# Introduction to ODEs

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March 7, 2014

# Differentiation Types

Ordinary Differentiation

- $y = f(x)$
  - $\frac{dy}{dx}$

Partial Differentiation

- $z = f(x, y)$
  - $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$
- $u = f(x, y, z)$
  - $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$

# ODEs and PDEs

Ordinary Differential Equation (ODE) Examples

- $y = f(x)$
  - $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = x$

Partial Differential Equation (ODE) Examples

- $u = f(x, y, z)$
  - $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

# ODEs in normal form

Ordinary Differential Equation (ODE) Examples

- $y = f(x)$
  - $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = x$
- A General Form
  - $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y - x = 0$ $\quad G(x, y, y', y'') = 0$
- A Normal Form
  - $\frac{d^2y}{dx^2} = -a_1 \frac{dy}{dx} - a_0 y + x$ $\quad y'' = g(x, y')$

# Linear and Non-linear ODEs

Examples of Linear ODEs

- $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = x$
- $a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$

Examples of Non-linear ODEs

- $y\frac{d^2y}{dx^2} + a_1(x)\left(\frac{dy}{dx}\right)^2 + a_0(x,y)y = x$

# Conditions of Linear ODEs

Examples of Linear ODEs

- $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = x$
- $a_2(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$

Conditions:

- the power of the dependent variable $y$ and all its derivatives $(y, y', y'', \cdots, y^{(n)})$ must be 1.
- the coefficients $a_i$ depend on at most on the independent variable $x$.

# Linear 1st and 2nd Order ODEs

Examples of Linear First Order ODEs

- $a_1 \frac{dy}{dx} + a_0 y = x$
- $a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$

Examples of Linear Second Order ODEs

- $a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = x$
- $a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$

# Solutions of ODEs

Ordinary Differential Equation

- $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y - x = 0$ $\qquad G(x, y, y', y'') = 0$

- A function $\Phi(x)$ is a solution of the ODE if and only if
  - $\Phi(x)$ is defined on an interval $I$
  - its derivative $\Phi'(x), \Phi''(x)$are continuous on an interval $I$
  - $G(x, \Phi', \Phi'') = 0$ for all $x$ in the interval $I$

- the interval of definition / validity / the solution

# Implicit and Explicit Solutions of ODEs

Ordinary Differential Equation

- $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y - x = 0$ $\qquad G(x, y, y', y'') = 0$

Explicit solutions

- $y = \Phi(x)$

Implicit solutions

- $H(x, y) = 0$

# Families of Solutions

Ordinary Differential Equations

- $a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y - x = 0$ $\quad$ $G(x, y, y', y'') = 0$

Algebraic Equations

- $a_2 X + a_1 Y + a_0 Z - x = 0$

- 3 unknowns

- for the unique solution, we need 2 more equations.

# Initial Conditions

Ordinary Differential Equations

- $a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y - x = 0$

$G(x, y, y', y'') = 0$

Initial Conditions

- $y(x_0) = c_0$
- $y'(x_0) = c_1$

# Initial Value Problem

Solve

- $y^{(n)} = g(x, y, y', \cdots, y^{(n-1)})$

$G(x, y, y', \cdots, y^{(n)}) = 0$

Subject to

- $y(x_0) = c_0$
- $y'(x_0) = c_1$
- ...
- $y^{(n-1)}(x_0) = c_{n-1}$

# Existence and Uniqueness of IVP

Existence

- Does the differential equation possess solutions?
- Does any of the solution curves pass through the point $(x_0, y_0)$

Uniqueness

- There is precisely one solution curve that pass through the point $(x_0, y_0)$

# Reference

[1] D. G. Zill and W. S. Wright , “Advanced Engineering Mathematics”, 4th ed.