Complex Integration (2A)

Young Won Lim
2/24/14

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Cauchy's Theorem and Integral

The integral of a complex function is path independent iff the integral over a closed contour always vanishes

The integral vanishes if $f(z)$ is **analytic** (differentiable) at the every interior point of a closed contour

For the closed curve **C** and the interior domain **A** of **C**

$$\int_C f(z)\,dz$$

if $f(z)$ is **analytic** (differentiable) everywhere inside and on **C**

i.e, if $\frac{df}{dz}$ exists

For any counterclockwise contour **C** that encloses z_0

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)}\,dz$$

if $f(z)$ is **analytic** (differentiable) everywhere inside and on **C**

i.e, if $\frac{df}{dz}$ exists

knowing $f(z)$ on **C** completely determines $f(z)$ everywhere **A** inside the contour.

– boundary integral method

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z}\,dw$$

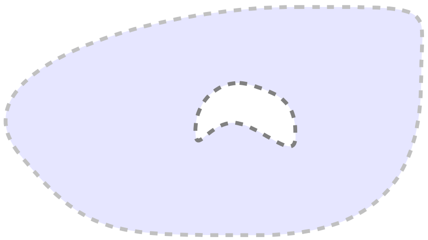
Young Won Lim
2/24/14

Domain and Region

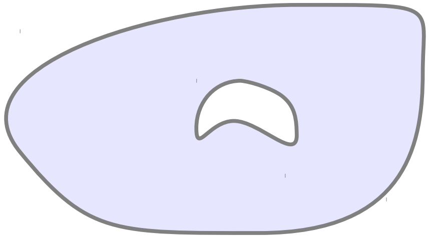
A **connected** set S

Any two of its points can be joined by a broken line
of finitely many straight-line segments
all of whose points belong to S

An **open connected** set S : a **domain**



An **open connected** set S +
some or all of its boundary points : a **region**



Simply and Multiply Connected

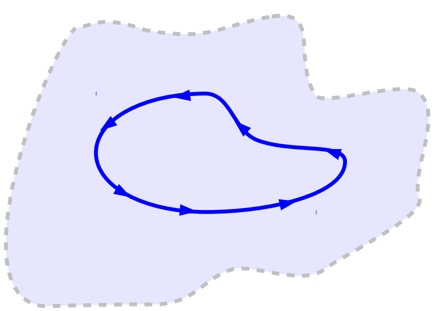
A **simply connected** domain D

If every simple closed contour C lying entirely in D
can be **shrunk to a point** without leaving D

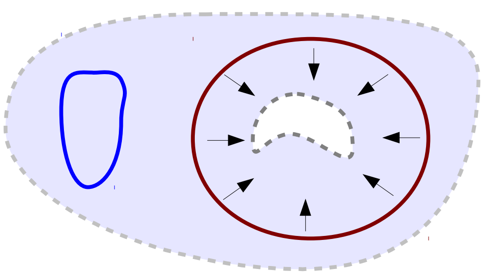
Every simple closed contour C lying entirely in D
encloses **only points** of D

No holes in D

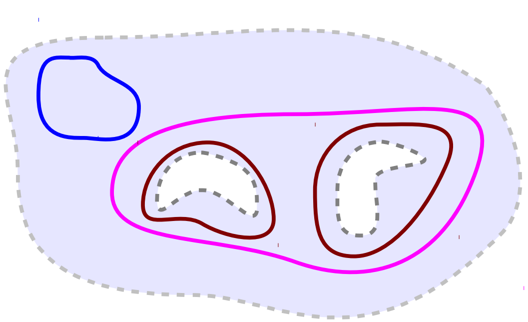
Simply Connected



Doubly Connected



Triply Connected

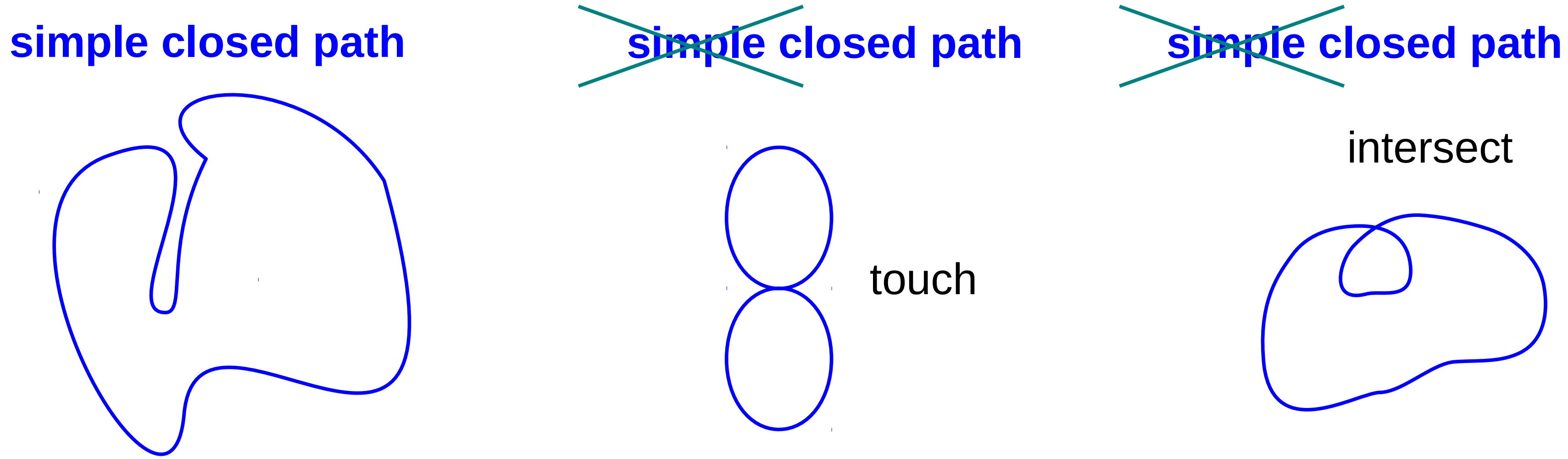


Simple Closed Path

A **simple closed path**

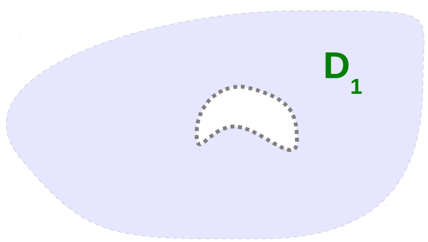
A closed path that does not intersect or touch itself

