ODE Background: Differential (1A)

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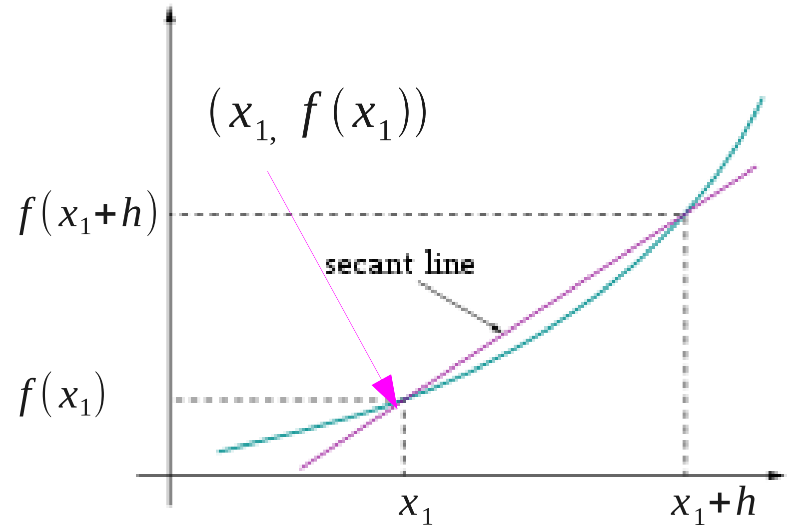
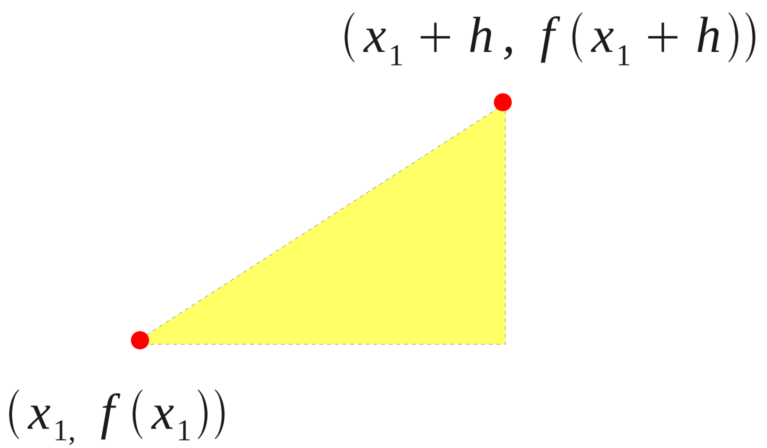
Differentials

A triangle and its slope

$$y = f(x)$$

$$\frac{f(x_1 + h) - f(x_1)}{h}$$





http://en.wikipedia.org/wiki/Derivative

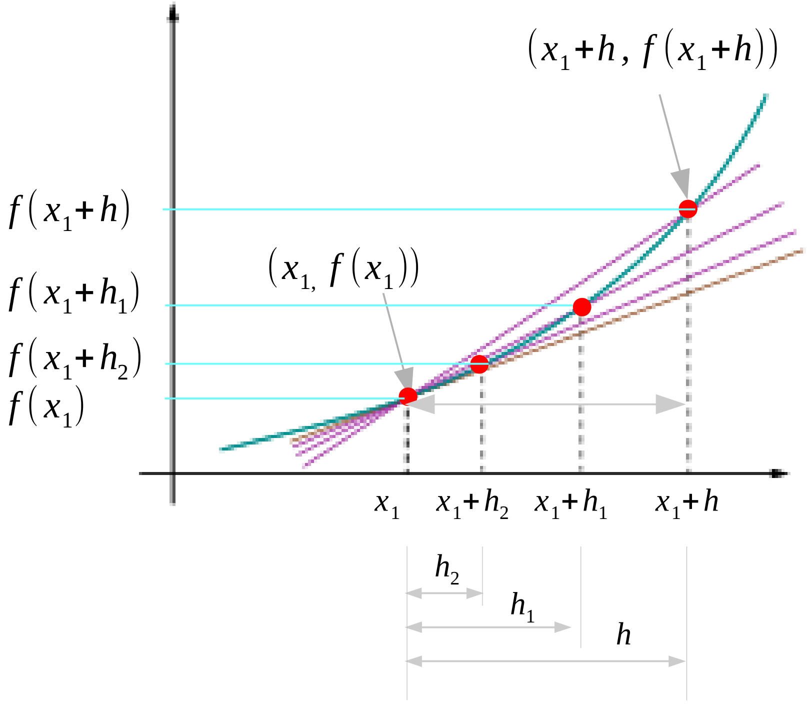
Many smaller triangles and their slopes

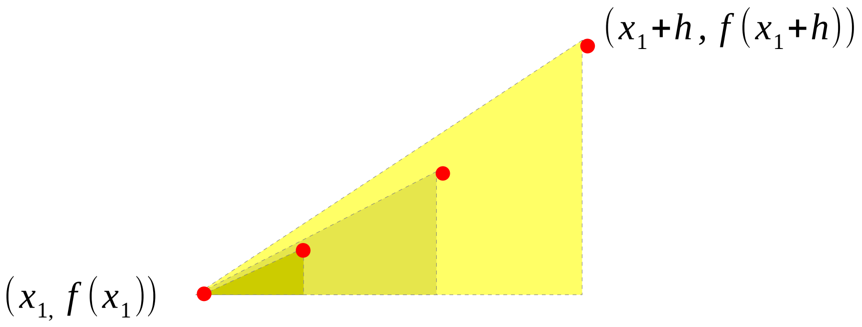
$$\frac{f(x_1 + h) - f(x_1)}{h}$$

$$\frac{f(x_1 + h_1) - f(x_1)}{h_1}$$

$$\frac{f(x_1 + h_2) - f(x_1)}{h_2}$$

$$\lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$





http://en.wikipedia.org/wiki/Derivative

The limit of triangles and their slopes

$$y = f(x)$$

The derivative of the function f at x_1

$$f'(x_1) = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

The derivative function of the function f

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$y' = f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x)$$

5. (*calculus*) The derived function of a function.

   The derivative of $f : f(x) = x^2$ *is* $f' : f'(x) = 2x$

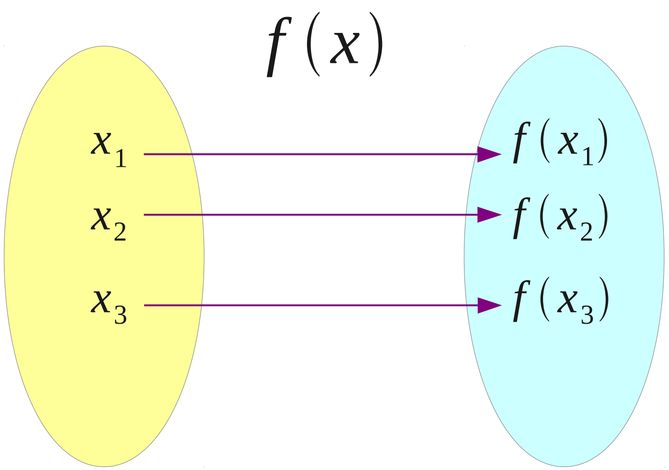
6. (*calculus*) The value of this function for a given value of its independent variable.

   The derivative of $f(x) = x^2$ *at x = 3 is* $f'(3) = 2 * 3 = 6$.

http://en.wiktionary.org/

The derivative as a function

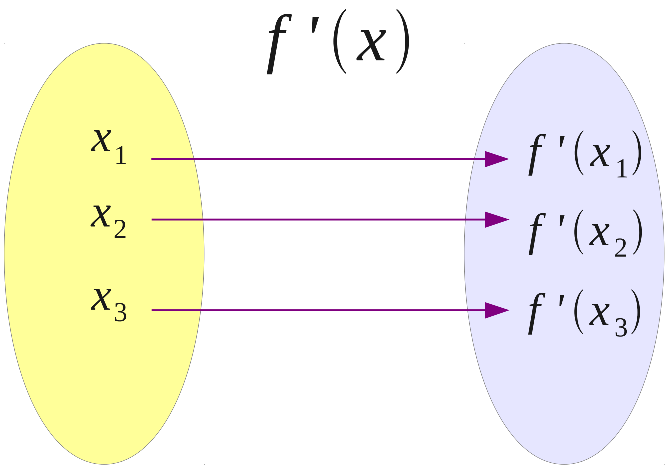
$$y = f(x)$$



Derivative Function

$$y' = f'(x)$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



The notations of derivative functions

Largrange's Notation

$$y' = f'(x)$$

Leibniz's Notation

$$\frac{dy}{dx} = \frac{d}{dx} f(x)$$

not a ratio.

