Laurent Series and z-Transform
- Geometric Series
Causality A

20191026 Sat

2 formulas of z

① $$\frac{3}{2}\frac{-1}{(z-0.5)(z-2)} = \left(\frac{1}{z-0.5} - \frac{1}{z-2}\right)$$

$$\Big\updownarrow z^{-1}$$

② $$\frac{3}{2}\frac{-z^2}{(z-2)(z-0.5)} = \left(\frac{0.5z}{(z-0.5)} - \frac{2z}{(z-2)}\right)$$

$$\boxed{\frac{3}{2}\frac{-1}{(z-0.5)(z-2)} \xleftrightarrow{z^{-1}} \frac{3}{2}\frac{-z^2}{(z-2)(z-0.5)}}$$

$$\frac{3}{2}\frac{-1}{(z-0.5)(z-2)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{1}{z-0.5} - \frac{1}{z-2}\right)$$

$$= \left(\frac{1}{z-0.5} - \frac{1}{z-2}\right)$$

z^{-1}

$$\frac{3}{2}\frac{-1}{(z^{-1}-0.5)(z^{-1}-2)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{1}{z^{-1}-0.5} - \frac{1}{z^{-1}-2}\right)$$

$$= \left(\frac{2}{2z^{-1}-1} - \frac{0.5}{0.5z^{-1}-1}\right)$$

$$= \left(\frac{2z}{2-z} - \frac{0.5z}{0.5-z}\right)$$

$$= \left(\frac{-2z}{z-2} + \frac{0.5z}{z-0.5}\right)$$

$$= z\left(\frac{-2}{z-2} + \frac{0.5}{z-0.5}\right)$$

$$= z\left(\frac{-\frac{3}{2}z}{(z-2)(z-0.5)}\right)$$

$$= \frac{3}{2}\frac{-z^2}{(z-2)(z-0.5)}$$

$$\frac{3}{2}\frac{-z^2}{(z-2)(z-0.5)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{0.5z}{(z-0.5)} - \frac{2z}{(z-2)}\right)$$

$f(z), g(z)$: causal form of $\boxed{\text{Laurent}}$ series

$$\frac{\text{nominator polynomial of } z}{\text{denominator polynomial of } z}$$

$f(z^{-1}), g(z^{-1})$: anti-causal form of $\boxed{\text{Laurent}}$ series

$$\frac{\text{nominator polynomial of } z^{-1}}{\text{denominator polynomial of } z^{-1}}$$

$X(z), Y(z)$: causal form of $\boxed{z\text{- Trans}}$

$$\frac{\text{nominator polynomial of } z^{-1}}{\text{denominator polynomial of } z^{-1}}$$

$X(z^{-1}), Y(z^{-1})$: anti-causal form of $\boxed{z\text{- Trans}}$

$$\frac{\text{nominator polynomial of } z}{\text{denominator polynomial of } z}$$

2 formulas Simple Pole Form

$$\boxed{\frac{1}{z - p}} \qquad \boxed{\frac{1}{z^{-1} - p}}$$

2 representations each Geometric Series Form

causal anti- causal

$$\boxed{\frac{1}{z-p}} \quad \begin{cases} \boxed{-\dfrac{p^{-1}}{1 - p^{-1} z}} \triangleq f(z) = X(z^{-1}) \\ \qquad\qquad\qquad\quad \| \qquad\quad \| \\ \boxed{\dfrac{z^{-1}}{1 - p z^{-1}}} \triangleq Y(z) = g(z^{-1}) \end{cases}$$

causal anti- causal

causal anti- causal

$$\boxed{\frac{1}{z^{-1}-p}} \quad \begin{cases} \boxed{-\dfrac{p^{-1}}{1 - p^{-1} z^{-1}}} \triangleq X(z) = f(z^{-1}) \\ \qquad\qquad\qquad\quad \| \qquad\quad \| \\ \boxed{\dfrac{z}{1 - p z}} \triangleq g(z) = Y(z^{-1}) \end{cases}$$

causal anti- causal

Simple Pole Form | Geometric Series Form

z polynomials

z^{-1} polynomials

causal	causal
$f(z)$	$g(z)$
$\Vert$	$\Vert$
$Y(z)$	$X(z)$
causal	causal

z polynomials

z^{-1} polynomials

z polynomials

z^{-1} polynomials

anti- causal		causal		causal		anti- causal
$X(z^{-1})$	$=$	$f(z)$		$g(z)$	$=$	$Y(z^{-1})$
$\Vert$		$\Vert$		$\Vert$		$\Vert$
$g(z^{-1})$	$=$	$Y(z)$		$X(z)$	$=$	$f(z^{-1})$
anti- causal		causal		causal		anti- causal

z polynomials

z^{-1} polynomials

$$\boxed{\frac{1}{z-p}} \xleftrightarrow{\;\frac{p^{-1}}{p^{-1}}\;} \boxed{-\frac{p^{-1}}{1-p^{-1}z}} = -(p^{-1})^{k+1} z^{k} \qquad k \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

① $-(p^{-1})^{n+1} z^{n}$ n $n = 0, 1, 2, \cdots$ causal

② $-(p^{-1})^{-m+1} z^{-m}$ $-m$ $m = 0, -1, -2, \cdots$ anti-causal

$$\boxed{\frac{1}{z-p}} \xleftrightarrow{\;\frac{z^{-1}}{z^{-1}}\;} \boxed{\frac{z^{-1}}{1-pz^{-1}}} = (p)^{k} z^{-k-1} \qquad -k-1 \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$

① $(p^{-1})^{n+1} z^{n}$ n $n = -1, -2, -3, \cdots$ anti-causal

② $(p^{-1})^{-m+1} z^{-m}$ $-m$ $m = 1, 2, 3, \cdots$ causal

$$\boxed{\frac{1}{z^{-1}-p}} \xleftrightarrow{\;\frac{p^{-1}}{p^{-1}}\;} \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} = -(p)^{-k-1} z^{-k} \qquad -k \qquad -(p^{1}, p^{2}, p^{3}, \cdots)$$

① $-(p)^{n-1} z^{n}$ n $n = 0, -1, -2, \cdots$ anti-causal

② $-(p)^{-m-1} z^{-m}$ $-m$ $m = 0, 1, 2, \cdots$ causal

$$\boxed{\frac{1}{z^{-1}-p}} \xleftrightarrow{\;\frac{z^{-1}}{z^{-1}}\;} \boxed{\frac{z}{1-pz}} = (p)^{k} (z)^{k+1} \qquad k+1 \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$