Closed Paths

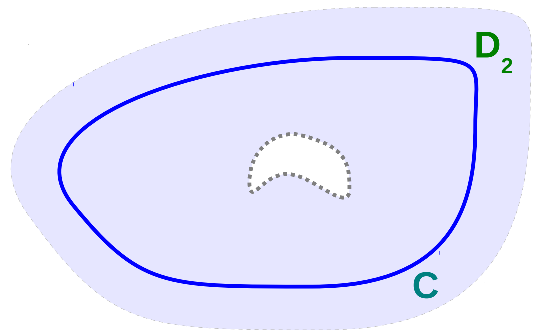


Domains and Regions

Doubly Connected Domain D_1

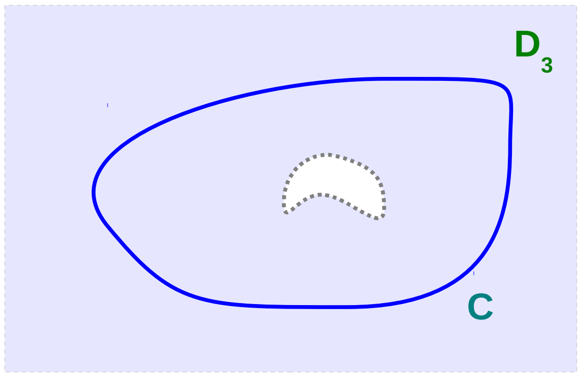


Doubly Connected Domain D_2



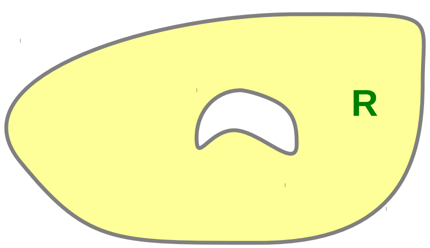
Simple Closed Path C

Doubly Connected Domain D_3



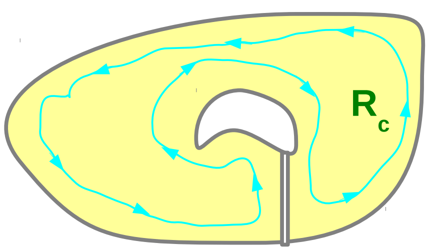
Simple Closed Path C

Can convert into
Simply Connected Regions R_c or R_a & R_b



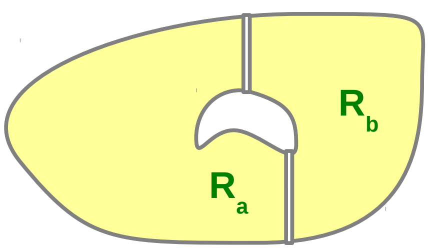
Boundary included

Simply Connected Region R_c



1 cut

Simply Connected Regions R_a & R_b



2 cuts

Contour Integrals

$f(z)$: defined at points of a smooth curve C

The contour integral of **f** along C

a smooth curve C is defined by

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad a \le t \le b$$

$$\int_C f(z)\, dz = \int_C (u + iv)(dx + i\,dy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$= \int_a^b [u\,x'(t) - v\,y'(t)]\,dt + i\int_C [v\,x'(t) + u\,y'(t)]\,dt$$

$$= \int_a^b (u + iv)(x'(t) + i\,y'(t))\,dt$$

$$z(t) = x(t) + i\,y(t)$$

$$z'(t) = x'(t) + i\,y'(t)$$

$$a \le t \le b$$

$$\int_C f(z)\, dz = \int_a^b f(z(t))\,z'(t)\,dt$$

Antiderivative

$f(z)$: continuous in a domain D $f(z) = F'(z)$ for every z in a domain D	$\triangleq$	$F(z)$: antiderivative of $f(z)$

$F(z)$: antiderivative of $f(z)$ for every z in a domain D

➡ $F(z)$ has a derivative at every z in a domain D : $f(z)$

➡ $F(z)$ analytic at every z in a domain D

➡ $F(z)$ continuous at every z in a domain D

Differentiability implies continuity

Fundamental Theorem (1)

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Fundamental Theorem for Contour Integrals

$f(z)$: **continuous** in a **domain** D

$F(z)$: **antiderivative** of $f(z)$ $F'(z) = f(z)$ **for every z** in a **domain** D

⇒

$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

for **any contour** C in D

with an initial point z_1 and a terminal point z_2 (**any point** z_1, z_2 in D)

Fundamental Theorem (2)

$f(z)$: **continuous** in a **domain** D

$F(z)$: **antiderivative** of $f(z)$ $F'(z) = f(z)$ **for every z** in a **domain** D

$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

for **any contour** C in D

with an initial point z_1 and a terminal point z_2 (**any point** z_1, z_2 in D)

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_a^b F'(z(t))\,z'(t)\,dt \quad = \int_a^b \frac{d}{dt} F(z(t))\,dt$$

$$= F(z(b)) - F(z(a))$$

$$= F(z_2) - F(z_1)$$

Fundamental Theorem (3)

$f(z)$: **continuous** in a **domain** D

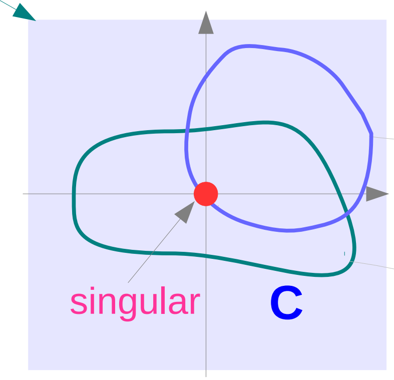
$F(z)$: **antiderivative** of $f(z)$ $F'(z) = f(z)$

$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

for **any contour** C in D

with an initial point z_1 and a terminal point z_2 (**any point** z_1, z_2 in D)

D: **multiply** connected domain



C: simple closed path

$$\int_{z_0}^{z_1} f(z)\,dz \;\cancel{=}\; F(z_1) - F(z_0)$$

for **any contour C** in D

we may not call F(z) an antiderivative of f(z) in D

we can still find a contour **C** such that

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

for a contour **C**

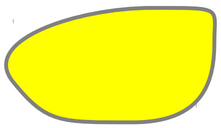
Contour Integration Evaluation (1)

(1) Indefinite Integration of Analytic Functions

$$f(z) = F'(z)$$

antiderivative

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$



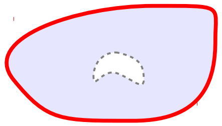
must have no
singularities in D

(2) Integration by the Use of the Path

$$z = z(t) \qquad (a \le t \le b)$$

parametric

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt$$



must be
continuous on C

Contour Integration Evaluation

(1) Indefinite Integration of Analytic Functions

$f(z)$: **analytic** in a **simply connected domain D** $f(z) = F'(z)$

➡ There exists an **indefinite integral** in D : an analytic function $F(z)$

➡ $$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0)$$

for every path in D between z_0 and z_1

(2) Integration by the Use of the Path

$f(z)$ a **continuous** function on a path C

C : a **piecewise smooth path** represented by $z = z(t) \quad (a \le t \le b)$

➡ $$\int_C f(z)\, dz = \int_a^b f[z(t)]\, z'(t)\, dt$$

Contour Integration Evaluation f(z) = 1/z

(1) Indefinite Integration of Analytic Functions

$z_1 = z_0$ ➡ $\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) = 0$

But $f(z) = \frac{1}{z}$ not analytic at $z = 0$ ➡ cannot apply this method

(2) Integration by the Use of the Path

C : the unit circle ➡ $z(t) = \cos t + i \sin t = e^{it}$ $(0 \le t \le 2\pi)$

$z'(t) = -\sin t + i \cos t = i e^{it}$

$$\int_C f(z)\,dz = \int_0^{2\pi} \frac{i e^{it}}{e^{it}}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

Contour Integration Evaluation $f(z) = z^m$

(1) Indefinite Integration of Analytic Functions

$z_1 = z_0$ ➡ $\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) = 0$

But $f(z) = z^m$ not analytic at $z = 0$ for $m < 0$ ➡ cannot apply this method

(2) Integration by the Use of the Path

C : the unit circle ➡ $z(t) = \cos t + i\sin t = e^{it}$ $\quad (0 \le t \le 2\pi)$

$z'(t) = -\sin t + i\cos t = ie^{it}$

$$\int_C f(z)\,dz = \int_0^{2\pi} e^{mit} ie^{it}\,dt = \int_0^{2\pi} ie^{i(m+1)t}\,dt = i\left[\int_0^{2\pi} \cos((m+1)t)\,dt + i\int_0^{2\pi} \sin((m+1)t)\,dt\right]$$

$$\int_C z^m\,dz = \begin{cases} 2\pi i & (m = -1) \\ 0 & (m \ne -1) \end{cases}$$