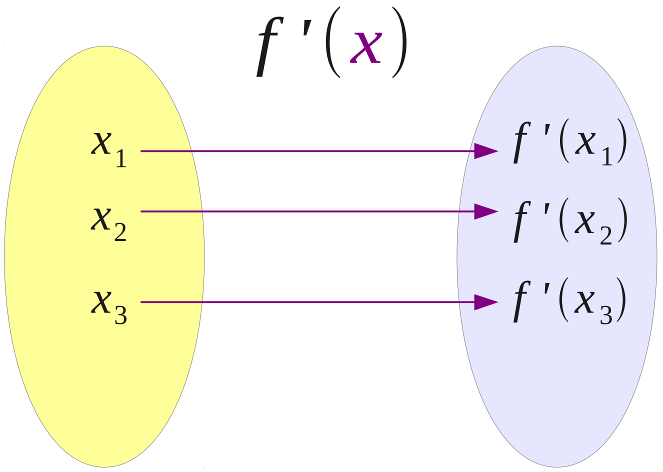
slope of a tangent line

Newton's Notation

$$\dot{y} = \dot{f}(x)$$

Euler's Notation

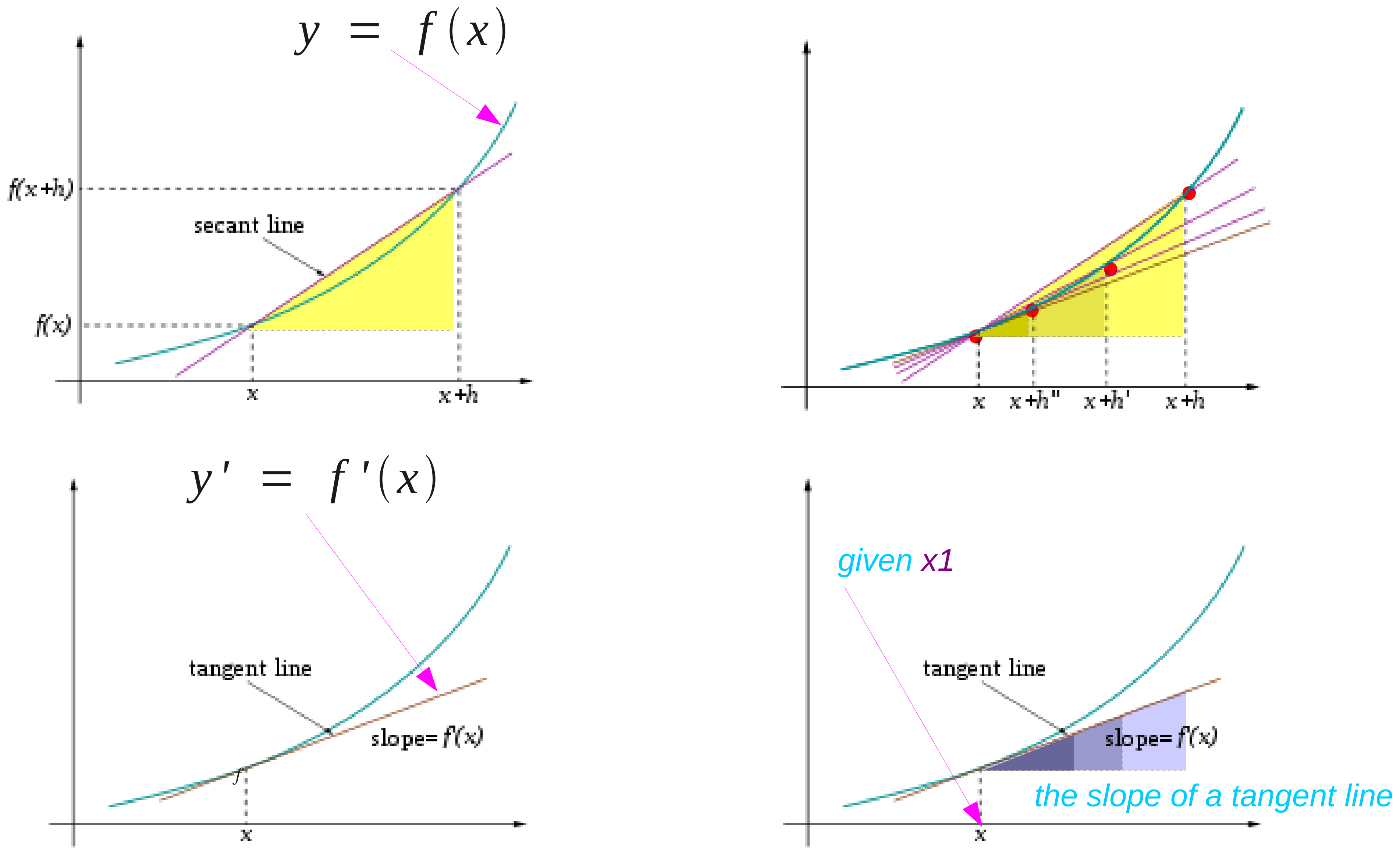
$$D_x y = D_x f(x)$$



$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- *derivative with respect to x*
- *x is an independent variable*

Another kind of triangles and their slope

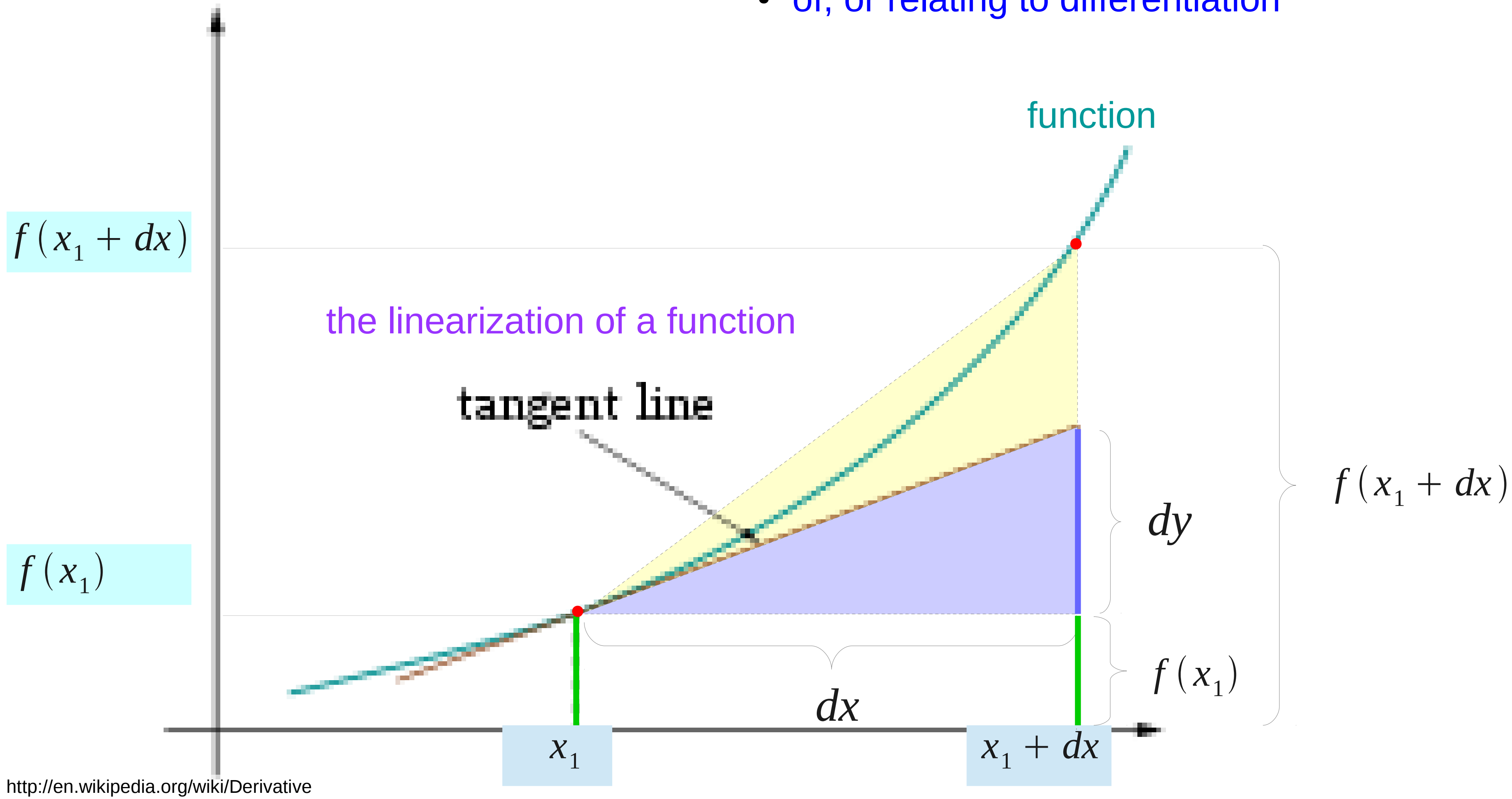


http://en.wikipedia.org/wiki/Derivative

Differential in calculus

Differential: dx, dy, …

- infinitesimals
- a change in the linearization of a function
- of, or relating to differentiation



