Bilateral Laplace Transform (6A)

Young Won Lim
2/23/15

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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An Improper Integration

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Complex Number

Real Number

$$s = \sigma + i\,\omega$$

$\Re\{s\}$ real part

$\Im\{s\}$ imag part

Real Number

t Integration Variable

The improper integral **converges** if the limit defining it exists.

Laplace transforms of 1 and exp(−at)

$1 \rightarrow \boxed{L} \rightarrow \dfrac{1}{s}$

$$F(s) = \int_0^{\infty} 1 \cdot e^{-st}\, dt = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s0}\right]$$

$$-s < 0 \Rightarrow \lim_{b\to\infty} e^{-sb} = 0 \qquad s > 0 \Rightarrow F(s) = \frac{1}{s}$$

$e^{-at} \rightarrow \boxed{L} \rightarrow \dfrac{1}{s+a}$

$$F(s) = \int_0^{\infty} e^{-at} \cdot e^{-st}\, dt = \lim_{b\to\infty}\left[-\frac{1}{(s+a)}e^{-(s+a)t}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{(s+a)}e^{-(s+a)b} + \frac{1}{(s+a)}e^{-(s+a)0}\right]$$

$$-(s+a) < 0 \Rightarrow \lim_{b\to\infty} e^{-(s+a)b} = 0 \qquad s > -a \Rightarrow F(s) = \frac{1}{(s+a)}$$

Laplace transforms of exp(+at) and exp(–at)

e^{-at} → **L** → $\dfrac{1}{s+a}$

$$F(s) = \int_0^{\infty} e^{-at} \cdot e^{-st}\, dt = \lim_{b\to\infty} \left[-\frac{1}{(s+a)} e^{-(s+a)t} \right]_0^b = \lim_{b\to\infty} \left[-\frac{1}{(s+a)} e^{-(s+a)b} + \frac{1}{(s+a)} e^{-(s+a)0} \right]$$

$$-(s+a) < 0 \Rightarrow \lim_{b\to\infty} e^{-(s+a)b} = 0 \qquad s > -a \Rightarrow F(s) = \frac{1}{(s+a)}$$

e^{+at} → **L** → $\dfrac{1}{s-a}$

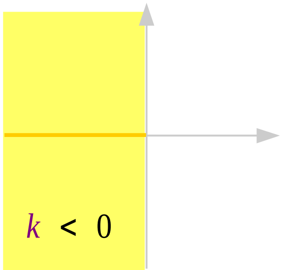
$$F(s) = \int_0^{\infty} e^{+at} \cdot e^{-st}\, dt = \lim_{b\to\infty} \left[-\frac{1}{(s-a)} e^{-(s-a)t} \right]_0^b = \lim_{b\to\infty} \left[-\frac{1}{(s-a)} e^{-(s-a)b} + \frac{1}{(s-a)} e^{-(s-a)0} \right]$$

$$-(s-a) < 0 \Rightarrow \lim_{b\to\infty} e^{-(s-a)b} = 0 \qquad s > +a \Rightarrow F(s) = \frac{1}{(s-a)}$$

Converging Improper Integrals

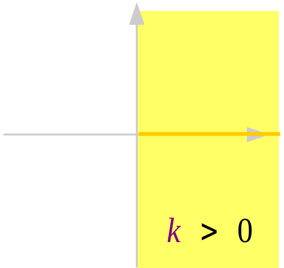
$$\int_0^\infty e^{+kt}\,dt = \left[+\frac{1}{k}e^{kt}\right]_0^\infty = +\frac{1}{k}\cdot\left(e^{k\cdot\infty} - e^{+k\cdot 0}\right) = -\frac{1}{k}$$

$k > 0$



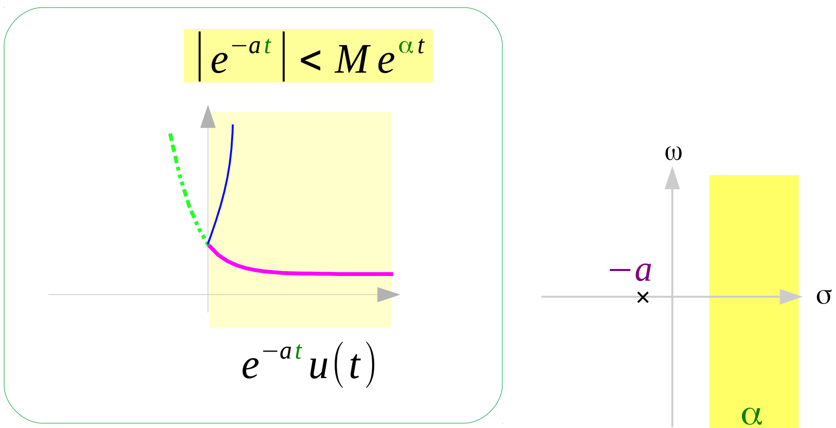
$$\int_{-\infty}^0 e^{-kt}\,dt = \left[-\frac{1}{k}e^{-kt}\right]_{-\infty}^0 = -\frac{1}{k}\cdot\left(e^{-k\cdot 0} - e^{-k\cdot\infty}\right) = -\frac{1}{k}$$