Contour Integration $f(z) = z^2,\ z^1,\ z^0,\ z^{-1},\ z^{-2},\ z^{-3}$

$$\int_C f(z)\,dz = \int_0^{2\pi} e^{mit}\, ie^{it}\,dt = \int_0^{2\pi} ie^{i(m+1)t}\,dt \qquad dz = ie^{it}dt$$

m=2	$\int_C z^2\,dz = \int_0^{2\pi} e^{i2t} ie^{it}\,dt = \int_0^{2\pi} ie^{i3t}\,dt = \left[\frac{1}{3}e^{i3t}\right]_0^{2\pi} = \frac{1}{3}(e^{6\pi} - e^0) = 0$	3
m=1	$\int_C z\,dz = \int_0^{2\pi} e^{it} ie^{it}\,dt = \int_0^{2\pi} ie^{i2t}\,dt = \left[\frac{1}{2}e^{i2t}\right]_0^{2\pi} = \frac{1}{2}(e^{4\pi} - e^0) = 0$	2
m=0	$\int_C 1\,dz = \int_0^{2\pi} ie^{it}\,dt = \int_0^{2\pi} ie^{it}\,dt = \left[e^{it}\right]_0^{2\pi} = (e^{2\pi} - e^0) = 0$	1
m=-1	$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} e^{-it} ie^{it}\,dt = \int_0^{2\pi} i\,dt = \left[i\right]_0^{2\pi} = i(2\pi - 0) = 2\pi i$	0
m=-2	$\int_C \frac{1}{z^2}\,dz = \int_0^{2\pi} e^{-i2t} ie^{it}\,dt = \int_0^{2\pi} ie^{-it}\,dt = \left[-e^{-it}\right]_0^{2\pi} = -(e^{-2\pi} - e^0) = 0$	-1
m=-3	$\int_C \frac{1}{z^3}\,dz = \int_0^{2\pi} e^{-i3t} ie^{it}\,dt = \int_0^{2\pi} ie^{-i2t}\,dt = \left[-\frac{1}{2}e^{-i2t}\right]_0^{2\pi} = -\frac{1}{2}(e^{-4\pi} - e^0) = 0$	-2

Integration by using an Antiderivative (1)

$z = e^w \quad (z \neq 0)$ ➡ $w = \ln z \quad (z \neq 0)$

$z = x + iy = e^{u+iv} = e^u(\cos v + i \sin v) = e^u \cos v + i e^u \sin v$

$e^u > 0$, $(\cos v + i\sin v) \neq 0$

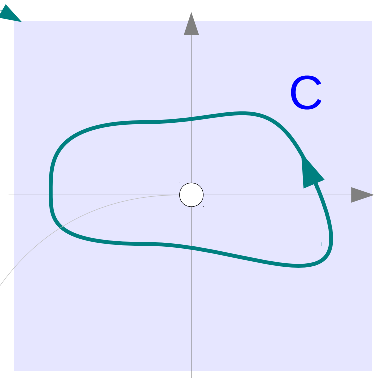
$$\frac{d}{dz} Ln\, z = \frac{1}{z}$$

principal value

D: **multiply** connected domain ➡ $Ln\, z$: not **analytic** in D

➡ $Ln\, z$ is not an antiderivative of $\frac{1}{z}$ in D

➡ ~~$\oint_C \frac{1}{z}\, dz = 0$~~ $\quad \oint_C \frac{1}{z}\, dz = 2\pi i$



C: simple closed path

~~$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0)$~~

$Ln\, z$ is not continuous on the negative real axis

➡ **branch cut**

$F(z)$: **antiderivative** of $f(z)$

➡ $F(z)$ **has a derivative at every z** in a **domain** D : $f(z)$

➡ $F(z)$ **analytic at every z** in a **domain** D

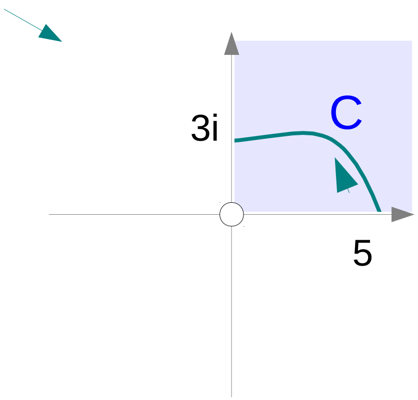
➡ $F(z)$ **continuous at every z** in a **domain** D

Integration by using an Antiderivative (2)

D: a **simply** connected domain ➡ $Ln\, z$: **analytic** in D

➡ $Ln\, z$ is an antiderivative of $\frac{1}{z}$ in D

$$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0)$$



$Ln\, z$ is continuous on the C

C: a simple path

$$\int_5^{3i} \frac{1}{z} dz = [Ln\, z]_5^{3i} = Ln\, 3i - Ln\, 5$$

$$Ln\, z = \ln|z| + i\, Arg(z)$$

$$= \ln 3 + i\frac{\pi}{2} - \ln 5 = \ln\frac{3}{5} + i\frac{\pi}{2}$$

$F(z)$: **antiderivative** of $f(z)$

➡ $F(z)$ **has a derivative at every z** in a **domain** D : $f(z)$

➡ $F(z)$ **analytic at every z** in a **domain** D

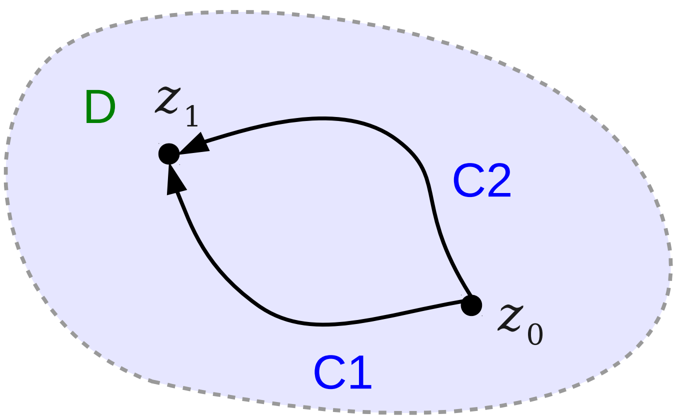
➡ $F(z)$ **continuous at every z** in a **domain** D

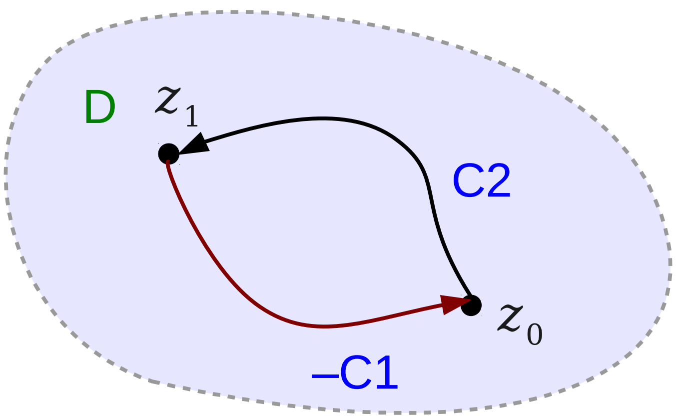
Independence of the Path

Independence of the path $\triangleq$ $z_0,\ z_1$: **points** in a domain D

for **all contours** C in D
with an initial point z_0 and a terminal point z_1

$\int_C f(z)\,dz$ The value of its **contour integral** is the same





$$\oint_{C1} f(z)\,dz = \oint_{C2} f(z)\,dz \quad \Leftrightarrow \quad \oint_{-C1} f(z)\,dz + \oint_{C2} f(z)\,dz = 0$$