http://en.wikipedia.org/wiki/Derivative

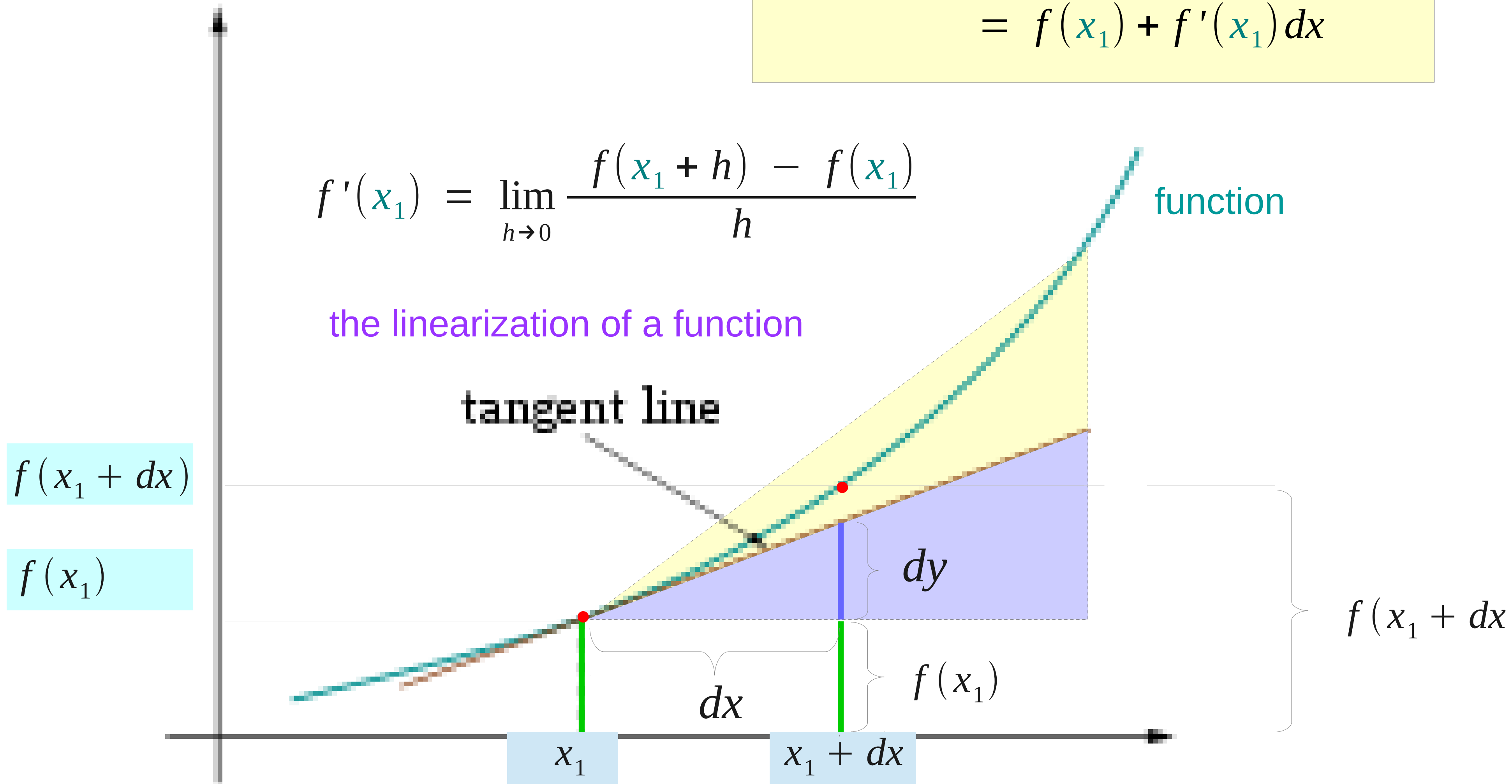
Approximation

Differential: dx, dy, …

$$f(x_1 + dx) \approx f(x_1) + dy$$
$$= f(x_1) + f'(x_1)\,dx$$

$$f'(x_1) = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

the linearization of a function

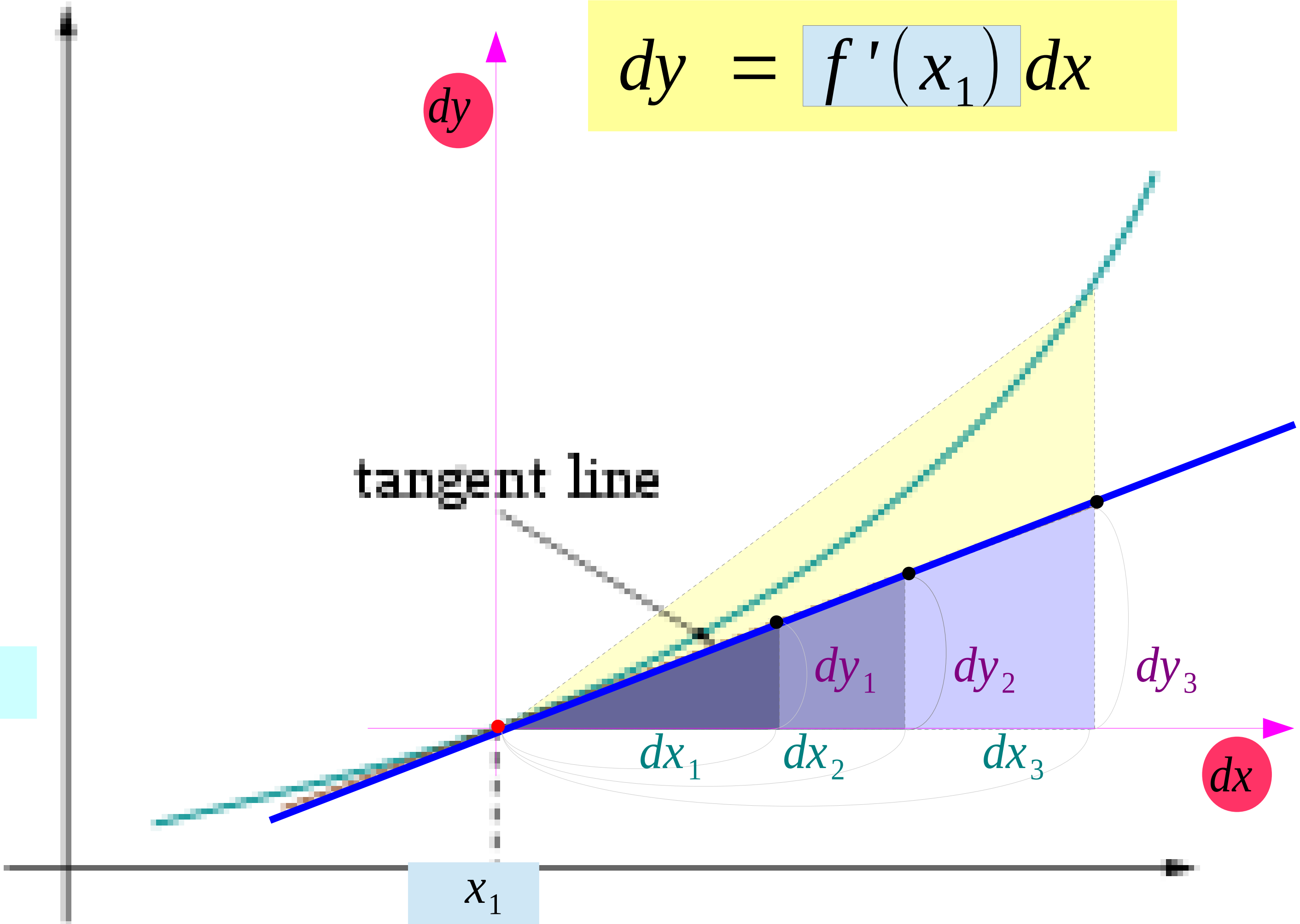


http://en.wikipedia.org/wiki/Derivative

Differential as a function

Line equation in the new coordinate.

$$dy = f'(x_1)\,dx$$



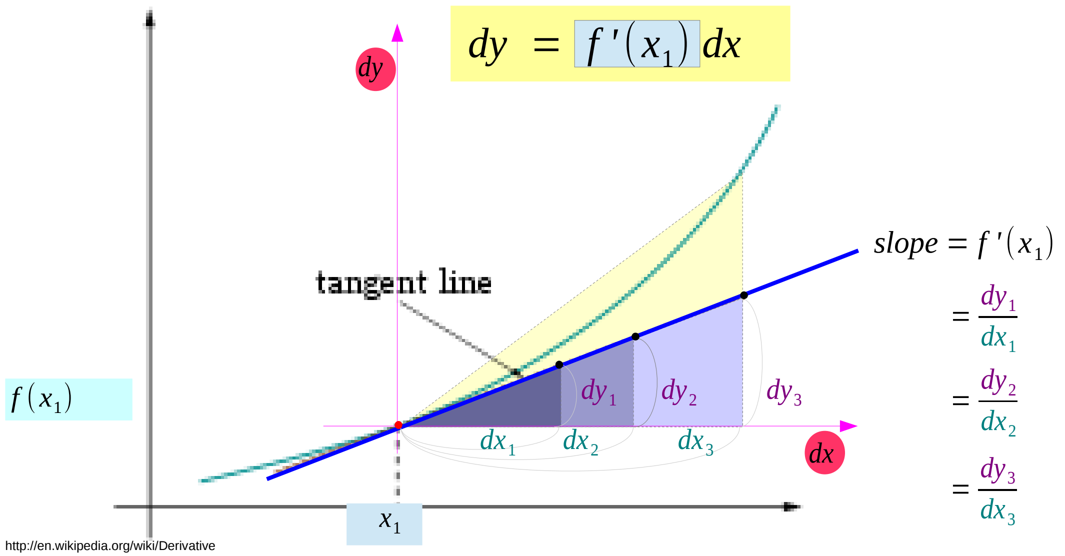
$$slope = f'(x_1) = \frac{dy_1}{dx_1} = \frac{dy_2}{dx_2} = \frac{dy_3}{dx_3}$$

http://en.wikipedia.org/wiki/Derivative

Differentials and Derivatives (1)

$$dy = f'(x)\,dx$$

$$dy = \frac{df}{dx}\,dx$$



differentials ***derivative***

$$\frac{dy}{dx} = f'(x)$$

ratio ***not a ratio***

Differentials and Derivatives (2)