$k < 0$

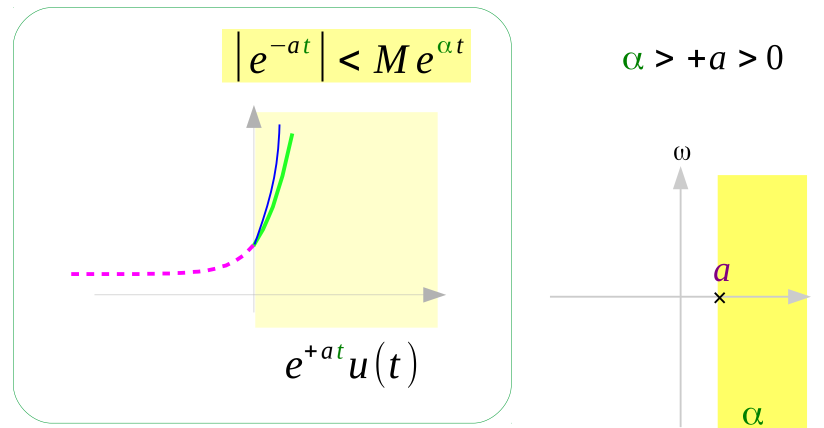


Existence of Laplace Transforms

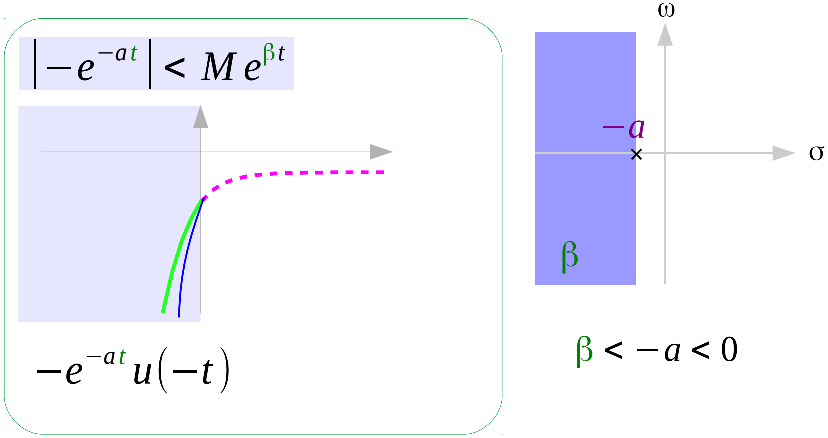
Right-sided function



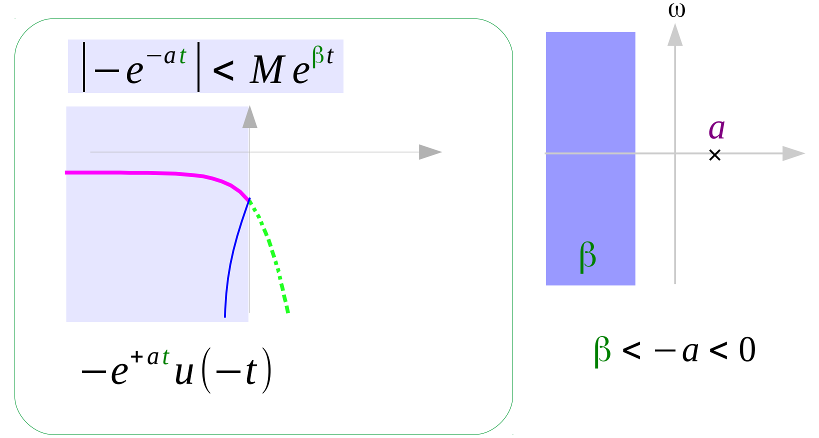
Right-sided function



Left-sided function

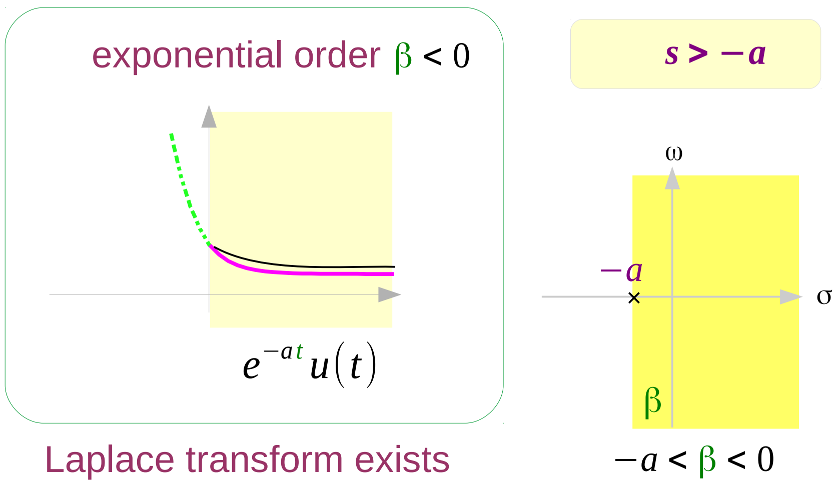


Left-sided function

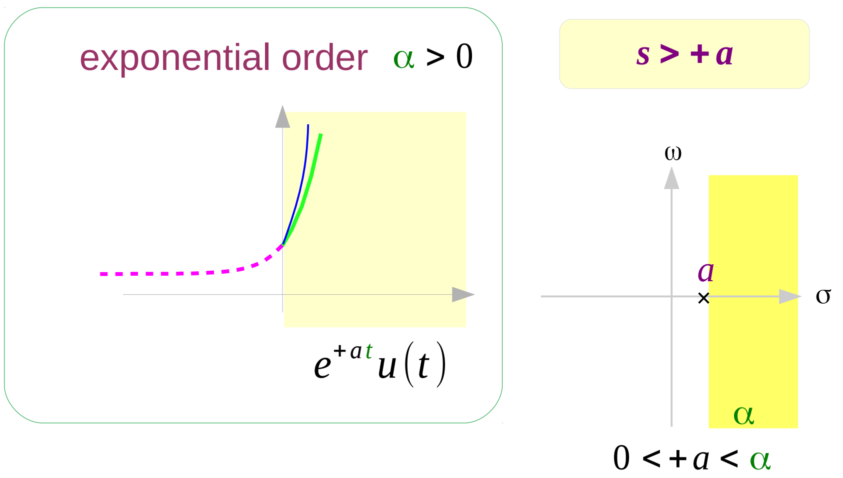


ROC and Exponential Order

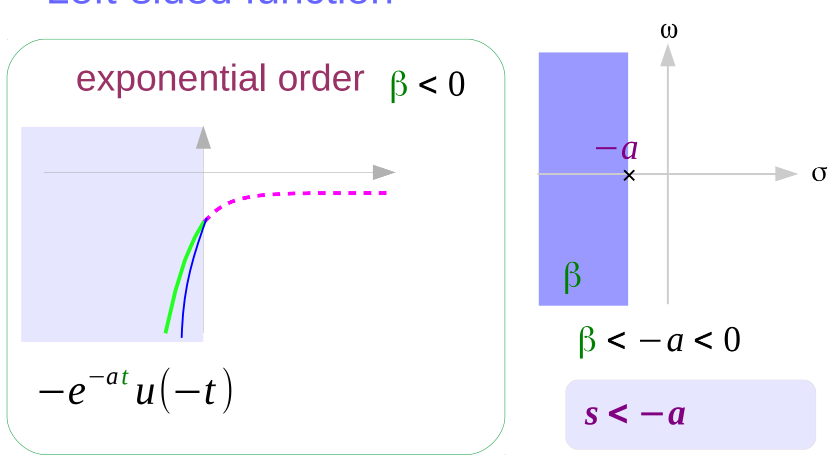
Right-sided function



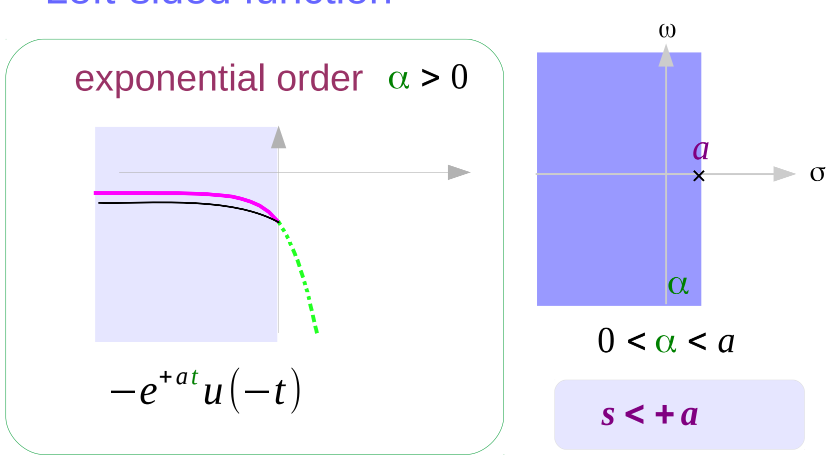
Right-sided function



Left-sided function



Left-sided function



Improper Integrals of f(t)u(+t) and f(t)u(−t)

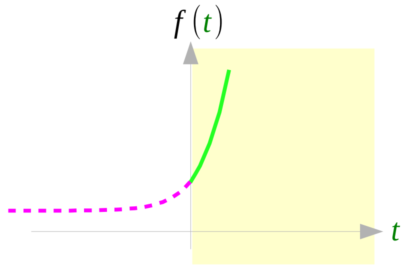
$f(t) \rightarrow \boxed{L} \rightarrow F(s)$

$\int_{0}^{+\infty} \cdot \, dt$

$$F(s) = \int_{0}^{\infty} f(t) \cdot e^{-st} \, dt$$

$$(0^{-} < t < +\infty)$$

$$f(t)u(+t)$$