Young Won Lim
2/24/14

Analyticity → Path Independence

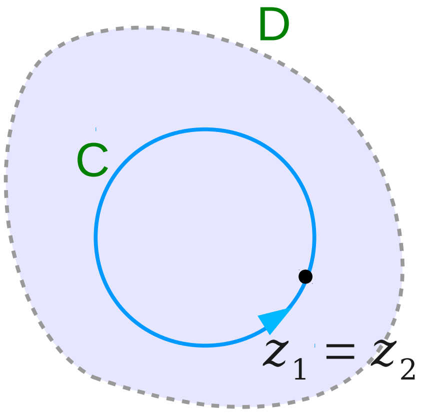
$f(z)$: **analytic** in a **simply connected domain** D

⇒ $\int_C f(z)\,dz$: **independent of the path**

analytic ⇒ **antiderivative**

⇒ $\int_C f(z)\,dz = F(z_2) - F(z_1)$

⇒ $z_2 = z_1$ → $\int_C f(z)\,dz = 0$



Young Won Lim
2/24/14

Antiderivative and Path Independence

$f(z)$: **continuous** in a **domain** D

$F(z)$: **antiderivative** of $f(z)$ $\quad F'(z) = f(z)$ **for every z** in a **domain** D

For **any contour** C in D with an initial point z_0 and a terminal point z

$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

$f(z)$: **continuous** in a **domain** D

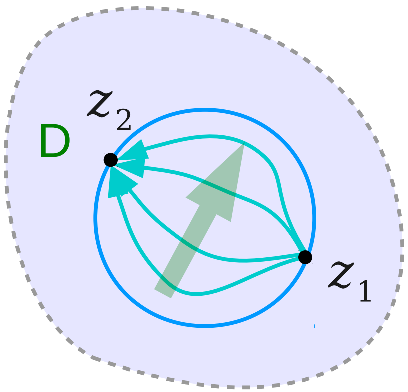
$F(z)$: **antiderivative** of $f(z)$ $\quad F'(z) = f(z)$ **for every z** in a **domain** D

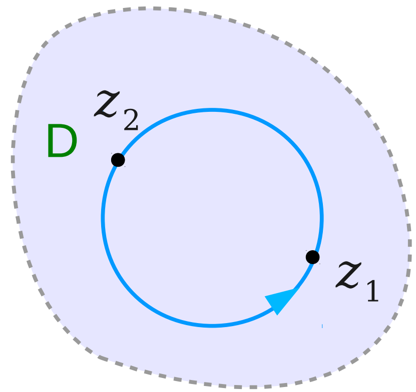
$\int_C f(z)\,dz$: **independent of the path**

Principle of Deformation of Path

Impose a continuous deformation of the path of an integral

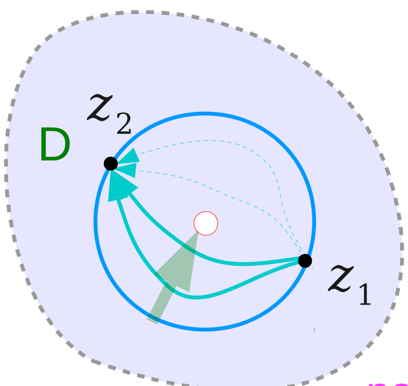
As long as deforming path always *contains only points*
at which f(z) is **analytic**, the integral retains the **same** value



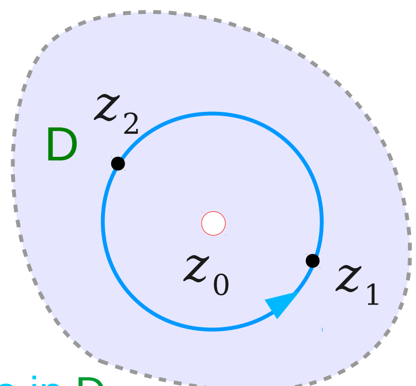


$m \geq 0$

$$\oint (z - z_0)^m \, dz = 0$$



continuous
deformation :
impossible



no anti-derivative in D

$m < 0$ not necessarily zero

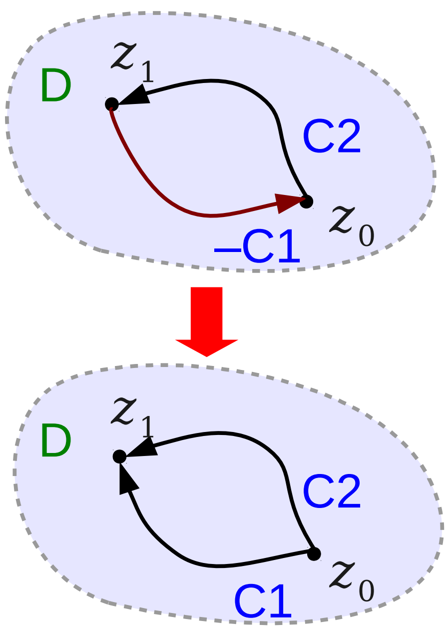
$$\oint (z - z_0)^m \, dz \neq 0$$

$$\oint (z - z_0)^{-1} \, dz = 2\pi i$$

$$\oint_{-C1+C2} f(z)\,dz = 0$$

$$\oint_{-C1+\Gamma+C2-\Gamma} f(z)\,dz = \oint_{-C1+C2} f(z)\,dz = 0$$

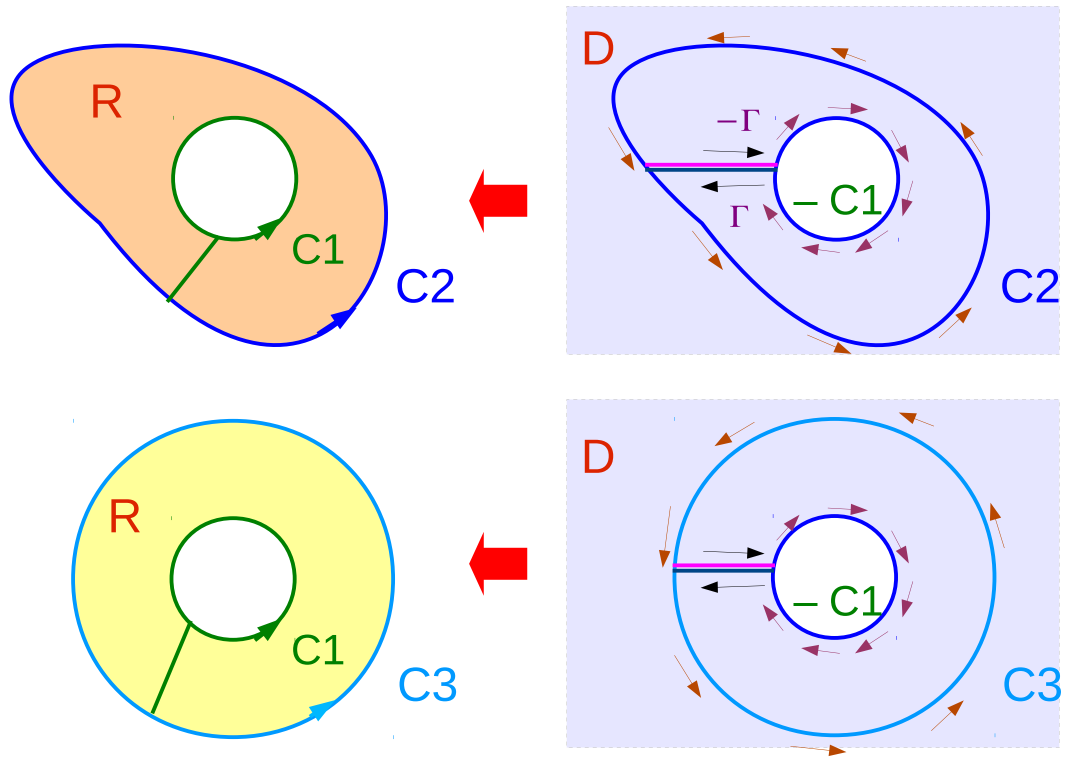
Deformation of Contours



simply connected domain

simply connected range

doubly connected domain



$$\oint_{C1} f(z)\,dz = \oint_{C2} f(z)\,dz$$

$$\oint_{C1} f(z)\,dz = \oint_{C2} f(z)\,dz = \oint_{C3} f(z)\,dz$$

Cauchy's Integral Theorem (1)