① $(p)^{n-1} z^{n}$ n $n = 1, 2, 3, \cdots$ causal

② $(p)^{-m-1} z^{-m}$ $-m$ $m = -1, -2, -3, \cdots$ anti-causal

$$\boxed{\frac{1}{z-p}}$$

$-\dfrac{p^{-1}}{1-p^{-1}z} = -(p^{-1})^{k+1} z^{k}$	k	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$		
① $-(p^{-1})^{n+1} z^{n}$	n	$n = 0, 1, 2, \cdots$	causal	$f(z)$
② $-(p^{-1})^{-m+1} z^{-m}$	$-m$	$m = 0, -1, -2, \cdots$	anti-causal	$X(z^{-1})$
$\dfrac{z^{-1}}{1-pz^{-1}} = (p)^{k} z^{-k-1}$	$-k-1$	$(p^{0}, p^{1}, p^{2}, \cdots)$		
① $(p^{-1})^{n+1} z^{n}$	n	$n = -1, -2, -3, \cdots$	anti-causal	$g(z^{-1})$
② $(p^{-1})^{-m+1} z^{-m}$	$-m$	$m = 1, 2, 3, \cdots$	causal	$Y(z)$

$$\boxed{\frac{1}{z^{-1}-p}}$$

$-\dfrac{p^{-1}}{1-p^{-1}z^{-1}} = -(p)^{-k-1} z^{-k}$	$-k$	$-(p^{1}, p^{2}, p^{3}, \cdots)$		
① $-(p)^{n-1} z^{n}$	n	$n = 0, -1, -2, \cdots$	anti-causal	$f(z^{-1})$
② $-(p)^{-m-1} z^{-m}$	$-m$	$m = 0, 1, 2, \cdots$	causal	$X(z)$
$\dfrac{z}{1-pz} = (p)^{k} (z)^{k+1}$	$k+1$	$(p^{0}, p^{1}, p^{2}, \cdots)$		
① $(p)^{n-1} z^{n}$	n	$n = 1, 2, 3, \cdots$	causal	$g(z)$
② $(p)^{-m-1} z^{-m}$	$-m$	$m = -1, -2, -3, \cdots$	anti-causal	$Y(z^{-1})$

$\boxed{\dfrac{1}{z-p}}$ causal $f(z)$, anti- causal $g(z^{-1})$

$\boxed{\dfrac{1}{z^{-1}-p}}$ anti- causal $f(z^{-1})$, causal $g(z)$

causal $f(z)$

$\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $\quad -(p^{-1})^{k+1}\, z^{k}$ $\quad -(p^{-1})^{n+1}\, z^{n}$

$-(p^{-1}, p^{-2}, p^{-3}, \dots)$ $\quad n = 0, 1, 2, \cdots$

$$\boxed{-(p^{-1}z^{0}+p^{-2}z^{1}+p^{-3}z^{2}+\cdots) = \sum_{n=0}^{\infty} -(p^{-1})^{n+1} z^{n} \quad (n \geq 0)}$$

$g(z^{-1})$ anti-causal

$\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $\quad (p)^{k}\, z^{-k-1}$ $\quad (p^{-1})^{n+1}\, z^{n}$

$(p^{0}, p^{1}, p^{2}, \dots)$ $\quad n = -1, -2, -3, \cdots$

$$\boxed{p^{0}z^{-1}+p^{1}z^{-2}+p^{2}z^{-3}+\cdots = \sum_{n=-1}^{-\infty} (p)^{-n-1} z^{n} \quad (n < 0)}$$

$f(z^{-1})$ anti-causal

$\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $\quad -(p^{-1})^{k+1}\, z^{-k}$ $\quad -(p)^{n-1}\, z^{n}$

$-(p^{-1}, p^{-2}, p^{-3} \dots)$ $\quad n = 0, -1, -2, \cdots$

$$\boxed{-(p^{-1}z^{0}+p^{-2}z^{-1}+p^{-3}z^{-2}+\cdots) = \sum_{n=0}^{-\infty} -(p)^{n-1} z^{n} \quad (n < 1)}$$

causal $g(z)$

$\boxed{\dfrac{z}{1-pz}}$ $\quad (p)^{k}\, z^{k+1}$ $\quad (p)^{n-1}\, z^{n}$

$(p^{0}, p^{1}, p^{2}, \dots)$ $\quad n = 1, 2, 3, \cdots$

$$\boxed{p^{0}z^{1}+p^{1}z^{2}+p^{2}z^{3}+\cdots = \sum_{n=1}^{\infty} (p)^{n-1} z^{n} \quad (n \geq 1)}$$

$$\begin{tikzcd}
\boxed{\dfrac{1}{z-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift left, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \arrow[l, shift left] \\
\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \arrow[u, shift left] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{\dfrac{1}{z^{-1}-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift left, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \arrow[l, shift left] \\
\boxed{\dfrac{z}{1-pz}} \arrow[u, shift left] &
\end{tikzcd}$$

$\boxed{\dfrac{1}{z-p}}$ anti-causal $X(z^{-1})$, causal $Y(z)$

$\boxed{\dfrac{1}{z^{-1}-p}}$ causal $X(z)$, anti-causal $Y(z^{-1})$

anti-causal $X(z^{-1})$: $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $\quad -(p^{-1})^{k+1}\, z^{k}$ $\quad -(p)^{m-1}\, z^{-m}$

$-(p^{-1}, p^{-2}, p^{-3}, \dots)$ $\quad m = 0, -1, -2, \cdots$

$$\boxed{-(p^{-1}z^{0} + p^{-2}z^{1} + p^{-3}z^{2} + \cdots) = -\sum_{n=0}^{-\infty} (p^{-1})^{-n+1} z^{-n} \qquad n < 1}$$

$Y(z)$ causal: $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $\quad (p)^{k}\, z^{-k-1}$ $\quad (p)^{m-1}\, z^{-m}$

$(p^{0}, p^{1}, p^{2}, \dots)$ $\quad m = 1, 2, 3, \cdots$

$$\boxed{p^{0}z^{-1} + p^{1}z^{-2} + p^{2}z^{-3} + \cdots = \sum_{n=1}^{\infty} (p)^{n-1} z^{-n} \qquad n \geq 1}$$

causal $X(z)$: $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $\quad -(p)^{-k-1}\, z^{-k}$ $\quad -(p^{-1})^{m+1}\, z^{-m}$

$-(p^{-1}, p^{-2}, p^{-3}, \dots)$ $\quad m = 0, 1, 2, \cdots$

$$\boxed{-(p^{-1}z^{0} + p^{-2}z^{-1} + p^{-3}z^{-2} + \cdots) = \sum_{n=0}^{\infty} -(p^{-1})^{n+1} z^{-n} \qquad (n \geq 0)}$$

$Y(z^{-1})$ anti-causal: $\boxed{\dfrac{z}{1-pz}}$ $\quad (p)^{k}(z)^{k+1}$ $\quad (p^{-1})^{m+1}\, z^{-m}$

$(p^{0}, p^{1}, p^{2}, \dots)$ $\quad m = -1, -2, -3, \cdots$

$$\boxed{p^{0}z^{1} + p^{1}z^{2} + p^{2}z^{3} + \cdots = \sum_{n=-1}^{-\infty} (p^{-1})^{n+1} z^{-n} \qquad (n < 0)}$$

$$\begin{tikzcd}
\boxed{\dfrac{1}{z-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift left] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \arrow[l, shift left] \\
\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \arrow[u, shift left, "\frac{z^{-1}}{z^{-1}}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{\dfrac{1}{z^{-1}-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift left] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \arrow[l, shift left] \\
\boxed{\dfrac{z}{1-pz}} \arrow[u, shift left, "\frac{z^{-1}}{z^{-1}}"] &
\end{tikzcd}$$