right side of f(t) is used

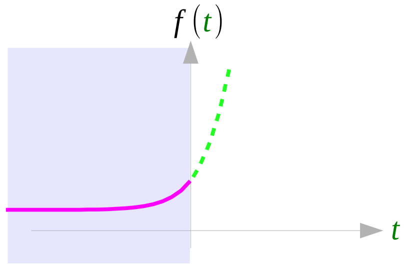
$f(t) \rightarrow \boxed{\hat{L}} \rightarrow G(s)$

$\int_{-\infty}^{0} \cdot \, dt$

$$G(s) = \int_{-\infty}^{0} f(t) \cdot e^{-st} \, dt$$

$$(-\infty < t < 0^{+})$$

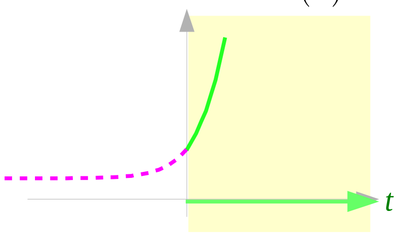
$$f(t)u(-t)$$



left side of f(t) is used

Improper Integrals of $e^{at}u(+t)$ and $e^{at}u(-t)$

$e^{at}u(t)$



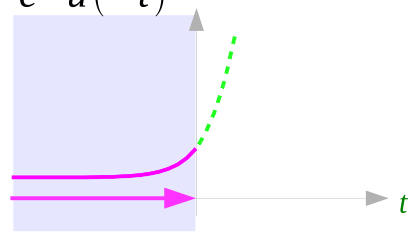
$$F(s) = \int_0^{\infty} e^{+at}\cdot e^{-st}\,dt$$

$$= \left[-\frac{1}{(s-a)}e^{-(s-a)t}\right]_0^{\infty}$$

$s > +a$ ⇒

$$\int_0^{\infty} e^{+at}\cdot e^{-st}\,dt = \frac{1}{(s-a)}$$

$e^{at}u(-t)$



$$G(s) = \int_{-\infty}^{0} e^{+at}\cdot e^{-st}\,dt$$

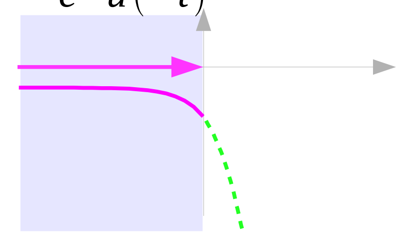
$$= \left[-\frac{1}{(s-a)}e^{-(s-a)t}\right]_{-\infty}^{0}$$