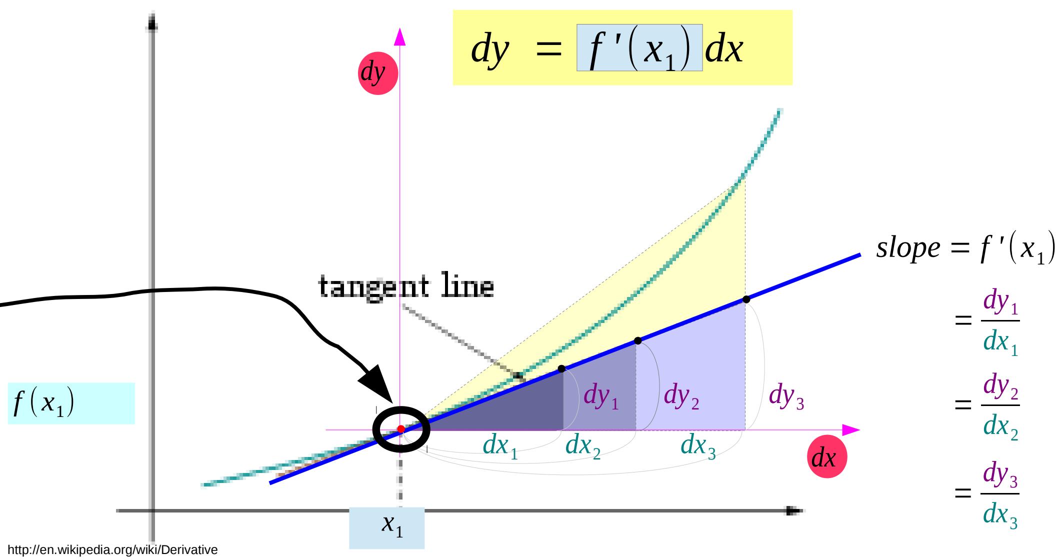
$$f(x_1 + dx) \approx f(x_1) + dy$$
$$= f(x_1) + f'(x_1)\,dx$$

for small enough dx

$$\lim_{dx \to 0}$$

$$f(x_1 + dx) = f(x_1) + dy$$
$$= f(x_1) + f'(x_1)\,dx$$

$$\lim_{dx \to 0} \frac{f(x_1 + dx) - f(x_1)}{dx} = f'(x_1)$$



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Differentials and Derivatives (3)

$$dy = f'(x)\,dx \quad \Rightarrow \quad \int dy = \int f'(x)\,dx$$

$$dy = \frac{df}{dx}\,dx \quad \Rightarrow \quad \int dy = \int \frac{df}{dx}\,dx$$

$$y = f(x)$$

$$dy = \dot{f}\,dx$$

$$dy = D_x f\,dx$$

$$\int dy = \int 1\,dy = y$$

Integration Constant C

place a constant

place another constant

$$\int dy = \int f'(x)\,dx$$

$$\int dy = \int \frac{df}{dx}\,dx$$

$$y + C_1 = f(x) + C_2$$

$$y = f(x) + C$$

differs by a constant

place only one constant from the beginning

$$\int dy = \int f'(x)\,dx + C$$

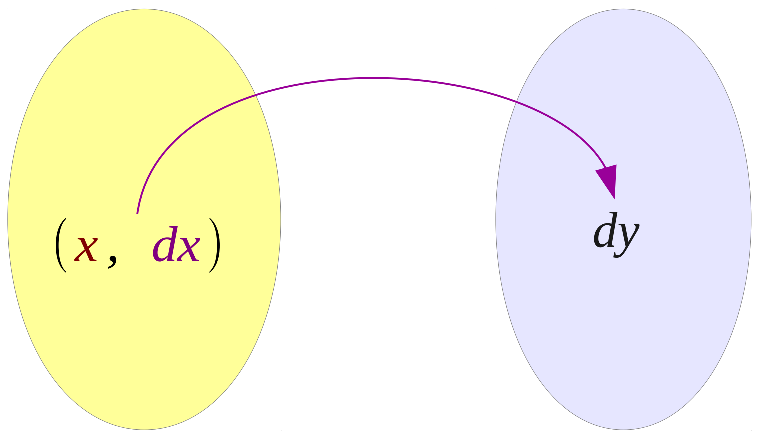
$$\int dy = \int \frac{df}{dx}\,dx + C$$

$$y = f(x) + C$$

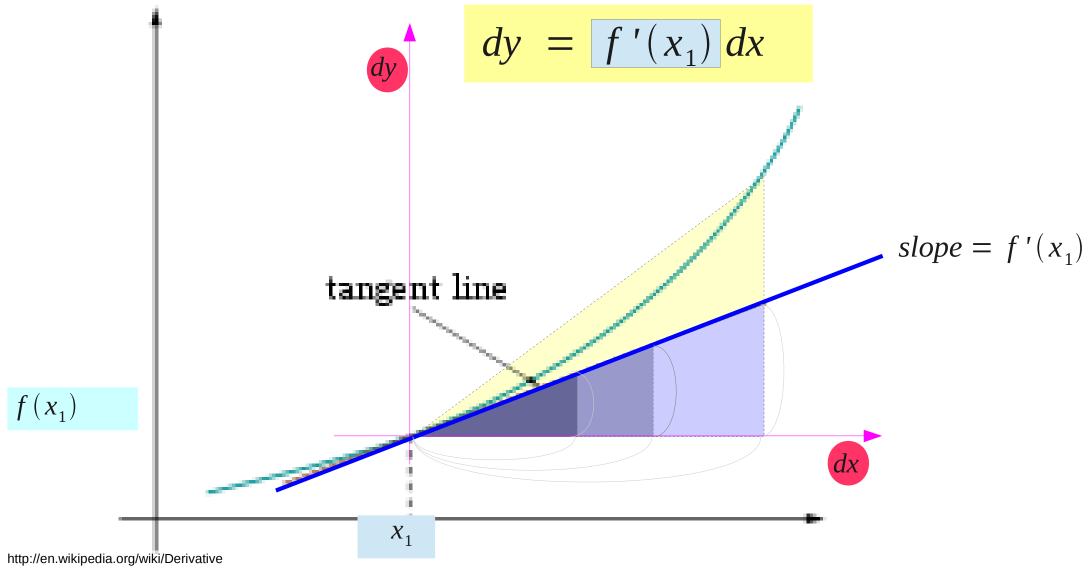
Differential as a function

*The **differential** of a function **f(x)** of a single real variable **x** is the function of two independent real variables **x** and **dx** given by*

$$dy = f'(x)\,dx$$



Line equation in the new coordinate.



Applications of Differentials (1)

Substitution Rule

$$\int f(\,g(x)\,)\cdot g'(x)\,dx = \int f(\,u\,)\,du$$

(I) $u = g(x)$ $\qquad du = g'(x)dx$ $\qquad du = \dfrac{dg}{dx}dx$

(II) $\displaystyle\int f(g)\frac{dg}{dx}\,dx = \int f(\,g\,)\,dg$

Applications of Differentials (2)

Integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$u = f(x) \qquad du = \underline{f'(x)\,dx} \qquad du = \frac{df}{dx}\,dx$$

$$v = g(x) \qquad dv = \underline{g'(x)\,dx} \qquad dv = \frac{dg}{dx}\,dx$$

$$\int f(x)\underline{g'(x)\,dx} = f(x)g(x) - \int \underline{f'(x)}g(x)\,\underline{dx}$$