$\boxed{\text{Laurent}}$ $f(z), g(z)$: causal , $f(z^{-1}), g(z^{-1})$: anti-causal

$\boxed{z\text{- Trans}}$ $X(z), Y(z)$: causal, $X(z^{-1}), Y(z^{-1})$: anti-causal

$$
\begin{tikzcd}
\boxed{\dfrac{1}{z-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift left, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \arrow[l, shift left] \\
\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \arrow[u, shift left] &
\end{tikzcd}
$$

$$-\frac{p^{-1}}{1-p^{-1}z} = f(z) \leftrightarrow -(p^{-1})^{n+1}$$ causal

$X(z^{-1})$ anti- causal

$$\frac{z^{-1}}{1-pz^{-1}} = g(z^{-1})$$ anti- causal

$$Y(z) \leftrightarrow (p)^{n-1}$$ causal

$$
\begin{tikzcd}
\boxed{\dfrac{1}{z^{-1}-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift left, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \arrow[l, shift left] \\
\boxed{\dfrac{z}{1-pz}} \arrow[u, shift left] &
\end{tikzcd}
$$

$$-\frac{p^{-1}}{1-p^{-1}z^{-1}} = f(z^{-1})$$ anti- causal

$X(z) \leftrightarrow -(p^{-1})^{n+1}$ causal

$$\frac{z}{1-pz} = g(z) \leftrightarrow (p)^{n-1}$$ causal

$Y(z^{-1})$ anti- causal

2 formulas of z — $f(z),\ g(z)$ — $X(z),\ Y(z)$

2 representations — $f(z^{-1}),\ g(z^{-1})$ — $X(z^{-1}),\ Y(z^{-1})$

* Simple Pole Forms

$$
\begin{array}{ccccccc}
p & \boxed{\dfrac{1}{z-p}} & \xleftrightarrow{z^{-1}} & \boxed{\dfrac{1}{z^{-1}-p}} & p^{-1} \\
 & \Big\updownarrow \ \dfrac{z^{-1}}{z^{-1}} \cdot \quad \cdot \dfrac{z}{z} & & \Big\updownarrow \ \dfrac{z}{z} \cdot \quad \cdot \dfrac{z^{-1}}{z^{-1}} & \\
p & \boxed{\dfrac{z^{-1}}{1-pz^{-1}}} & \xleftrightarrow{z^{-1}} & \boxed{\dfrac{z}{1-pz}} & p^{-1}
\end{array}
$$

* Geometric Series Forms

$n \geq 0$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n \geq 0$
$-(p^{-1})^{n+1}$ causal f(z)	$-\dfrac{p^{-1}}{1-p^{-1}z}$	$\xleftrightarrow{z^{-1}}$	$-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}$	causal X(z) $-(p^{-1})^{n+1}$
	$\cdot \dfrac{p}{p}\dfrac{z^{-1}}{z^{-1}}$		$\dfrac{p}{p}\dfrac{z}{z} \cdot$	
causal X(z) $(p)^{n-1}$	$\dfrac{z^{-1}}{1-pz^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\dfrac{z}{1-pz}$	$(p)^{n-1}$ causal f(z)
$n \geq 1$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n \geq 1$

Laurent Series

anti- causal

causal

causal $f(z) \quad (|z| < p) \quad \longleftrightarrow \quad a_n \quad (n \geq 0) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

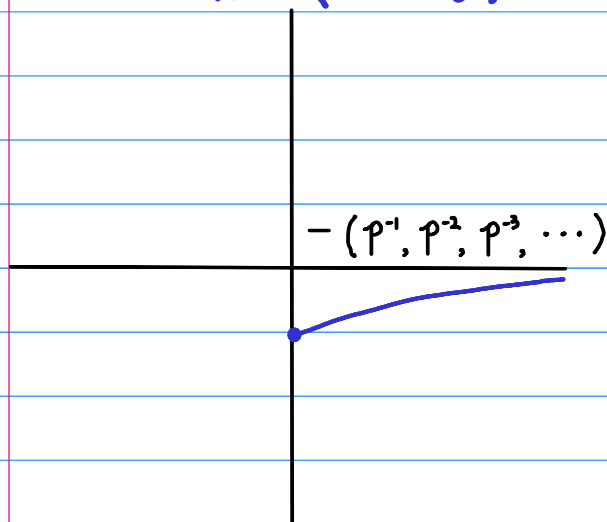
anti- causal $f(z^{-1}) \quad (|z| > p^{-1}) \quad \longleftrightarrow \quad a_{-n} \quad (n < 1) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

anti- causal $f(z) \quad (|z| > p) \quad \longleftrightarrow \quad -a_n \quad (n < 0) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$

causal $f(z^{-1}) \quad (|z| < p^{-1}) \quad \longleftrightarrow \quad -a_{-n} \quad (n \geq 1) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$