$f(z)$: **analytic** in a **simply connected domain** D

$f'(z)$: **continuous** in a **simply connected domain** D

⇒ for every **simple closed contour** C in D

$$\oint_C f(z)\, dz = 0$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Green's Theorem

line integration vs double integration

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA = 0$$

Cauchy-Riemann Eq

A necessary condition for analyticity

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

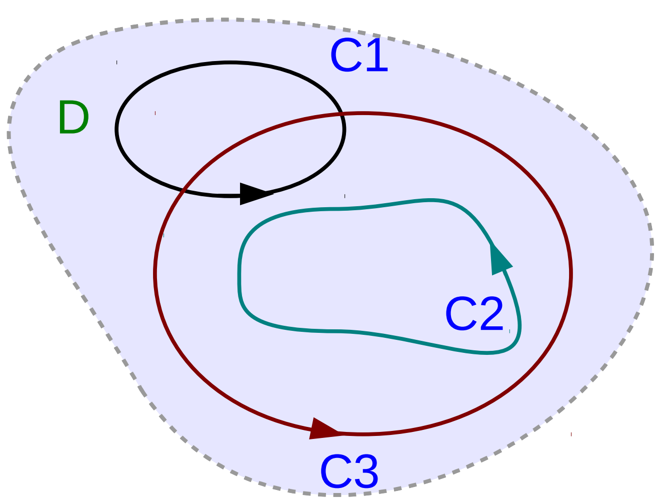
Cauchy's Integral Theorem (2)

$f(z)$: **analytic** in a **simply connected domain** D

$f'(z)$: **continuous** in a **simply connected domain** D

⇒ for every **simple[*] closed contour** C in D

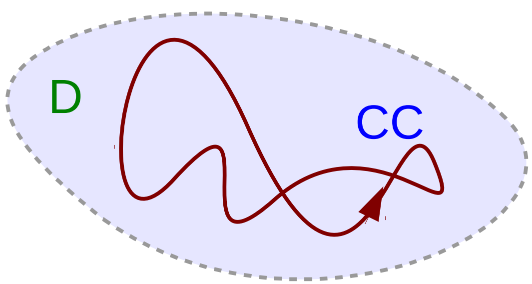
$$\oint_C f(z)\,dz = 0$$



$$\oint_{C1} f(z)\,dz = 0$$

$$\oint_{C2} f(z)\,dz = 0$$

$$\oint_{C3} f(z)\,dz = 0$$



Also for **any closed contour**

$$\oint_{CC} f(z)\,dz = 0$$

Cauchy-Goursat Theorem

$f(z)$: **analytic** in a **simply connected domain** D

⇒ for every **simple closed contour** C in D

$$\oint_C f(z)\,dz = 0$$

Cauchy Theorem

$f(z)$: **analytic** in a **simply connected domain** D

$f'(z)$: **continuous** in a **simply connected domain** D

⇒ for every **simple closed contour** C in D

$$\oint_C f(z)\,dz = 0$$

Cauchy-Goursat Theorem (2)

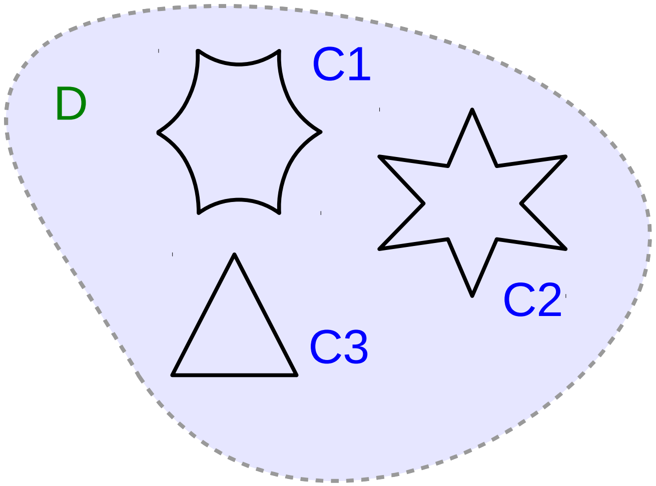
$f(z)$: **analytic** in a **simply connected domain** D

for every **simple closed contour** C in D

$$\oint_C f(z)\, dz = 0$$

$f'(z)$: **continuous** in a simply connected domain D

- simple closed curve
- a continuously turning tangent
- **except** possibly at a finite number of points
- allow a finite number of corners (not smooth)



Cauchy-Goursat Theorem (3)

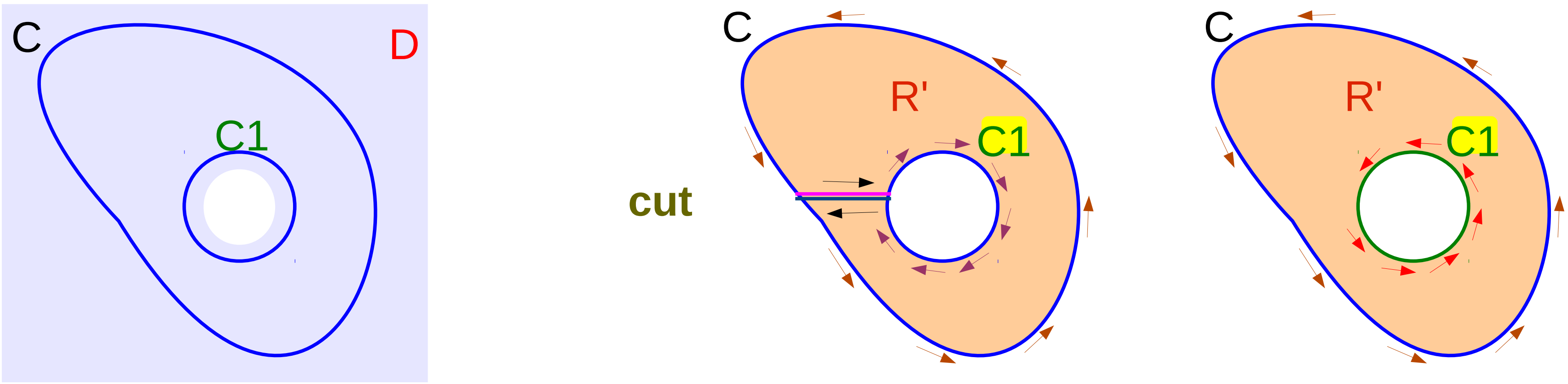
$f(z)$: **analytic** in a **multiply connected domain** D

for every **simple closed contour** C in D

$$\oint_C f(z)\,dz \neq 0$$

* not necessarily zero

doubly connected domain D → **simply connected region** R' **contour C & C1**

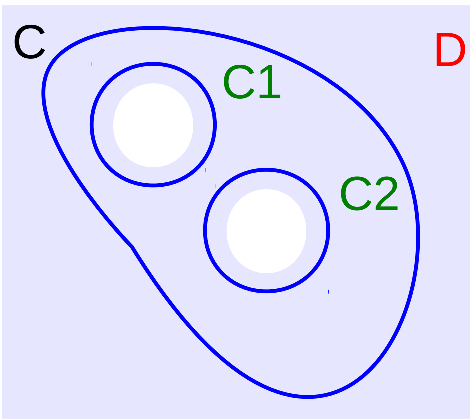


$$\oint_{ccw\ C} f(z)\,dz + \oint_{cw\ C1} f(z)\,dz = 0$$

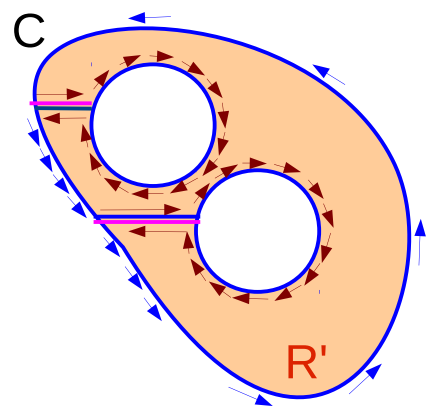
$$\oint_{ccw\ C} f(z)\,dz = \oint_{ccw\ C1} f(z)\,dz$$