$s < +a$ ⇒

$$\int_{-\infty}^{0} e^{+at}\cdot e^{-st}\,dt = -\frac{1}{(s-a)}$$

$$G(s) = -F(s)$$

$-e^{at}u(-t)$



$$F(s) = \int_{-\infty}^{0} -e^{+at}\cdot e^{-st}\,dt$$

$$= \left[\frac{1}{(s-a)}e^{-(s-a)t}\right]_{-\infty}^{0}$$

$s < +a$ ⇒

$$\int_{-\infty}^{0} -e^{+at}\cdot e^{-st}\,dt = \frac{1}{(s-a)}$$

$$\int_{-\infty}^{0} -e^{+at}\cdot e^{-st}\,dt = F(s)$$

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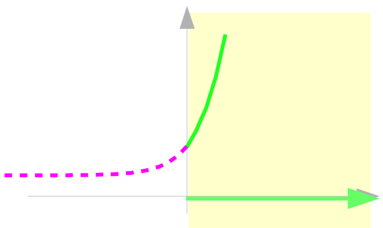
Two functions of s : G(s) = -F(s)

$f(t)u(t)$



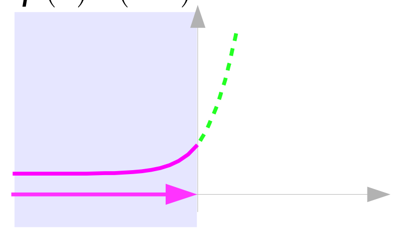
$$F(s) = \int_{0}^{\infty} f(t)\cdot e^{-st}\, dt$$

$s > +a$

$f(t)$ → $\mathcal{L}$ → $F(s)$, $s > +a$

$\int_{0}^{\infty} \cdot\, dt$

$f(t)u(-t)$



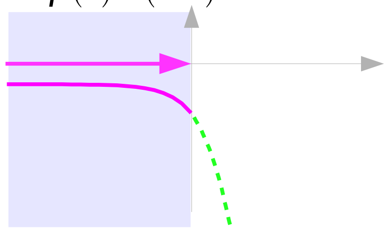
$$G(s) = \int_{-\infty}^{0} f(t)\cdot e^{-st}\, dt$$

$s < +a$

$f(t)$ → $\hat{\mathcal{L}}$ → $G(s)$

$\int_{-\infty}^{0} \cdot\, dt$

$-f(t)u(-t)$



$$F(s) = \int_{-\infty}^{0} -f(t)\cdot e^{-st}\, dt$$

$s < +a$

$-f(t)$ → $\hat{\mathcal{L}}$ → $F(s)$, $s < +a$

$\int_{-\infty}^{0} \cdot\, dt$

the same function of s $F(s)$

the different ROC's $s > +a$ $s < +a$

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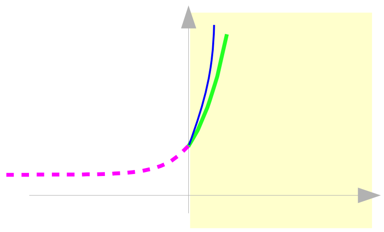
Improper Integrals of One-Sided Functions

Right-sided function

exponential order $\alpha > 0$



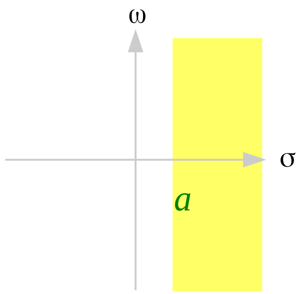
$e^{+at}u(t)$

$$e^{+at}u(+t) \rightarrow \boxed{\mathcal{L}} \rightarrow \frac{1}{s-a}$$

$$F(s) = \int_0^{\infty} e^{+at} \cdot e^{-st}\, dt$$

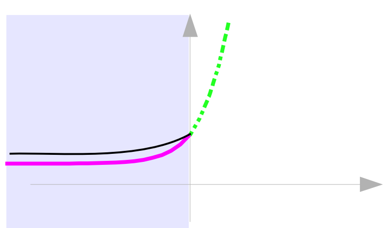
$s > +a$ ⇒ $F(s) = \dfrac{1}{(s-a)}$

$a > 0$



Left-sided function

exponential order $\alpha > 0$



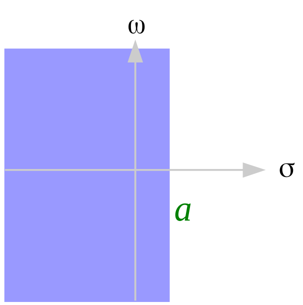
$+e^{+at}u(-t)$

$$e^{+at}u(-t) \rightarrow \boxed{\hat{\mathcal{L}}} \rightarrow \frac{-1}{s-a}$$