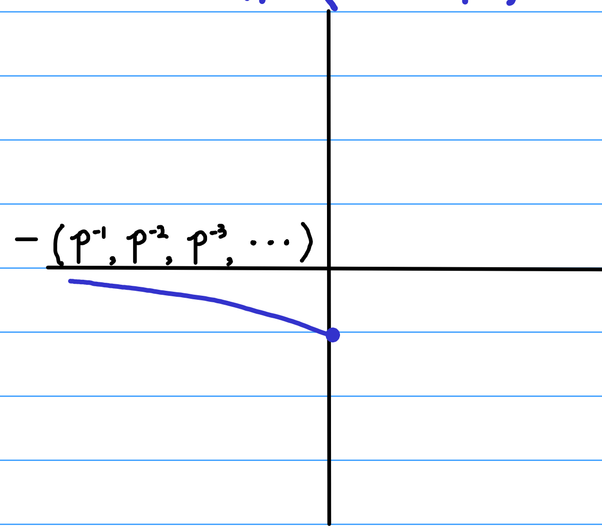
$f(z)$ $(|z| < p)$

a_n $(n \geq 0)$



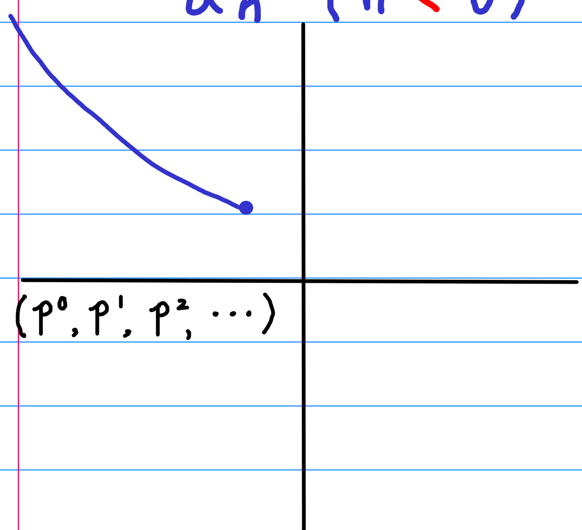
$f(z^{-1})$ $(|z| > p^{-1})$

a_{-n} $(n < 1)$



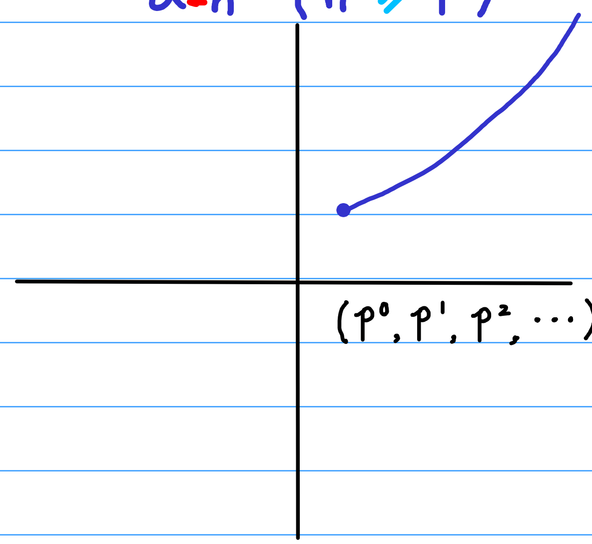
$f(z)$ $(|z| > p)$

$-a_n$ $(n < 0)$



$f(z^{-1})$ $(|z| < p^{-1})$

$-a_{-n}$ $(n \geq 1)$



Ⓐ $f(z)$ for $\boxed{|z|<p}$, $g(z)$ for $\boxed{|z|<p^{-1}}$ Laurent S

Geometric Series Forms

$p \qquad \boxed{f(z)} = \underset{}{\overset{|z|<p}{\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}}} \qquad \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$

$\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \qquad \underset{|z|<p^{-1}}{\boxed{\dfrac{z}{1-pz}}} = \boxed{g(z)} \quad p^{-1}$

Ⓑ $f(z^{-1})$ for $\boxed{|z|>p^{-1}}$, $g(z^{-1})$ for $\boxed{|z|>p}$

Geometric Series Forms

$f(z) = \overset{|z|<p}{\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}} \xleftrightarrow{z^{-1}} \overset{|z|>p^{-1}}{\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}} = \boxed{f(z^{-1})}$

$\boxed{g(z^{-1})} = \underset{|z|>p}{\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}} \xleftrightarrow{z^{-1}} \underset{|z|<p^{-1}}{\boxed{\dfrac{z}{1-pz}}} = g(z)$

Laurent Series $\boxed{a_n} \leftrightarrow f(z)$

symmetric ranges

$[0, \infty)$	$\xleftrightarrow{z^{-1}}$	$(-\infty, 0]$
$a_n \ (n \geq 0)$		$a_{-n} \ (n < 1)$

symmetric ranges

$(-\infty, -1]$	$\xleftrightarrow{z^{-1}}$	$[1, \infty)$
$b_{-n} \ (n < 0)$		$b_n \ (n \geq 1)$

$[0, \infty)$ $a_n \ (n \geq 0)$		$(-\infty, 0]$ $-b_n \ (n < 1)$	
‖	complementary ranges	‖	complementary ranges
$(-\infty, -1]$ $-a_n \ (n < 0)$		$[1, \infty)$ $b_n \ (n \geq 1)$	

$[0, \infty)$ $a_n \ (n \geq 0)$	$\xleftrightarrow{z^{-1}}$	$(-\infty, 0]$ $a_{-n} \ (n < 1)$
‖		‖
$(-\infty, -1]$ $-a_n \ (n < 0)$	$\xleftrightarrow{z^{-1}}$	$[1, \infty)$ $-a_{-n} \ (n \geq 1)$

Laurent Series $a_n \leftrightarrow \boxed{f(z)}$

ROC's with reciprocal poles

$$\boxed{f(z)\ (|z|<p)} \xleftrightarrow{z^{-1}} \boxed{f(z^{-1})\ (|z^{-1}|<p)}$$

ROC's with reciprocal poles

$$\boxed{g(z^{-1})\ (|z^{-1}|<p^{-1})} \xleftrightarrow{z^{-1}} \boxed{g(z)\ (|z|<p^{-1})}$$

$$\boxed{f(z)\ (|z|<p)} \qquad \boxed{f(z^{-1})\ (|z|>p^{-1})}$$

complementary ROC

complementary ROC

$$\boxed{g(z^{-1})\ (|z|>p)} \qquad \boxed{g(z)\ (|z|<p^{-1})}$$

$$\boxed{f(z)\ (|z|<p)} \xleftrightarrow{z^{-1}} \boxed{f(z^{-1})\ (|z|>p^{-1})}$$

$$\boxed{f(z)\ (|z|>p)} \xleftrightarrow{z^{-1}} \boxed{f(z^{-1})\ (|z|<p^{-1})}$$

Laurent Series $a_n \leftrightarrow f(z)$ $b_n \leftrightarrow g(z)$

Geometric Series Forms

$$\underset{|z|<p}{f(z) = \boxed{-\frac{p^{-1}}{1-p^{-1}z}}} \quad \xleftrightarrow{z^{-1}} \quad \underset{|z|>p^{-1}}{\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} = f(z^{-1})}$$

$$\Big\Vert \cdot \frac{p}{p}\frac{z^{-1}}{z^{-1}} \qquad\qquad\qquad \cdot \frac{p}{p}\frac{z}{z} \Big\Vert$$

$$\underset{|z|>p}{g(z^{-1}) = \boxed{\frac{z^{-1}}{1-pz^{-1}}}} \quad \xleftrightarrow{z^{-1}} \quad \underset{|z|<p^{-1}}{\boxed{\frac{z}{1-pz}} = g(z)}$$

	$f(z)$ $(\lvert z\rvert<p)$ a_n $(n \geq 0)$	$\xleftrightarrow{z^{-1}}$	$f(z^{-1})$ $(\lvert z^{-1}\rvert<p)$ a_{-n} $(n<1)$	$g(z)$ $(\lvert z\rvert>p^{-1})$ $-b_n$ $(n<1)$
$f(z)$ $(\lvert z\rvert>p)$ $-a_n$ $(n<0)$	$g(z^{-1})$ $(\lvert z^{-1}\rvert<p^{-1})$ b_{-n} $(n<0)$	$\xleftrightarrow{z^{-1}}$	$g(z)$ $(\lvert z\rvert<p^{-1})$ b_n $(n\geq 1)$	