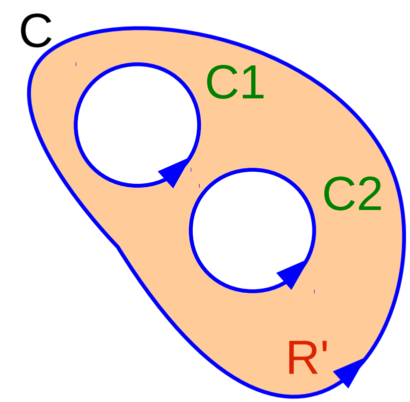
Cauchy-Goursat Theorem (4)

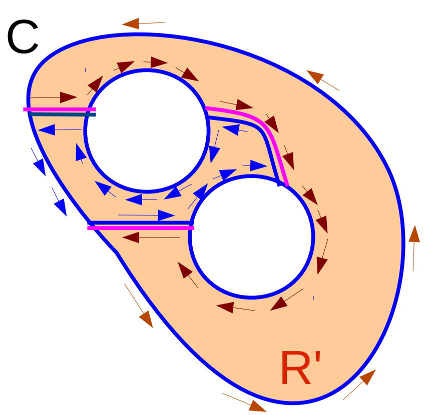
triply connected domain D ➡ **simply connected region** R' **contour C, C1, C2**



cuts





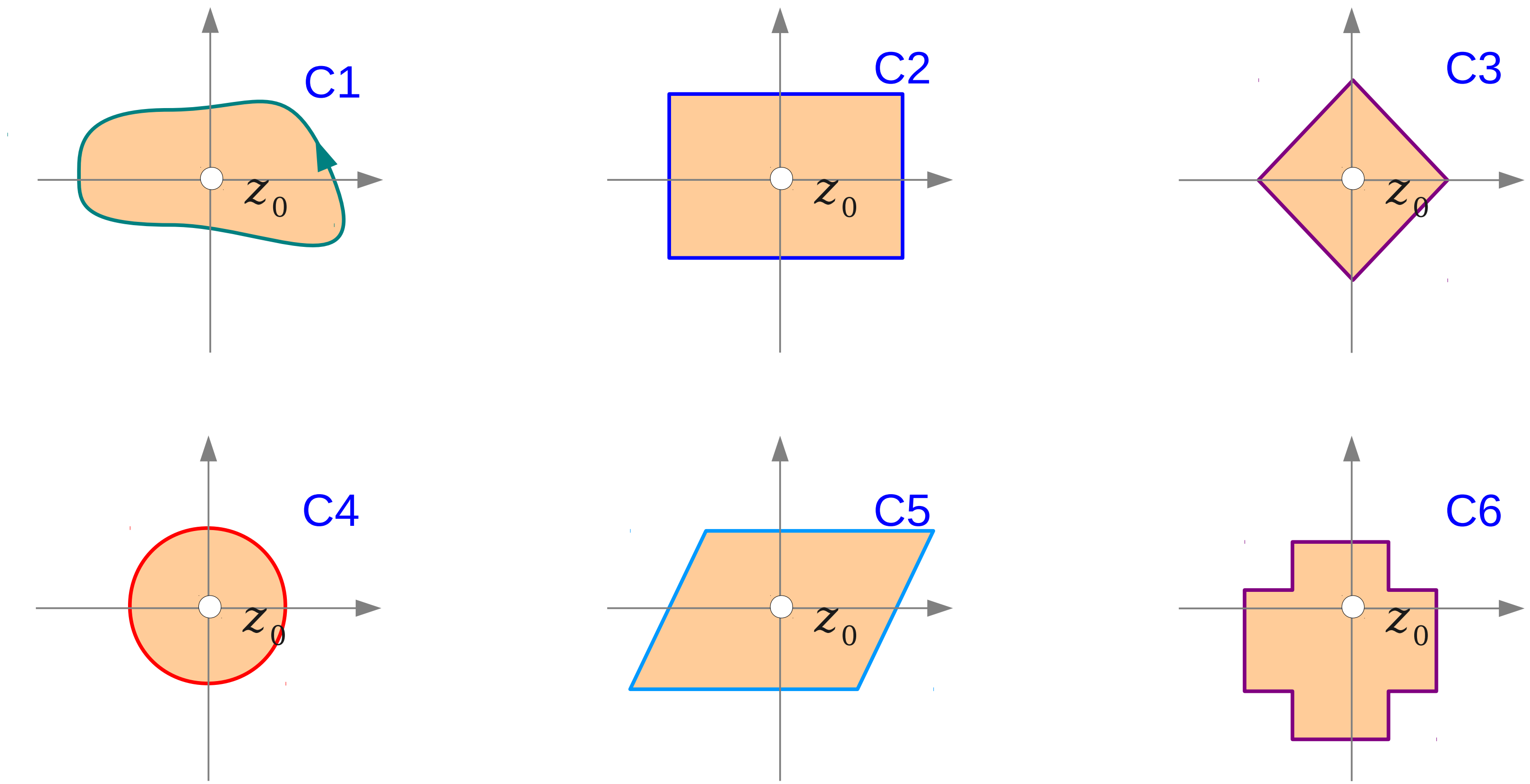


$$\oint_{ccw\ C} f(z)\,dz + \oint_{cw\ C1} f(z)\,dz + \oint_{cw\ C2} f(z)\,dz = 0$$

$$\oint_{C} f(z)\,dz = \oint_{C1} f(z)\,dz + \oint_{C2} f(z)\,dz$$

Integration of f(z) = 1/z

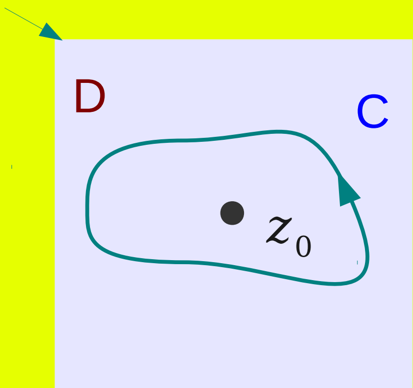
$$\int_C f(z)\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = \int_0^{2\pi} i\,dt = 2\pi i \qquad C \in \{C1,\ C2,\ C3,\ C4,\ C5,\ C6\}$$