$$G(s) = \int_{-\infty}^{0} e^{+at} \cdot e^{-st}\, dt$$

$s < +a$ ⇒ $G(s) = \dfrac{-1}{(s-a)} = -F(s)$

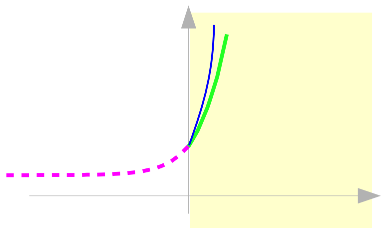
$a > 0$



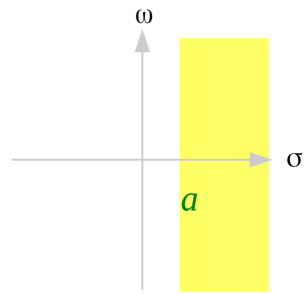
The Same Formula with Different ROCs

Right-sided function

exponential order $\alpha > 0$



$e^{+at}u(t)$



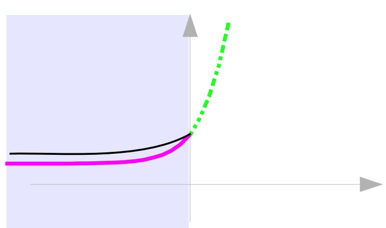
$\frac{1}{(s-a)}$ $s > +a$

$\frac{1}{(s-a)}$ $s < +a$

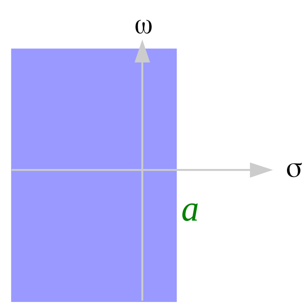
$a > 0$

Left-sided function

exponential order $\alpha > 0$

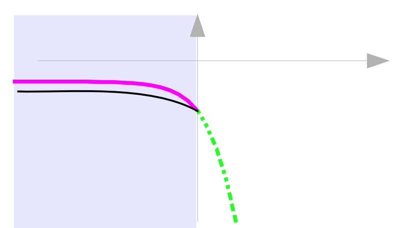


$+e^{+at}u(-t)$

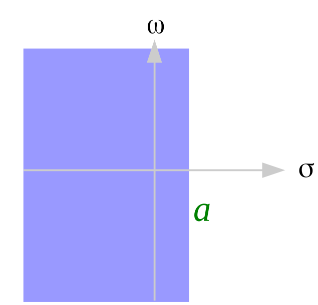


Left-sided function

exponential order $\alpha > 0$



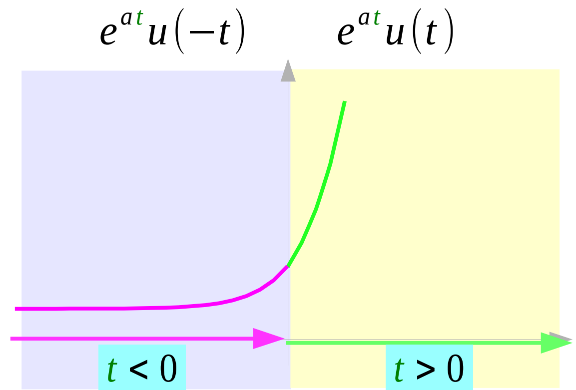
$-e^{+at}u(-t)$



ROCs and one-sided functions

$e^{at} \quad (t < 0)$

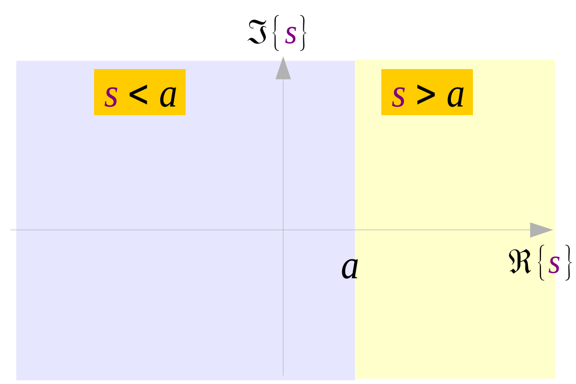
$\int_{-\infty}^{0} e^{+at} \cdot e^{-st} \, dt$



$e^{at} \quad (t > 0)$

$\int_{0}^{\infty} e^{+at} \cdot e^{-st} \, dt$

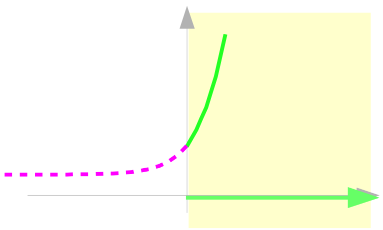
$\frac{-1}{(s-a)} \quad s < a$



$\frac{1}{(s-a)} \quad s > a$

Improper Integrals of f(t)u(+t) and f(-t)u(+t)

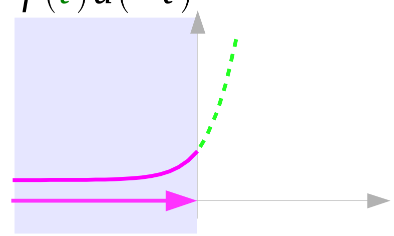
$f(t)u(t)$



$$\mathbf{L}\{f(t)u(t)\} = F(s)$$

$$= \int_0^{+\infty} f(t)\cdot e^{-st}\, dt$$

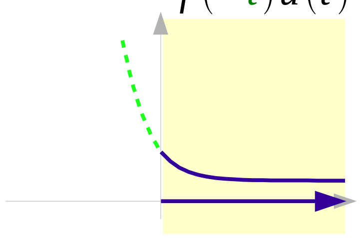
$f(t)u(-t)$



$$\hat{\mathbf{L}}\{f(t)u(-t)\} = G(s)$$

$$= \int_{-\infty}^{0} f(t)\cdot e^{-st}\, dt$$

$f(-t)u(t)$



$$\mathbf{L}\{f(-t)u(t)\} = H(s)$$

$$= \int_0^{+\infty} f(-t)\cdot e^{-st}\, dt$$

$$= -\int_0^{-\infty} f(v)\cdot e^{-s(-v)}\, dv$$

$$= \int_{-\infty}^{0} f(v)\cdot e^{sv}\, dv$$

$$= G(-s)$$

$$\mathbf{L}\{f(+v)\} = F(+s)$$

$$\mathbf{L}\{f(-v)\} = G(-s)$$

$$\mathbf{L}\{f(-t)u(t)\} = H(s)$$

$$= G(-s)$$

$$G(s) = H(-s)$$

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Improper Integrals : $e^{at}u(+t)$, $e^{at}u(-t)$, $e^{-at}u(+t)$