$$\int u\,dv = uv - \int v\,du$$

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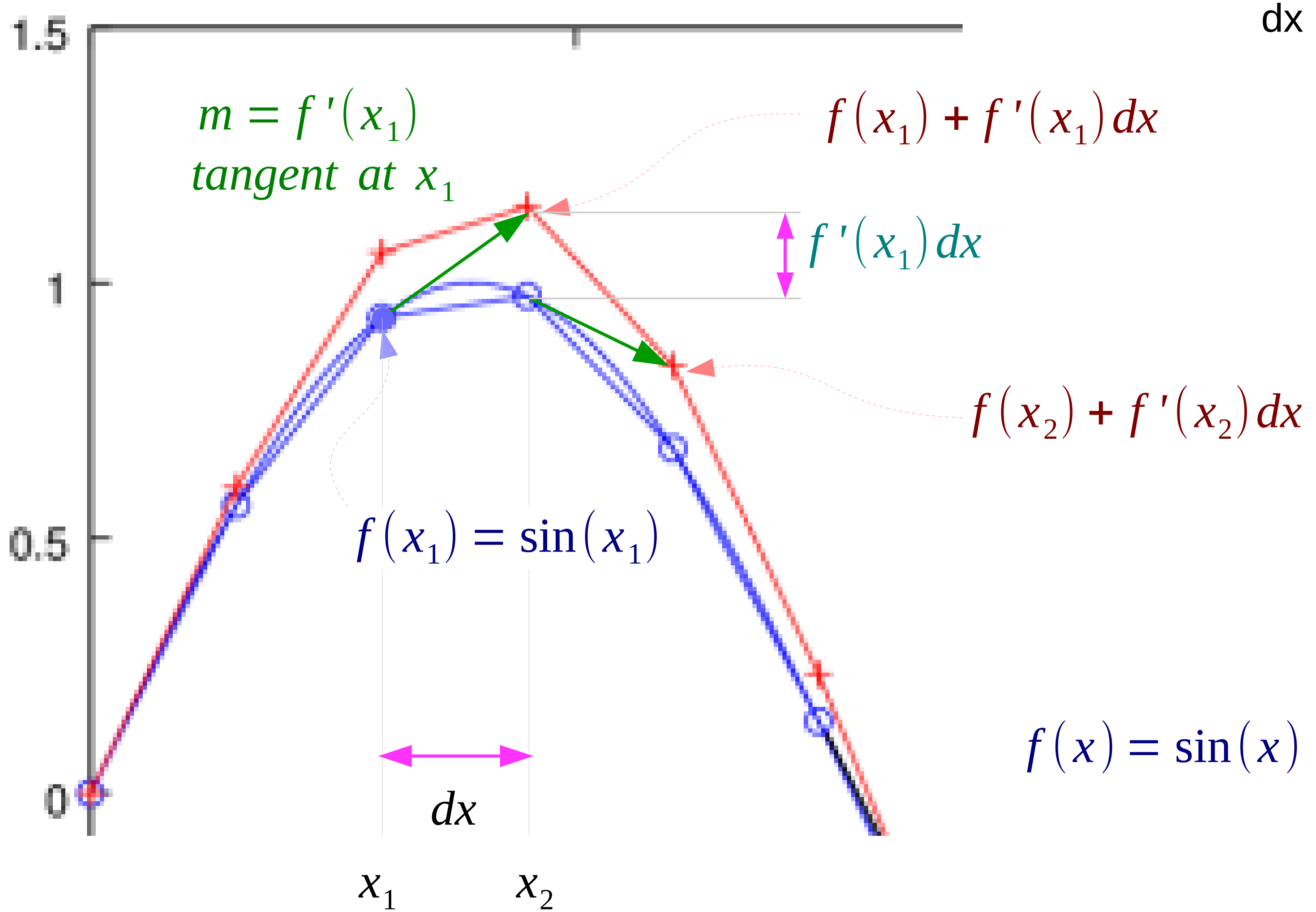
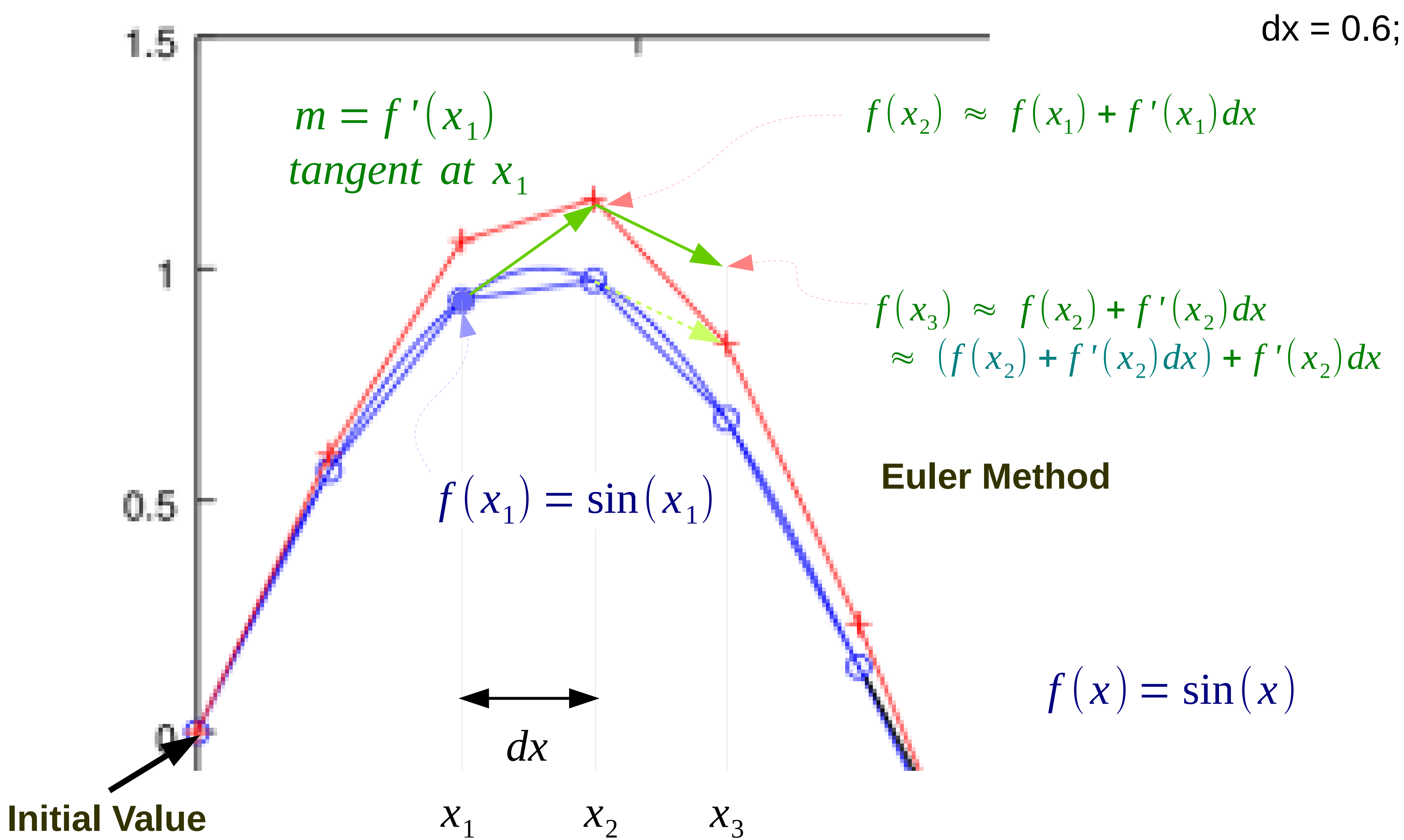
Derivatives and Differentials (large *dx*)

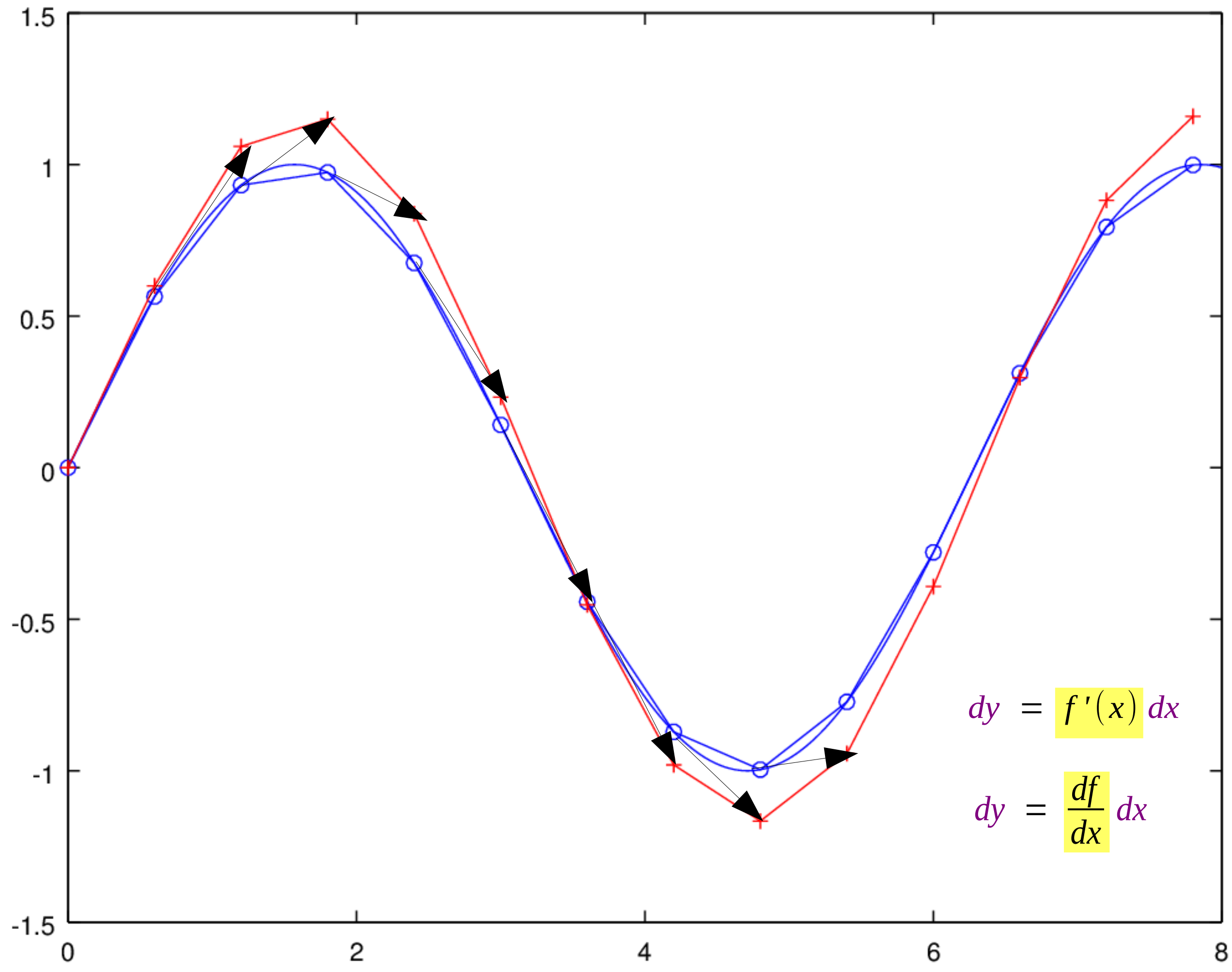
dx = 0.6;