Contour Integration for *f(z)/(z-z$_0$)*

$f(z)$ **analytic** in D

D: **simply** connected domain

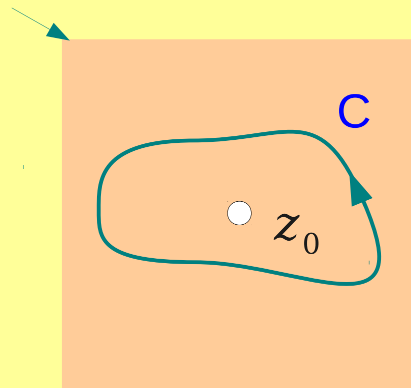


$$\oint_C f(z)\,dz = 0$$

$f(z_0)$

$\dfrac{f(z)}{(z-z_0)}$ not **analytic** at z_0 in D

D': **multiply** connected domain



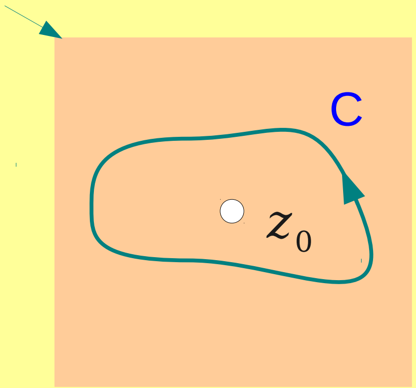
$$\oint_C \frac{f(z)}{(z-z_0)}\,dz \neq 0$$

not necessarily zero

Simply Connected Region R

$\dfrac{f(z)}{(z-z_0)}$ not **analytic** at z_0 in D

D': **multiply** connected domain

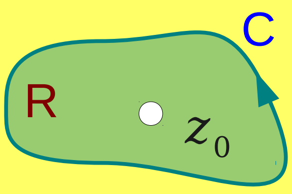


$\Rightarrow$ $\displaystyle\oint_C \frac{f(z)}{(z-z_0)}\,dz \neq 0$ not necessarily zero

$\dfrac{f(z)}{(z-z_0)}$ **analytic** at z_0 in a region R

R: all points within and on C

C: any simple closed path



$\Rightarrow$ $\displaystyle\oint_C \frac{f(z)}{(z-z_0)}\,dz \neq 0$ not necessarily zero

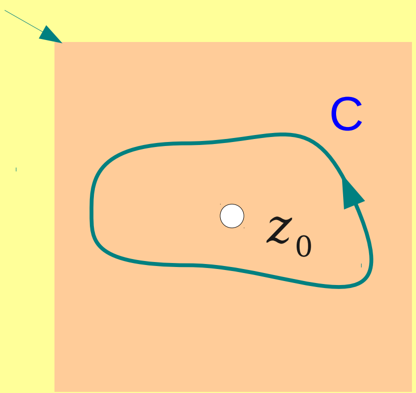
$$\oint_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i \boxed{f(z_0)}$$

The Function Value $f(z_0)$

$$\frac{f(z)}{(z-z_0)}$$

not **analytic** at z_0 in D

D': <u>**multiply**</u> connected domain

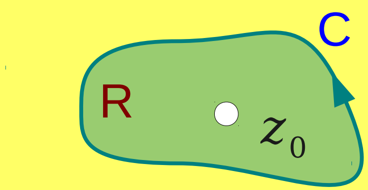


$$\frac{f(z)}{(z-z_0)}$$

analytic at z_0 in a reginon R

R: all points within and on C

C: any simple closed path



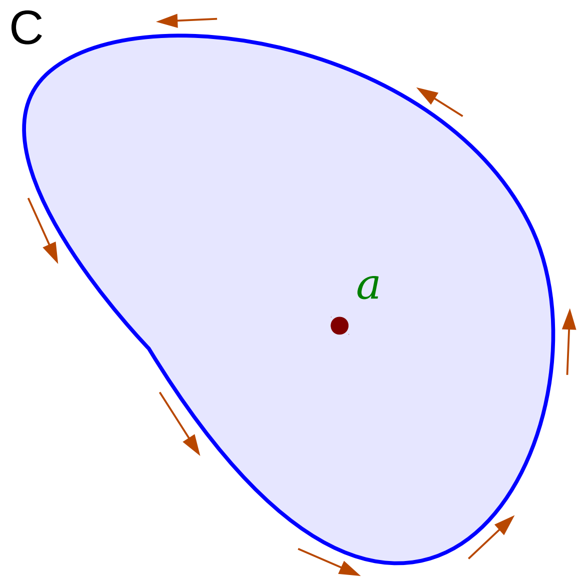
The value of an **analytic** function f
at any point z_0 in a **simply** connected domain
can be represented by a **contour integral**

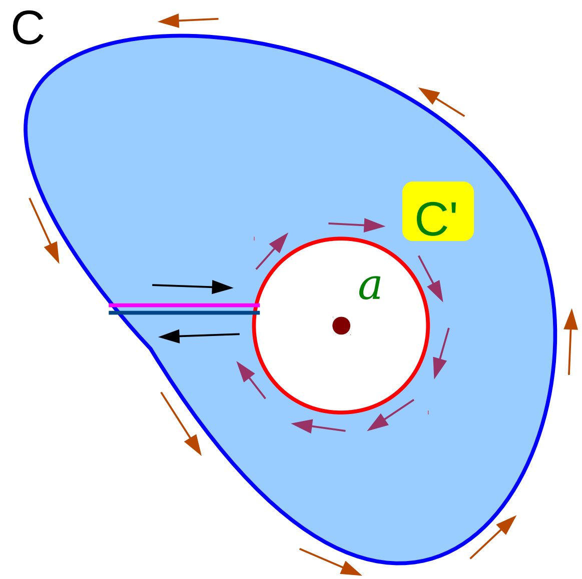
$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)}\, dz$$

Contour Integration in R

$$\oint_C f(z)\,dz = 0$$

$$\oint_{ccw\ C} \frac{f(z)}{z-a}\,dz + \oint_{cw\ C'} \frac{f(z)}{z-a}\,dz = 0$$





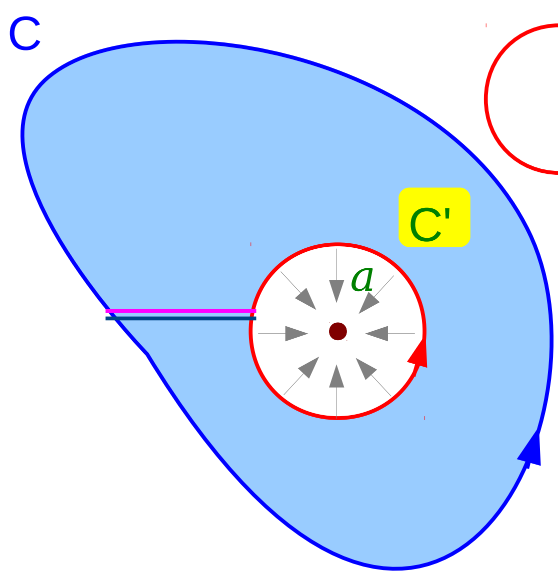
$$\oint_{ccw\ C} \frac{f(z)}{z-a}\,dz = \oint_{ccw\ C'} \frac{f(z)}{z-a}\,dz$$