Laurent Series using only $\boxed{a_n} \leftrightarrow \boxed{f(z)}$

$$
\begin{tikzcd}
 & \boxed{\begin{array}{l} f(z) \quad (|z|<p) \\ a_n \quad (n \geq 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{l} f(z^{-1}) \ (|z^{-1}|<p) \\ a_{-n} \quad (n<1) \end{array}} \arrow[d, equal] & \boxed{\begin{array}{l} g(z) \ (|z|>p^{-1}) \\ -b_n \quad (n<1) \end{array}} \\
\boxed{\begin{array}{l} f(z) \ (|z|>p) \\ -a_n \quad (n<0) \end{array}} & \boxed{\begin{array}{l} g(z^{-1}) \ (|z^{-1}|<p^{-1}) \\ b_{-n} \quad (n<0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{l} g(z) \ (|z|<p^{-1}) \\ b_n \quad (n \geq 1) \end{array}} &
\end{tikzcd}
$$

$$a_{-n} = -b_n \qquad\qquad -a_{-n} = b_n$$

$$
\begin{tikzcd}
\boxed{\begin{array}{l} f(z) \quad (|z|<p) \\ a_n \quad (n \geq 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{l} f(z^{-1}) \ (|z|>p^{-1}) \\ a_{-n} \quad (n<1) \end{array}} \arrow[d, equal] \\
\boxed{\begin{array}{l} f(z) \ (|z|>p) \\ -a_n \quad (n<0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{l} f(z^{-1}) \ (|z|<p^{-1}) \\ -a_{-n} \quad (n \geq 1) \end{array}}
\end{tikzcd}
$$

Z - Transform

anti- causal $X(z^{-1})\ (|z| < p) \leftrightarrow x_{-n}\ (n < 1)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

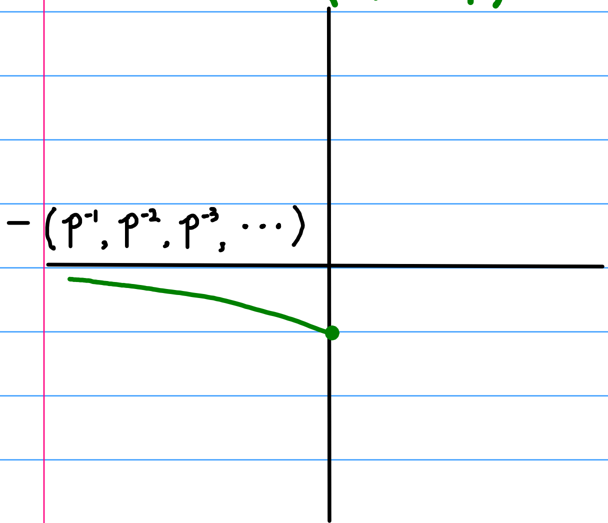
causal $X(z)\ (|z| > p^{-1}) \leftrightarrow x_{n}\ (n \geq 0)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

causal $X(z^{-1})\ (|z| > p) \leftrightarrow -x_{-n}\ (n \geq 1)\quad (p^{0}, p^{1}, p^{2}, \cdots)$

anti- causal $X(z)\ (|z| < p^{-1}) \leftrightarrow -x_{n}\ (n < 0)\quad (p^{0}, p^{1}, p^{2}, \cdots)$

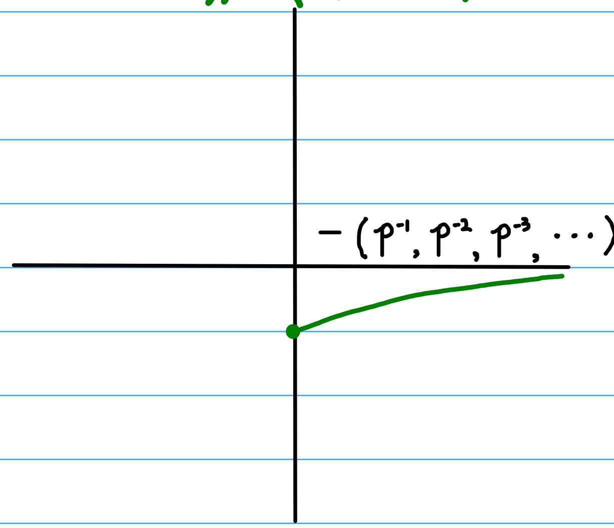
$X(z^{-1})$ $(|z| < p)$

x_{-n} $(n < 1)$



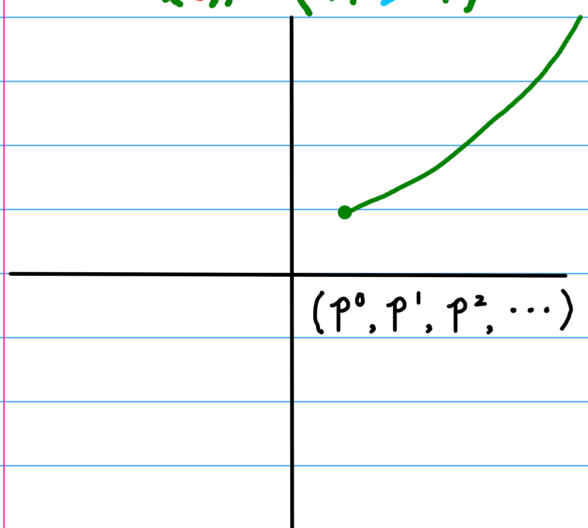
$X(z)$ $(|z| > p^{-1})$

x_n $(n \geqslant 0)$



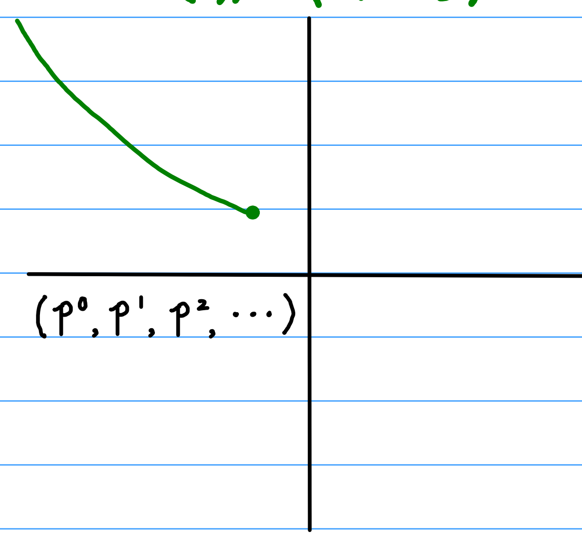
$X(z^{-1})$ $(|z| > p)$

$-x_{-n}$ $(n \geqslant 1)$



$X(z)$ $(|z| < p^{-1})$

$-x_n$ $(n < 0)$



Ⓐ $X(z)$ for $\boxed{|z|>p^{-1}}$, $Y(z)$ for $\boxed{|z|>p}$ z-Transform

Geometric Series Forms

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z}} \qquad \overset{|z|>p^{-1}}{\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}} = \boxed{X(z)} \quad p^{-1}$$

$$p \quad \boxed{Y(z)} = \overset{|z|>p}{\boxed{\frac{z^{-1}}{1-pz^{-1}}}} \qquad \boxed{\frac{z}{1-pz}}$$