$e^{+at}u(+t)$ → L → $\dfrac{1}{s-a}$

$\int_0^{\infty} \cdot\, dt$

$$F(s) = \int_0^{\infty} e^{+at} \cdot e^{-st}\, dt = \lim_{b\to\infty}\left[-\frac{1}{(s-a)}e^{-(s-a)t}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{(s-a)}e^{-(s-a)b} + \frac{1}{(s-a)}e^{-(s-a)0}\right]$$

$$-(s-a) < 0 \Rightarrow \lim_{b\to\infty} e^{-(s-a)b} = 0 \qquad s > +a \Rightarrow F(s) = \frac{1}{(s-a)}$$

$b > 0$

$e^{+at}u(-t)$ → $\hat{L}$ → $\dfrac{-1}{s-a}$

$\int_{-\infty}^{0} \cdot\, dt$

$$G(s) = \int_{-\infty}^{0} e^{+at} \cdot e^{-st}\, dt = \lim_{b\to-\infty}\left[-\frac{1}{(s-a)}e^{-(s-a)t}\right]_b^0 = \lim_{b\to-\infty}\left[-\frac{1}{(s-a)}e^{-(s-a)0} + \frac{1}{(s-a)}e^{-(s-a)b}\right]$$

$$-(s-a) < 0 \Rightarrow \lim_{b\to-\infty} e^{-(s-a)b} = 0 \qquad s < +a \Rightarrow G(s) = \frac{-1}{(s-a)} = -F(s)$$

$b < 0$

$$L\{f(+v)\} = F(+s)$$

$$L\{f(-v)\} = G(-s)$$

$$L\{e^{+at}\} = F(+s) = \frac{1}{(s-a)} \qquad \hat{L}\{e^{+at}\} = G(+s) = -\frac{1}{(s-a)}$$

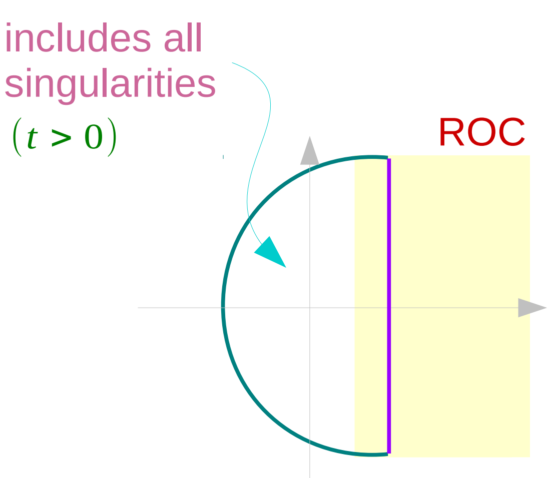
$$L\{e^{-at}\} = G(-s) = \frac{1}{(s+a)} \qquad G(+s) = \frac{1}{(-s+a)} = -\frac{1}{(s-a)}$$

Unilateral and Bilateral Laplace Transform

Unilateral Laplace Transform

$$F(s) = \int_{0}^{+\infty} f(t) e^{-st}\, dt$$

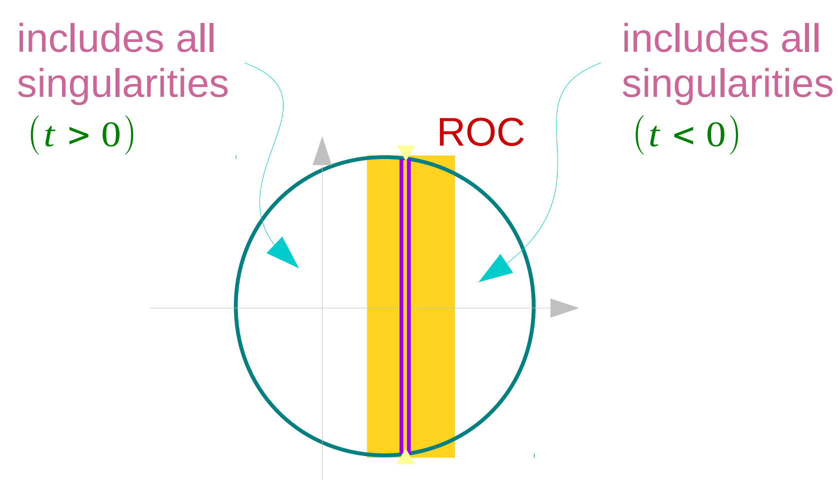
$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{+st}\, ds$$



Bilateral Laplace Transform

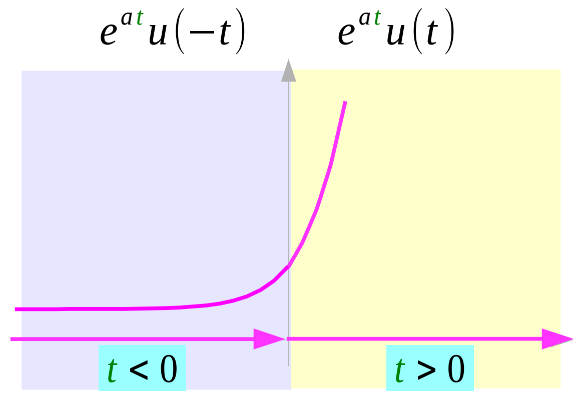
$$F(s) = \int_{-\infty}^{+\infty} f(t) e^{-st}\, dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{+st}\, ds$$