Euler Method of Approximation (large *dx*)



Derivatives and Differentials (large *dx = 0.6*)



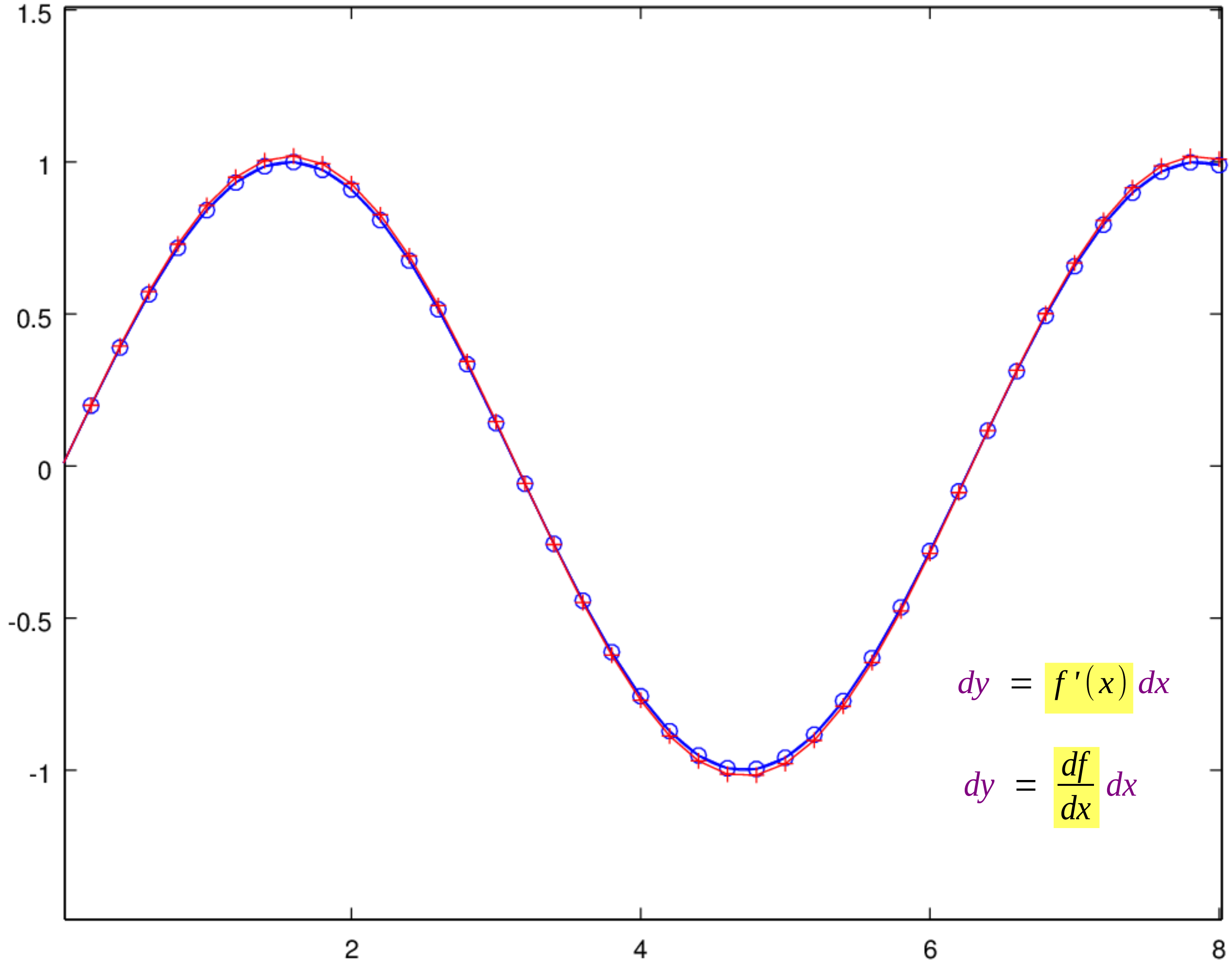
dx = 0.6;

$$dy = f'(x)\,dx \quad \Rightarrow \quad \int dy = \int f'(x)\,dx$$

$$dy = \frac{df}{dx}\,dx \quad \Rightarrow \quad \int dy = \int \frac{df}{dx}\,dx$$

$$y = f(x)$$

Derivatives and Differentials (small *dx = 0.2*)



dx = 0.2;

$$dy = f'(x)\,dx \quad\Rightarrow\quad \int dy = \int f'(x)\,dx$$

$$dy = \frac{df}{dx}\,dx \quad\Rightarrow\quad \int dy = \int \frac{df}{dx}\,dx$$

$$y = f(x)$$

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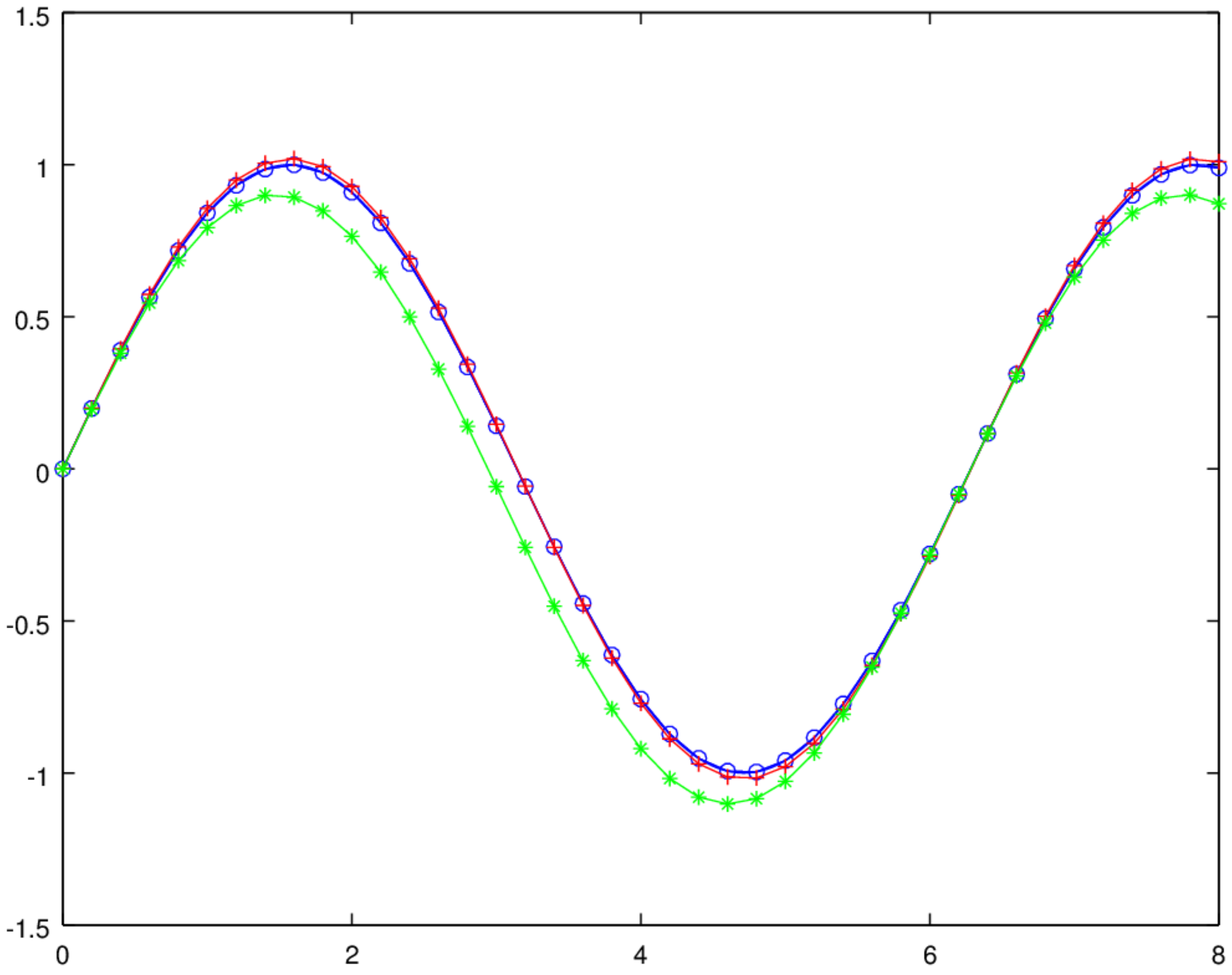
Euler's Method of Approximation