Ⓑ $X(z^{-1})$ for $\boxed{|z|<p}$, $Y(z^{-1})$ for $\boxed{|z|<p^{-1}}$

Geometric Series Forms

$$\boxed{X(z^{-1})} = \overset{|z|<p}{\boxed{-\frac{p^{-1}}{1-p^{-1}z}}} \xleftrightarrow{z^{-1}} \overset{|z|>p^{-1}}{\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}} = X(z)$$

$$Y(z) = \underset{|z|>p}{\boxed{\frac{z^{-1}}{1-pz^{-1}}}} \xleftrightarrow{z^{-1}} \underset{|z|<p^{-1}}{\boxed{\frac{z}{1-pz}}} = \boxed{Y(z^{-1})}$$

z - Transform $\boxed{x_n} \leftrightarrow X(z)$

symmetric ranges

$(-\infty, 0]$ x_{-n} $(n < 1)$ $\xleftrightarrow{z^{-1}}$ $[0, \infty)$ x_n $(n \geq 0)$

symmetric ranges

$[1, \infty)$ y_n $(n \geq 1)$ $\xleftrightarrow{z^{-1}}$ $(-\infty, -1]$ y_{-n} $(n < 0)$

$(-\infty, 0]$ $-y_n$ $(n < 1)$ $\qquad$ $[0, \infty)$ x_n $(n \geq 0)$

complementary ranges $\qquad$ complementary ranges

$[1, \infty)$ y_n $(n \geq 1)$ $\qquad$ $(-\infty, -1]$ $-x_n$ $(n < 0)$

$(-\infty, 0]$ x_{-n} $(n < 1)$ $\xleftrightarrow{z^{-1}}$ $[0, \infty)$ x_n $(n \geq 0)$

$[1, \infty)$ $-x_{-n}$ $(n \geq 1)$ $\xleftrightarrow{z^{-1}}$ $(-\infty, -1]$ $-x_n$ $(n < 0)$

z - Transform $x_n \leftrightarrow \boxed{X(z)}$

ROC's with reciprocal poles

$X(z^{-1})$ $(|z^{-1}| > p^{-1})$ $\xleftrightarrow{z^{-1}}$ $X(z)$ $(|z| > p^{-1})$

ROC's with reciprocal poles

$Y(z)$ $(|z| > p)$ $\xleftrightarrow{z^{-1}}$ $Y(z^{-1})$ $(|z^{-1}| > p)$

$Y(z)$ $(|z| < p)$

complementary ROC

$Y(z)$ $(|z| > p)$

$X(z)$ $(|z| > p^{-1})$

complementary ROC

$X(z)$ $(|z| < p)$

$X(z^{-1})$ $(|z| < p)$ $\xleftrightarrow{z^{-1}}$ $X(z)$ $(|z| > p^{-1})$

$X(z^{-1})$ $(|z| > p)$ $\xleftrightarrow{z^{-1}}$ $X(z)$ $(|z| < p)$

Z- Transform $\quad X(z) \leftrightarrow x_n \qquad Y(z) \leftrightarrow y_n$

Geometric Series Forms

$$
\begin{tikzcd}[column sep=huge, row sep=large]
X(z^{-1}) = \underset{|z|<p}{\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal, "\cdot \frac{p}{p}\frac{z^{-1}}{z^{-1}}"] & \underset{|z|>p^{-1}}{\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}} = X(z) \arrow[d, equal, "\cdot \frac{p}{p}\frac{z}{z}"'] \\
Y(z) = \underset{|z|>p}{\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}} \arrow[r, leftrightarrow, "z^{-1}"] & \underset{|z|<p^{-1}}{\boxed{\dfrac{z}{1-pz}}} = Y(z^{-1})
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large, row sep=large]
\boxed{\begin{array}{c} Y(z)\ (|z|<p) \\ -y_n\ (n<1) \end{array}} & \boxed{\begin{array}{c} X(z^{-1})\ (|z^{-1}|>p^{-1}) \\ x_{-n}\ (n<1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{c} X(z)\ (|z|>p^{-1}) \\ x_n\ (n\geq 0) \end{array}} \arrow[d, equal] & \\
& \boxed{\begin{array}{c} Y(z)\ (|z|>p) \\ y_n\ (n\geq 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{c} Y(z^{-1})\ (|z^{-1}|>p) \\ y_{-n}\ (n<0) \end{array}} & \boxed{\begin{array}{c} X(z)\ (|z|<p^{-1}) \\ -x_n\ (n<0) \end{array}}
\end{tikzcd}
$$

z-Transform using only $\boxed{x_n} \longleftrightarrow \boxed{X(z)}$

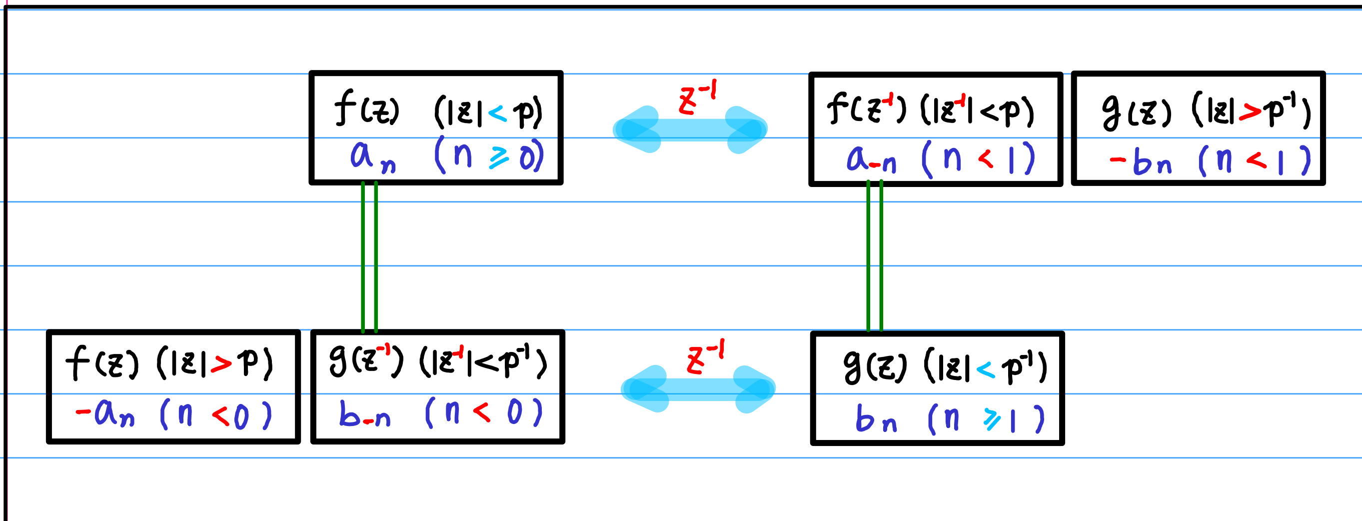
$Y(z)\ (\lvert z\rvert < p)$ $-y_n\ (n<1)$	$X(z^{-1})\ (\lvert z^{-1}\rvert > p^{-1})$ $x_{-n}\ (n<1)$	$\xleftrightarrow{z^{-1}}$	$X(z)\ (\lvert z\rvert > p^{-1})$ $x_n\ (n \geq 0)$	
	$Y(z)\ (\lvert z\rvert > p)$ $y_n\ (n \geq 1)$	$\xleftrightarrow{z^{-1}}$	$Y(z^{-1})\ (\lvert z^{-1}\rvert > p)$ $y_{-n}\ (n<0)$	$X(z)\ (\lvert z\rvert < p^{-1})$ $-x_n\ (n<0)$