ROCs and two-sided functions

$e^{at} \quad (t < 0)$



$e^{at} \quad (t > 0)$

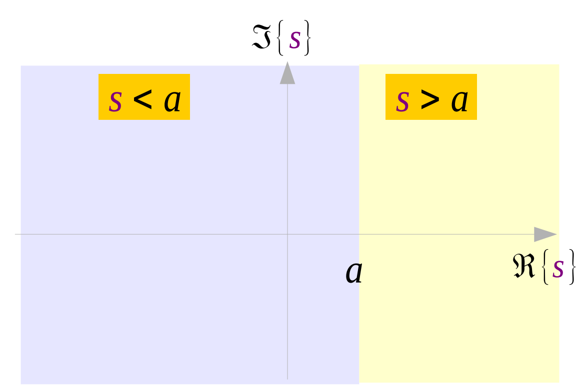
Bilateral Laplace Transform $\int_{-\infty}^{+\infty} e^{+at} \cdot e^{-st}\, dt$ ✕

no overlapping ROC

➡ No Convergence

$\int_{-\infty}^{0} e^{+at} \cdot e^{-st}\, dt$

$\frac{-1}{(s-a)}$ $s < a$



$\int_{0}^{\infty} e^{at} \cdot e^{-st}\, dt$

$\frac{1}{(s-a)}$ $s > -a$

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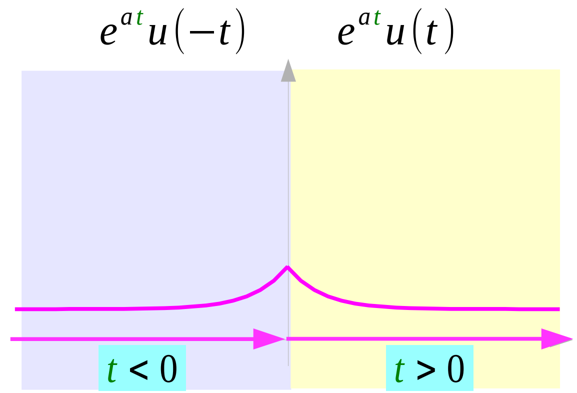
ROCs and two-sided functions

e^{at} $(t < 0)$



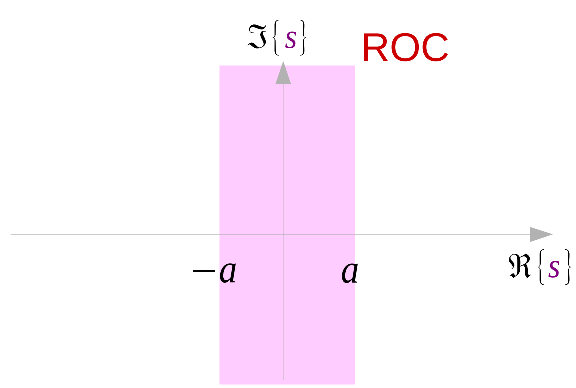
e^{-at} $(t > 0)$

Bilateral Laplace Transform

$$\int_{-\infty}^{+\infty} e^{+at} \cdot e^{-st}\, dt = \frac{1}{(s+a)} - \frac{1}{(s-a)} = -\frac{2a}{s^2 - a^2}$$

$$\int_{-\infty}^{0} e^{+at} \cdot e^{-st}\, dt$$

$$\frac{-1}{(s-a)} \qquad s < a$$

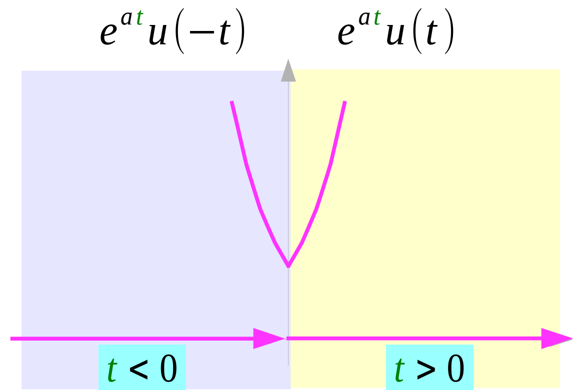


$$\int_{0}^{\infty} e^{-at} \cdot e^{-st}\, dt$$

$$\frac{1}{(s+a)} \qquad s > -a$$

ROCs and two-sided functions

$e^{-at} \quad (t < 0)$



$e^{at} \quad (t > 0)$

Bilateral Laplace Transform

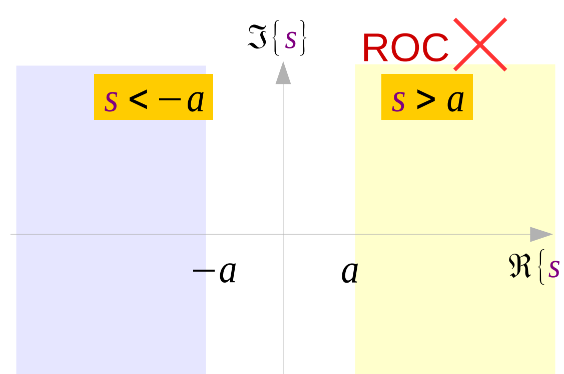
$$\int_{-\infty}^{+\infty} e^{+at} \cdot e^{-st}\, dt$$ ✕