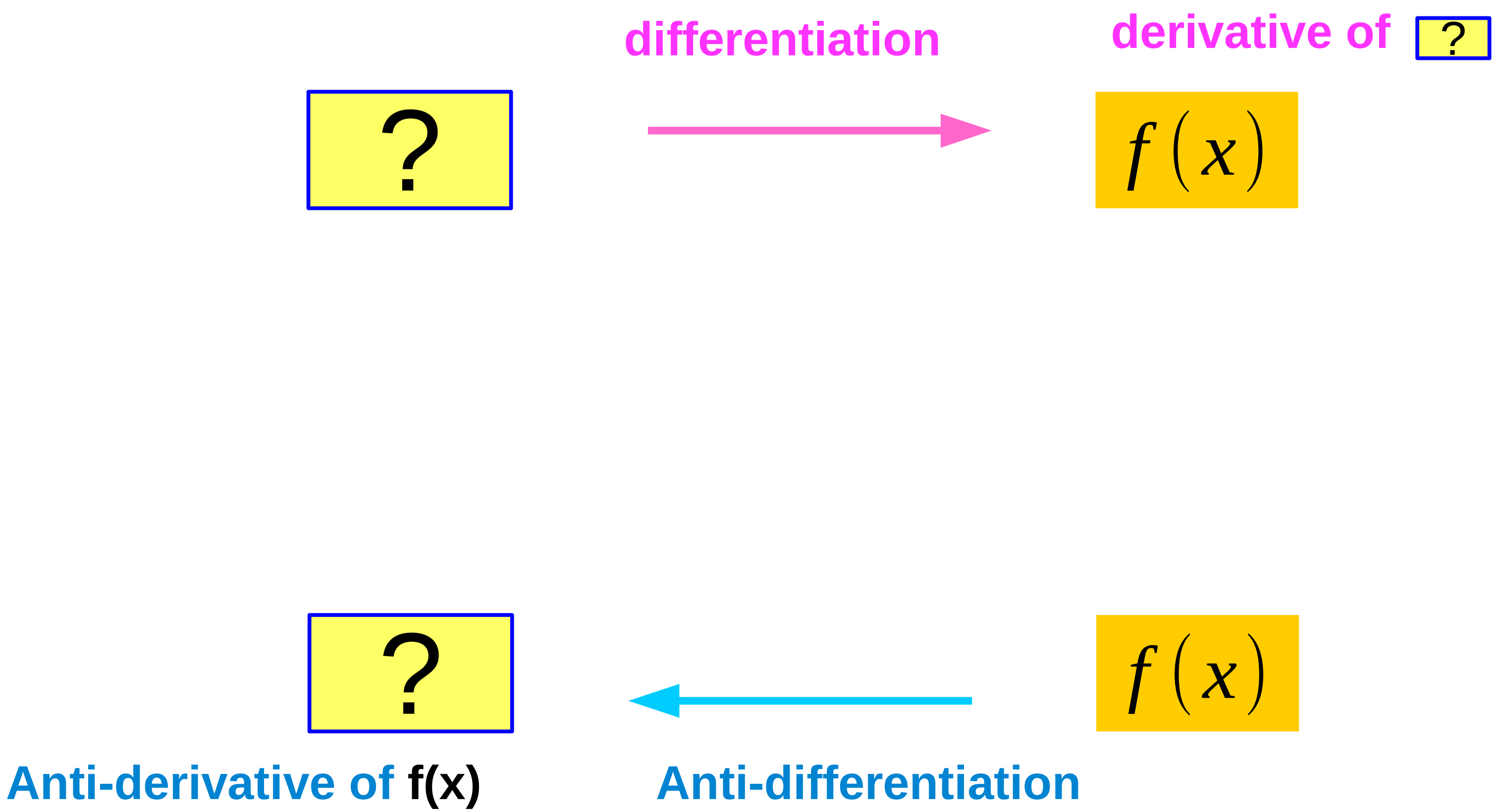


Octave Code

```
clf; hold off;
dx = 0.2;

x = 0 : dx : 8;
y = sin(x);
plot(x, y);
t = sin(x) + cos(x)*dx ;
y1 = [y(1), t(1:length(y)-1)];

y2 = [0];
y2(1) = y(1);
for i=1:length(y)-1
  y2(i+1) = y2(i) + cos((i)*dx)*dx;
endfor

hold on
t = 0:0.01:8;
plot(t, sin(t), "color", "blue");
plot(x, y,   "color", 'blue',  "marker", 'o');
plot(x, y1,  "color", 'red',   "marker", '+');
plot(x, y2,  "color", 'green',  "marker", '*');
```

Anti-derivatives

Anti-derivative



Anti-derivative and Indefinite Integral

$$F'(x) = f(x)$$

$F(x)$ — **Anti-derivative without constant**
the most *simple* anti-derivative

$F(x) + C$ — **the most *general* anti-derivative**

$\int f(x)\,dx$ — **Indefinite Integral** **: a function of x**

$$\int f(x)\,dx = F(x) + C$$

Anti-derivative Examples

All are Anti-derivative of f(x)

$$\begin{cases} F_1(x) = \frac{1}{3}x^3 \\ F_2(x) = \frac{1}{3}x^3 + 100 \\ F_3(x) = \frac{1}{3}x^3 - 49 \end{cases}$$

differentiation →

← **Anti-differentiation**

$f(x) = x^2$

the most *general* anti-derivative of f(x) $\frac{1}{3}x^3 + C$

indefinite Integral of f(x) $\equiv \int x^2 \, dx$

Indefinite Integrals

$\int_a^{x_1} 1\, dx$	$\int_a^{x} 1\, dx$	$\int dx$	$\int dy$
||	||	||	||
$x_1 - a$	$x - a$	$x + C$	$y + C$
given x_1	a variable x	indefinite integral	

$\int_a^{x_1} \frac{df}{dx}\, dx$	$\int_{-c}^{x} \frac{df}{dx}\, dx$	$\int \frac{df}{dx}\, dx$
||	||	||
$f(x_1) - f(a)$	$f(x) - f(a)$	$f(x) + C$
given x_1	a variable x	indefinite integral

definite integral $\int_a^x f(t)\,dt$ ← **anti-derivative** $f(x)$

indefinite integral $\int f(x)\,dx$ ← **anti-derivative** $f(x)$

$$\int f(x)\,dx = F(x) + C$$

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

a common reference point : arbitrary

definite integral	$\int_a^x f(t)\,dt$	← **anti-derivative**	$f(x)$
indefinite integral	$\int f(x)\,dx$	← **anti-derivative**	$f(x)$

$$\int_{x_1}^{x_2} f(t)\,dt = \int_a^{x_1} f(t)\,dt + \int_a^{x_2} f(t)\,dt$$

a common reference point : arbitrary

$\left[F(x) + c\right]_{x_1}^{x_2} = F(x_2) - F(x_1)$ ⟷ $\left[F(x)\right]_{x_1}^{x_2} = F(x_1) - F(x_2)$

Anti-derivative without constant

Indefinite Integral Examples

$$\int_0^x f(x)\,dx = \left[\frac{1}{3}x^3\right]_0^x = \frac{1}{3}x^3$$

$f(x) = x^2$

$$\int_a^x f(x)\,dx = \left[\frac{1}{3}x^3\right]_a^x = \frac{1}{3}x^3 - \frac{1}{3}a^3$$

$$\int_a^x f(t)\,dt = \left[\frac{1}{3}t^3\right]_a^x = \frac{1}{3}x^3 - \frac{1}{3}a^3$$

anti-derivative by the definite integral of f(x)

$$\int_a^x t^2\,dt = \frac{1}{3}x^3 - \frac{1}{3}a^2$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) = x^2$$

indefinite integral of f(x)

$$\int x^2\,dx = \frac{1}{3}x^3 + C$$

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Definite Integrals on [a, x_1]

$\int_a^{x_1} 1 \, dx$ ⇨ $\int_a^{x_1} f'(x) \, dx$ $f'(x) = 1$ view (I)

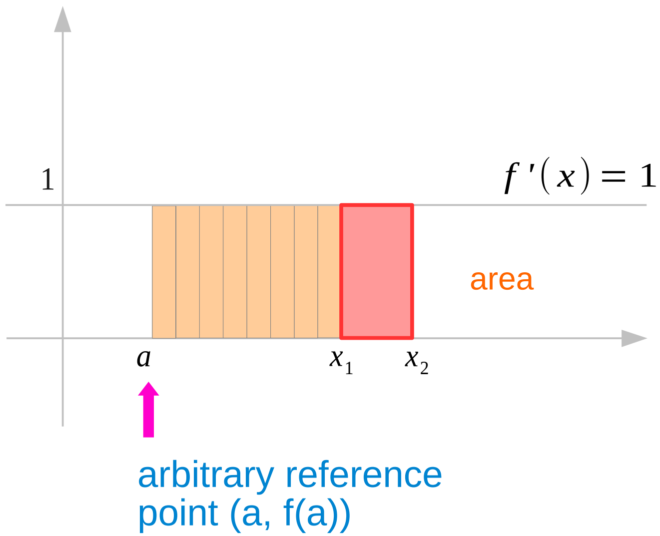
$\int_a^{x_1} 1 \, dx$ ⇨ $\int_a^{x_1} g(x) \, dx$ $g(x) = 1$ view (II)