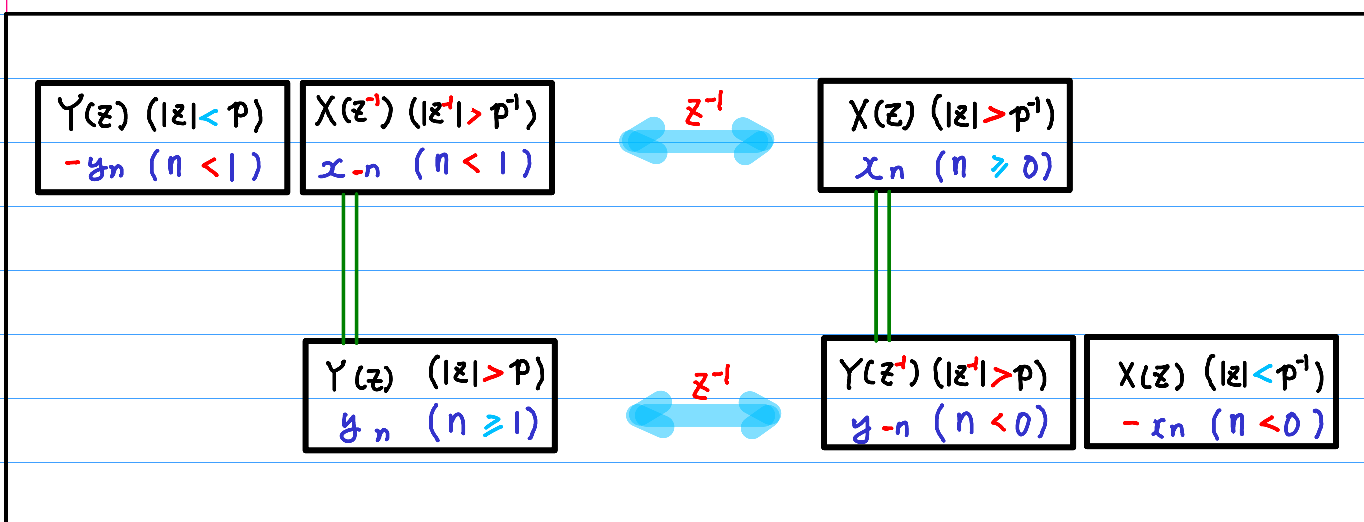
$$x_{-n} = -y_n \qquad\qquad -x_{-n} = y_n$$

$X(z^{-1})\ (\lvert z\rvert < p)$ $x_{-n}\ (n<1)$	$\xleftrightarrow{z^{-1}}$	$X(z)\ (\lvert z\rvert > p^{-1})$ $x_n\ (n \geq 0)$
$X(z^{-1})\ (\lvert z\rvert > p)$ $-x_{-n}\ (n \geq 1)$	$\xleftrightarrow{z^{-1}}$	$X(z)\ (\lvert z\rvert < p^{-1})$ $-x_n\ (n<0)$

Laurent Series and z-Transform

with the same ROC





$\boxed{\text{Laurent Series}}$ and $\boxed{z\text{-Transform}}$ $a_n,\ x_n$

with the same ROC

$$
\begin{tikzcd}
\boxed{\begin{array}{l} f(z) \quad (|z|<p) \\ a_n \quad (n \geq 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{l} f(z^{-1}) \quad (|z|>p^{-1}) \\ a_{-n} \quad (n < 1) \end{array}} \arrow[d, equal] \\
\boxed{\begin{array}{l} f(z) \quad (|z|>p) \\ -a_n \quad (n < 0) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{l} f(z^{-1}) \quad (|z|<p^{-1}) \\ -a_{-n} \quad (n \geq 1) \end{array}}
\end{tikzcd}
$$

$a_n = x_{-n}$ $\qquad\qquad$ $a_{-n} = x_n$

$$
\begin{tikzcd}
\boxed{\begin{array}{l} X(z^{-1}) \quad (|z|<p) \\ x_{-n} \quad (n < 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{\begin{array}{l} X(z) \quad (|z|>p^{-1}) \\ x_n \quad (n \geq 0) \end{array}} \arrow[d, equal] \\
\boxed{\begin{array}{l} X(z^{-1}) \quad (|z|>p) \\ -x_{-n} \quad (n \geq 1) \end{array}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{\begin{array}{l} X(z) \quad (|z|<p^{-1}) \\ -x_n \quad (n < 0) \end{array}}
\end{tikzcd}
$$

Laurent Series and z-Transform Examples

with the same ROC

	$n \geq 0$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n < 1$	
a_n	causal f(z) $-(p^{-1})^{n+1}$	$-\dfrac{p^{-1}}{1-p^{-1}z}$	$\xleftrightarrow{z^{-1}}$	$-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}$	anti-causal f(z) $-(p^{-1})^{-n+1}$	a_{-n}
	$\updownarrow \cdot(-1)$	$= \quad \cdot\dfrac{p}{p}\dfrac{z^{-1}}{z^{-1}}$		$\cdot\dfrac{p}{p}\dfrac{z}{z} \quad =$	$\updownarrow \cdot(-1)$	
b_{-n}	$(p^{-1})^{n+1}$ anti-causal f(z)	$\dfrac{z^{-1}}{1-pz^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\dfrac{z}{1-pz}$	$(p^{-1})^{-n+1}$ causal f(z)	b_n
	$n < 0$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n \geq 1$	

z-Transform

	$n < 1$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n \geq 0$	
x_{-n}	anti-causal X(z) $-(p^{-1})^{-n+1}$	$-\dfrac{p^{-1}}{1-p^{-1}z}$	$\xleftrightarrow{z^{-1}}$	$-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}$	causal X(z) $-(p^{-1})^{n+1}$	x_n
	$\updownarrow \cdot(-1)$	$= \quad \cdot\dfrac{p}{p}\dfrac{z^{-1}}{z^{-1}}$		$\cdot\dfrac{p}{p}\dfrac{z}{z} \quad =$	$\updownarrow \cdot(-1)$	
y_n	$(p^{-1})^{-n+1}$ causal X(z)	$\dfrac{z^{-1}}{1-pz^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\dfrac{z}{1-pz}$	$(p^{-1})^{n+1}$ anti-causal X(z)	y_{-n}
	$n \geq 1$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n < 0$	

$\boxed{\text{Laurent Series}}$ and $\boxed{z\text{-Transform}}$

$$f(z)\ (|z|<p) \longleftrightarrow a_n\ (n \geq 0) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$
$$X(z^{-1})\ (|z|<p) \longleftrightarrow x_{-n}\ (n < 1) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$f(z^{-1})\ (|z|>p^{-1}) \longleftrightarrow a_{-n}\ (n < 1) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$
$$X(z)\ (|z|>p^{-1}) \longleftrightarrow x_n\ (n \geq 0) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$f(z)\ (|z|>p) \longleftrightarrow -a_n\ (n < 0) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$
$$X(z^{-1})\ (|z|>p) \longleftrightarrow -x_{-n}\ (n \geq 1) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$