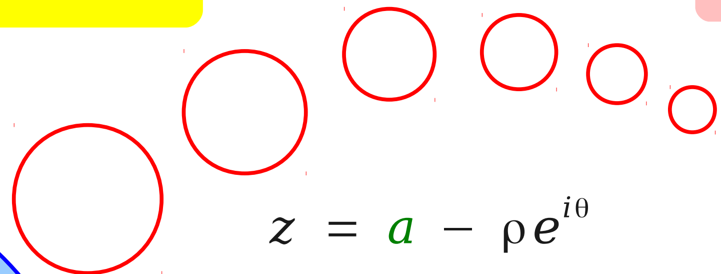
Young Won Lim
2/24/14

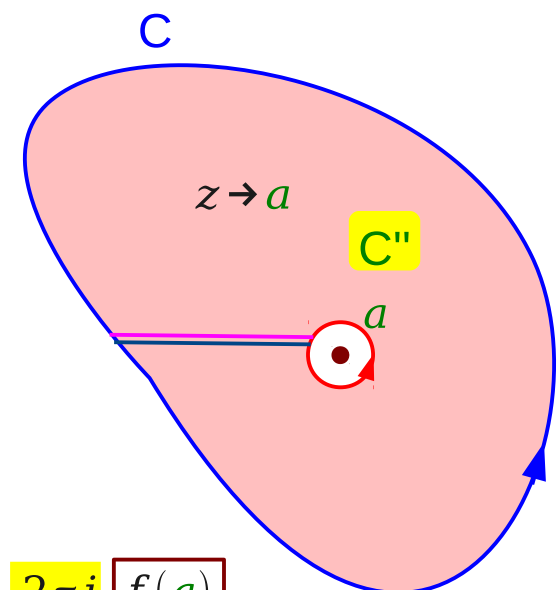
As z approaches to a

along C' $\quad z - a = \rho e^{i\theta}$

$z \to a \;\Rightarrow\; \rho \to 0,\; f(z) \to f(a)$







$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta} d\theta$$

$$\frac{dz}{z-a} = \frac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \int_0^{2\pi} f(z)\, i\, d\theta = 2\pi i\, \boxed{f(a)}$$

$f(z) \to f(a)$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$

$$= 2\pi i\; f(a)$$

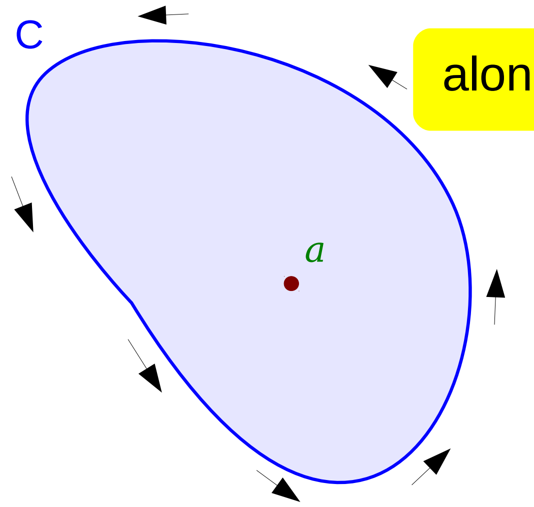
Other Contour Integration in R

$$\frac{dz}{(z-a)^2} = \frac{i\rho e^{i\theta}\,d\theta}{(\rho e^{i\theta})^2}$$

$$\oint_{ccw\ C} \frac{f(z)}{(z-a)^2}\,dz = \int_0^{2\pi} \frac{f(z)i}{\rho e^{i\theta}}\,d\theta$$

$$= \int_0^{2\pi} \frac{f(z)}{\rho} i e^{-i\theta}\,d\theta = \left[-\frac{f(z)}{\rho} e^{-i\theta}\right]_0^{2\pi}$$

$$= -\frac{f(z)}{\rho}(e^{-i2\pi} - e^{-i0}) = 0$$



along C' $\quad z - a = \rho e^{i\theta}$

$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta}\,d\theta$$

$$dz = i\rho e^{i\theta}\,d\theta$$

$$\oint_{ccw\ C} f(z)\,dz = \int_0^{2\pi} f(z) i\rho e^{i\theta}\,d\theta$$

$$= \left[f(z)\rho e^{i\theta}\right]_0^{2\pi}$$

$$= f(z)\rho(e^{-i2\pi} - e^{-i0}) = 0$$

$$(z-a)\,dz = \rho e^{i\theta} i\rho e^{i\theta}\,d\theta$$

$$\oint_{ccw\ C} (z-a)f(z)\,dz = \int_0^{2\pi} f(z) i(\rho e^{i\theta})^2\,d\theta$$

$$= \int_0^{2\pi} f(z)\rho^2 i e^{i2\theta}\,d\theta = \left[f(z)\frac{\rho}{2} e^{i2\theta}\right]_0^{2\pi}$$

$$= f(z)\frac{\rho}{2}(e^{-i4\pi} - e^{-i0}) = 0$$

Cauchy's Integral Formula I

$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

the value of $f(z)$ at a point $z = a$ inside C

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$

if $f'(z)$ exists
in the neighborhood of a point a

→ $f(z)$ is *infinitely differentiable* in that neighborhood

→ $f(z)$ can be expanded in a Taylor series about a that converges inside a disk whose radius is equal to the distance between a and the *nearest singularity* of $f(z)$

Cauchy's Integral Formula II

$f(z)$: **analytic** on and inside simple close curve C

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$

the value of $f(z)$ at a point $z = a$ inside C

$$\frac{d}{dz} f(z) = \frac{d}{dz} \left\{ \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw \right\} \qquad f'(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{(w-z)^2} dw$$

$$\frac{d}{dz} f(z) = \frac{d}{dz} \left\{ \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw \right\} \qquad f''(z) = \frac{2}{2\pi i} \oint \frac{f(w)}{(w-z)^3} dw$$

$$\frac{d}{dz} f(z) = \frac{d}{dz} \left\{ \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw \right\} \qquad f^{(3)}(z) = \frac{3!}{2\pi i} \oint \frac{f(w)}{(w-z)^4} dw$$

• • •

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

$f(z)$ is *infinitely differentiable* in that neighborhood

Cauchy's Integral Formula I & II

$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

the value of $f(z)$ at a point $z = a$ inside C

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$

$f(z)$ is *infinitely differentiable* in that neighborhood

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

Complex Analytic Functions

f(z) is **Analytic** (Holomorphic) ⟺ *f(z)* is differentiable in a neighborhood of z (if *df/dz* exists) ⟺ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Cauchy's Integration Formula I

f(z) is differentiable in a neighborhood of z (if *df/dz* exists) → then *f(z)* can be represented by a contour integral

Cauchy's Integration Formula II

→ then *f(z)* is **infinitely** differentiable in that neighborhood.

Generalized Cauchy's Formula

→ *f(z)* can be expanded in a Taylor series

Complex Analytic Functions

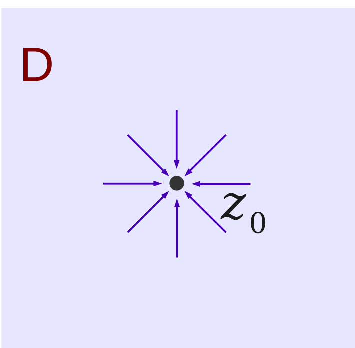
f(z) is **Analytic** (Holomorphic)

⟺

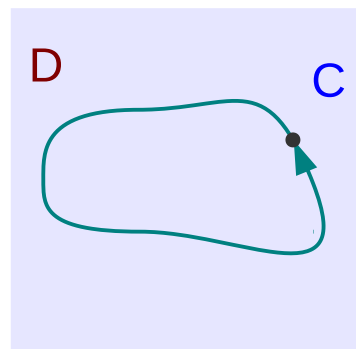
f(z) is differentiable in a neighborhood of z (if *df/dz* exists)

⟺

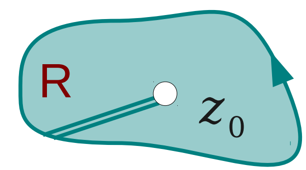
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



$$f(z)$$



$$\frac{f(z)}{(z-z_0)}$$



Cauchy's Theorem

$$\int_C f(z)\, dz = 0$$

Cauchy's Integration Formula I

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0} dz$$

Cauchy's Integration Formula II

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] E. Kreyszig, “Advanced Engineering Mathematics”
[5] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[6] T. J. Cavicchi, “Digital Signal Processing”
[7] F. Waleffe, Math 321 Notes, UW 2012/12/11
[8] R.E. Norton, “Complex Analysis for Scientists and Engineers An Introduction”