view (I) $\int_a^{x_1} f'(x) \, dx$ $\left[f(x)\right]_a^{x_1} = f(x_1) - f(a)$

view (II) $\int_a^{x_1} g(x) \, dx$ $\left[G(x)\right]_a^{x_1} = G(x_1) - G(a)$

Definite Integrals over an interval $[x_1, x_2]$

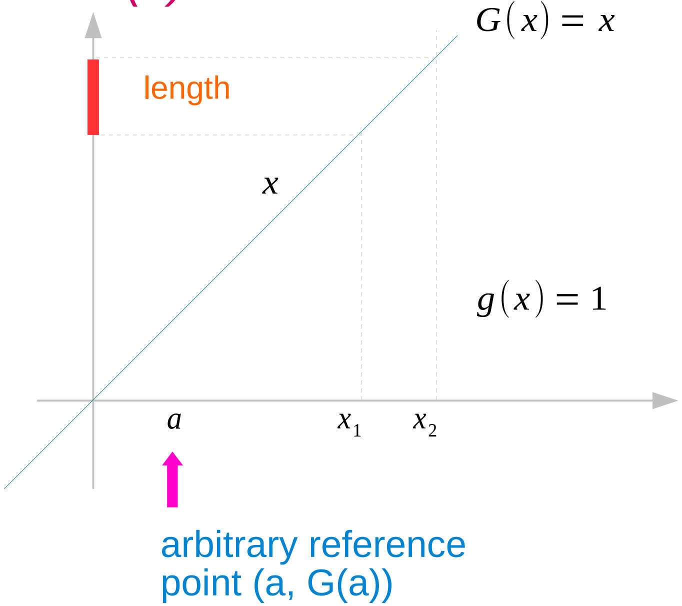
view (I)



$$\int_{x_1}^{x_2} f'(x)\, dx =$$

$$\left[f(x)\right]_{x_1}^{x_2} = f(x_2) - f(x_1)$$

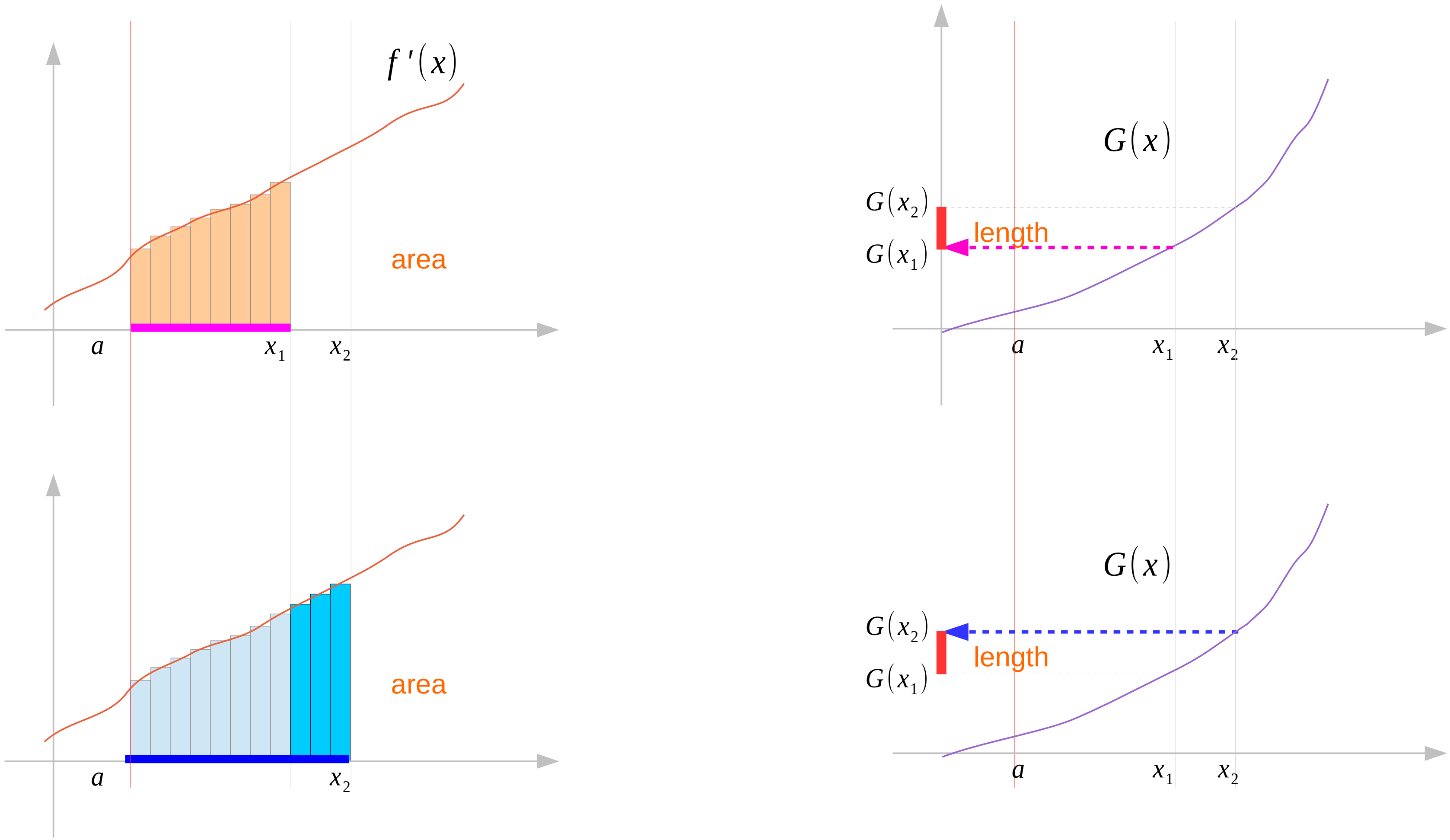
view (II)



$$\int_{x_1}^{x_2} g(x)\, dx =$$

$$\left[G(x)\right]_{x_1}^{x_2} = G(x_2) - G(x_1)$$

Definite Integrals on [a, x_1] and [a, x_2]



$$\int_c^{x_2} f'(x)\,dx - \int_c^{x_1} f'(x)\,dx$$

$$\int_c^{x_2} g(x)\,dx - \int_c^{x_1} g(x)\,dx$$

Derivative Function and Indefinite Integrals

$$f'(x_1) \Rightarrow \lim_{h\to 0}\frac{f(x_1+h)-f(x_1)}{h}$$

$$f'(x_2) \Rightarrow \lim_{h\to 0}\frac{f(x_2+h)-f(x_2)}{h}$$

$$f'(x_3) \Rightarrow \lim_{h\to 0}\frac{f(x_3+h)-f(x_3)}{h}$$

$$\int_{x_1}^{x_2} f(x)\,dx$$

$$\int_{x_3}^{x_4} f(x)\,dx$$

$$\int_{x_5}^{x_6} f(x)\,dx$$

$x_1,\ x_2,\ x_3$

$$f'(\mathbf{x}) = \lim_{h\to 0}\frac{f(\mathbf{x}+h)-f(\mathbf{x})}{h}$$

$f'(x_1),\ f'(x_2),\ f'(x_3)$

function of x

$[x_1,\ x_2],[x_3,\ x_4],[x_5,\ x_6]$

$$F(\mathbf{x})+C = \int_a^{\mathbf{x}} f(x)\,dx$$

$[\,F(x)\,]_{x_1}^{x_2},\ [\,F(x)\,]_{x_3}^{x_4},\ [\,F(x)\,]_{x_5}^{x_6}$

function of x

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a

Differential Equation

$$f(x) = e^{3x}$$

$$f'(x) = 3e^{3x}$$

$$f'(x) - 3f(x) = 3e^{3x} - 3e^{3x} = 0$$

$$f(x) \quad ?$$

$$f'(x) - 3f(x) = 0$$

First Order Examples (y=f(x))

$$f'(x) = f(x)$$

$$y' = y$$

$$f(x) \quad ?$$

$$f(x) = ce^x$$

$$for \quad all \quad x$$

$$y \quad ?$$

$$y = ce^x$$

$$I: (-\infty, +\infty)$$

An Example of A First Order Differential Equation

$$f'(x) = f(x)$$
$$f(0) = 3$$

$$y' = y$$
$$f(0) = 3$$

$$f(x) \quad ?$$

$$f(x) = 3e^x$$

$$for \quad all \quad x$$

$$y \quad ?$$

$$y = 3e^x$$

$$I: (-\infty, +\infty)$$

An Example of A First Order Initial Value Problem

Second Order Examples (y=f(x))

$$f''(x) = f(x)$$

$$y'' = y$$

An Example of A Second Order Differential Equation

$f(x) =$

$c_1 e^{+x} + c_2 e^{-x}$

for all x

$y =$

$c_1 e^{+x} + c_2 e^{-x}$

$I: (-\infty, +\infty)$

$$f''(x) = f(x)$$

$$f'(0) = 0$$

$$f(0) = 1$$

$$y'' = y$$

$$y'(0) = 0$$

$$f(0) = 1$$

An Example of A Second Order Initial Value Problem

$f(x) =$

$+1 \cdot e^{+x} - 1 \cdot e^{-x}$

for all x

$y =$

$+1 \cdot e^{+x} - 1 \cdot e^{-x}$

$I: (-\infty, +\infty)$

Guess the possible solution.

General First & Second Order IVPs (y=f(x))

First Order Initial Value Problem

$$f'(x) = f(x) \qquad f(0) = 3$$

$$y' = y \qquad f(0) = 3$$

$$\frac{dy}{dx} = g(x, y) \qquad y(x_0) = y_0$$

$$y' = g(x, y) \qquad y(x_0) = y_0$$

Second Order Initial Value Problem

$$f''(x) = f(x) \qquad f'(0) = 0 \qquad f(0) = 1$$

$$y'' = y \qquad y'(0) = 0 \qquad f(0) = 1$$

$$\frac{d^2y}{dx^2} = g(x, y, y') \qquad y(x_0) = y_0 \qquad y'(x_0) = y_1$$

$$y'' = g(x, y, y') \qquad y(x_0) = y_0 \qquad y'(x_0) = y_1$$

Guess the possible solution.

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”