no overlapping ROC

➡ No Convergence

$$\int_{-\infty}^{0} e^{-at} \cdot e^{-st}\, dt$$

$$\frac{-1}{(s+a)} \quad s < -a$$

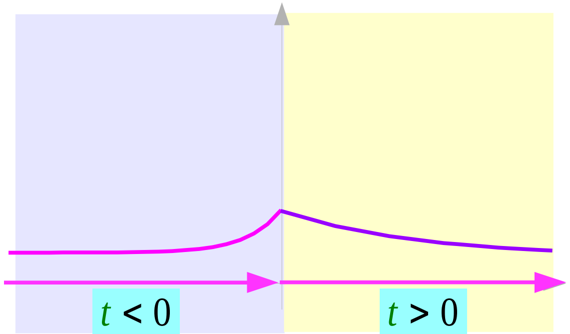


$$\int_{0}^{\infty} e^{at} \cdot e^{-st}\, dt$$

$$\frac{1}{(s-a)} \quad s > a$$

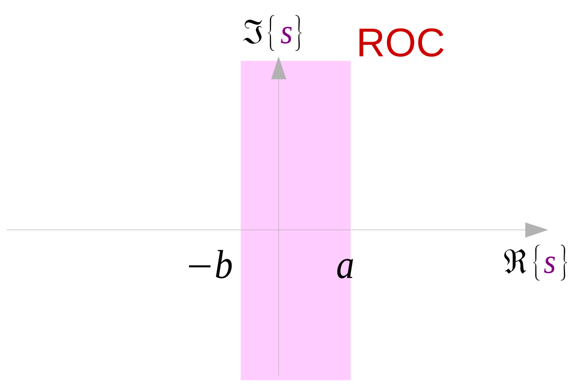
ROCs and two-sided functions

$e^{at}u(-t)$ $\quad$ $e^{-bt}u(t)$



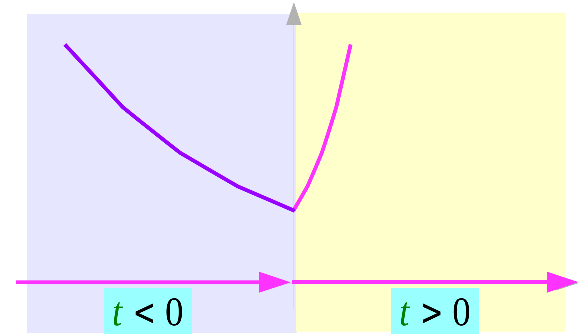
$$\frac{-1}{(s-a)} \qquad \frac{1}{(s+b)}$$

$s < a$ $\quad$ $s > -b$



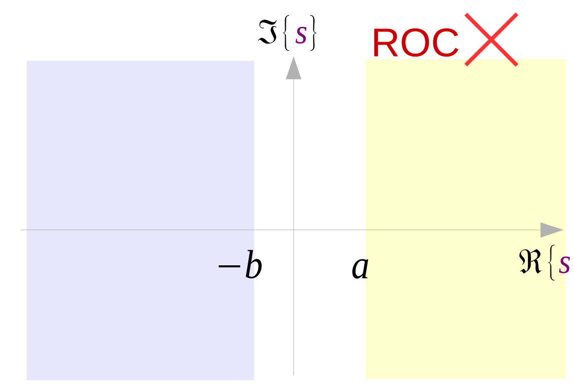
$0 < b < a$

$e^{-bt}u(-t)$ $\quad$ $e^{at}u(t)$



$$\frac{-1}{(s+b)} \qquad \frac{1}{(s-a)}$$

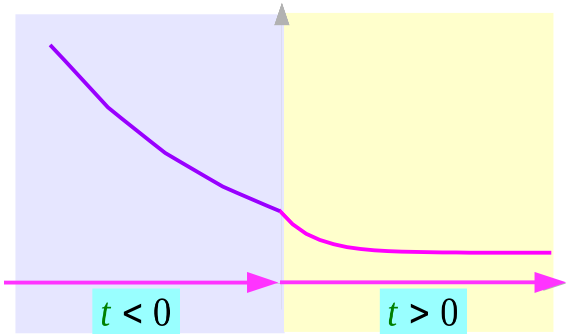
$s < -b$ $\quad$ $s > a$



$0 < b < a$

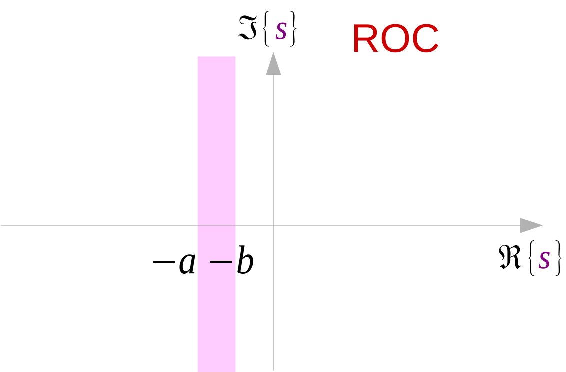
ROCs and two-sided functions

$e^{-bt}u(-t)$ $\quad$ $e^{-at}u(t)$



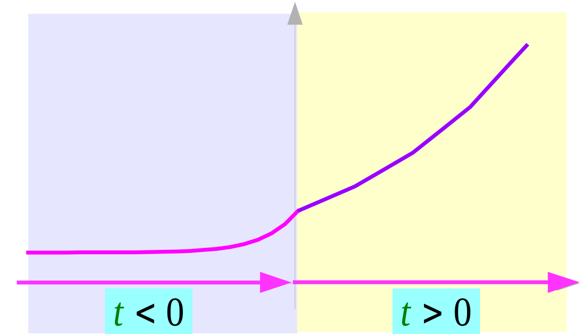
$\frac{-1}{(s+b)}$ $\quad$ $\frac{1}{(s+a)}$

$s < -b$ $\quad$ $s > -a$



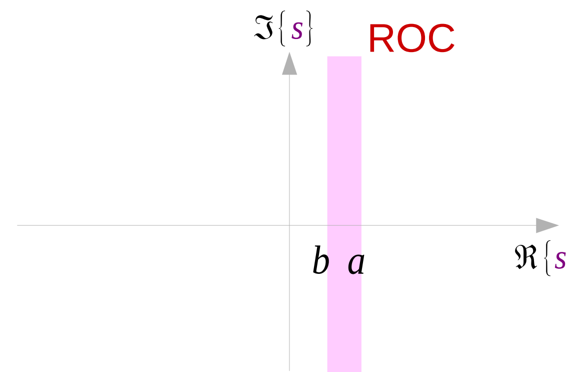
$0 < b < a$

$e^{at}u(-t)$ $\quad$ $e^{bt}u(t)$



$\frac{-1}{(s-a)}$ $\quad$ $\frac{1}{(s-b)}$

$s < a$ $\quad$ $s > b$



$0 < b < a$

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