}{dx} = \sin(y)$$

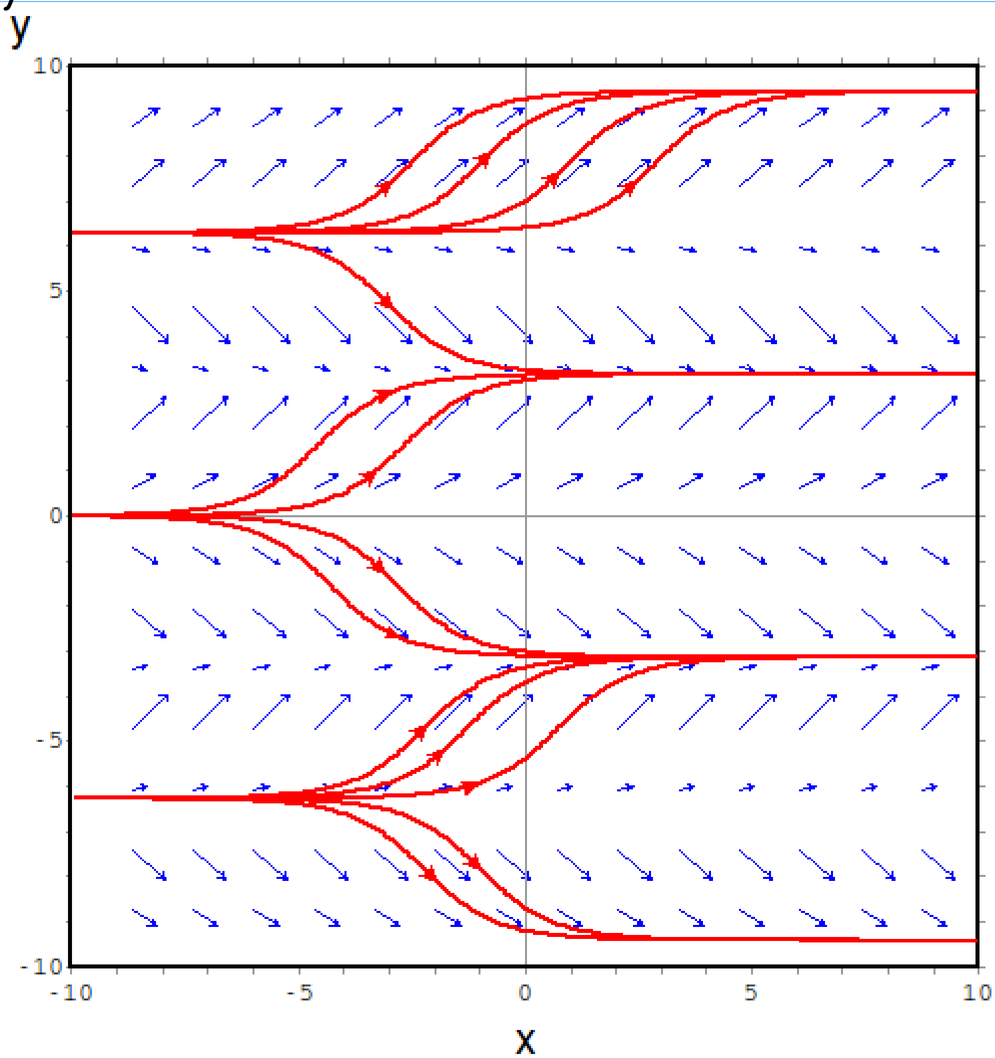
(x,y) tangent slope

$y=\pi$: tangent slope $= \frac{dy}{dx} = \sin(y) = 0$ $\rightarrow$

$y=\frac{\pi}{2}$: tangent slope $= \frac{dy}{dx} = \sin(y) = 1$ $\nearrow$

$y=0$: tangent slope $= \frac{dy}{dx} = \sin(y) = 0$ $\rightarrow$

$y=-\frac{\pi}{2}$: tangent slope $= \frac{dy}{dx} = \sin(y) = -1$ $\searrow$



$\pi = 3.14$

$\frac{\pi}{2} = \frac{3.14}{2}$