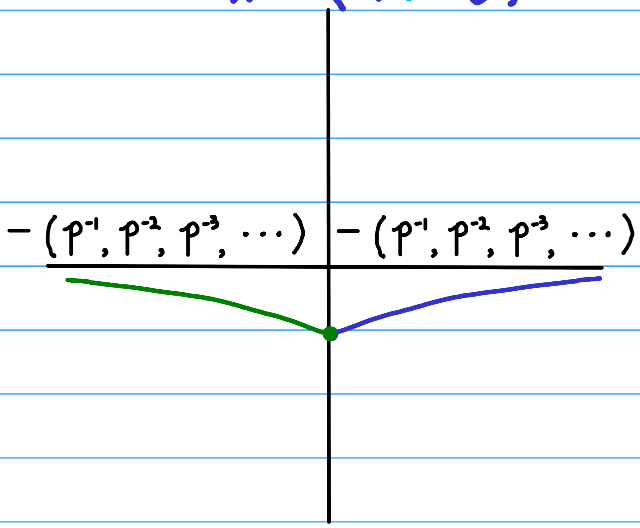
$$f(z^{-1})\ (|z|<p^{-1}) \longleftrightarrow -a_{-n}\ (n \geq 1) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$
$$X(z)\ (|z|<p^{-1}) \longleftrightarrow -x_n\ (n < 0) \qquad (p^{0}, p^{1}, p^{2}, \cdots)$$

$X(z^{-1})$ $(|z| < p)$

x_{-n} $(n < 1)$

$f(z)$ $(|z| < p)$

a_n $(n \geq 0)$

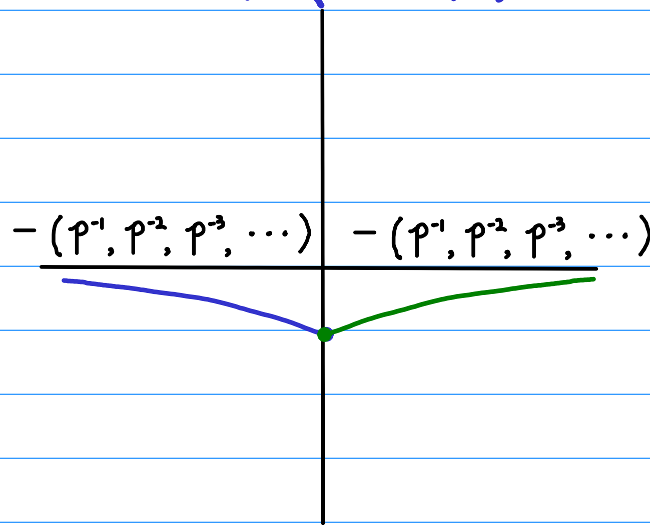


$X(z)$ $(|z| > p^{-1})$

x_n $(n \geq 0)$

$f(z^{-1})$ $(|z| > p^{-1})$

a_{-n} $(n < 1)$

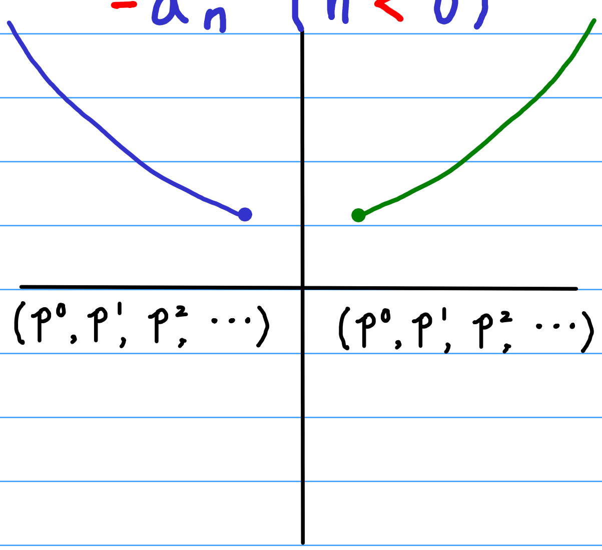


$X(z^{-1})$ $(|z| > p)$

$-x_{-n}$ $(n \geq 1)$

$f(z)$ $(|z| > p)$

$-a_n$ $(n < 0)$

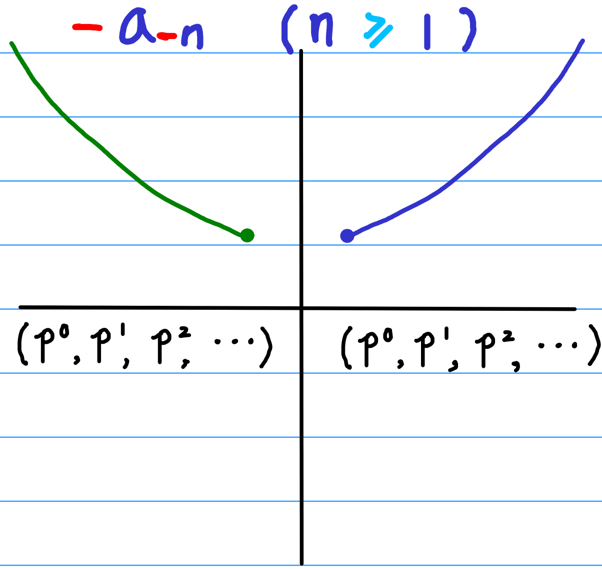


$X(z)$ $(|z| < p^{-1})$

$-x_n$ $(n < 0)$

$f(z^{-1})$ $(|z| < p^{-1})$

$-a_{-n}$ $(n \geq 1)$



$$\boxed{\begin{matrix} f(z) & f(z^{-1}) \\ g(z^{-1}) & g(z) \end{matrix}} \qquad \boxed{\begin{matrix} X(z^{-1}) & X(z) \\ Y(z) & Y(z^{-1}) \end{matrix}} \qquad \boxed{\begin{matrix} a_n & a_{-n} \\ b_{-n} & b_n \end{matrix}} \qquad \boxed{\begin{matrix} x_{-n} & x_n \\ y_n & y_{-n} \end{matrix}}$$

$$\boxed{\begin{matrix} f(z) & f(z^{-1}) \\ f(z) & f(z^{-1}) \end{matrix}} \qquad \boxed{\begin{matrix} X(z^{-1}) & X(z) \\ X(z^{-1}) & X(z) \end{matrix}}$$

$$\boxed{\begin{matrix} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^0, p^1, p^2, \dots) & (p^0, p^1, p^2, \dots) \end{matrix}} \qquad \boxed{\begin{matrix} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^0, p^1, p^2, \dots) & (p^0, p^1, p^2, \dots) \end{matrix}}$$

$$\boxed{\begin{matrix} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{matrix}} \qquad \boxed{\begin{matrix} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{matrix}}$$

$f(z)$	$g(z)$
$f(z)$	$g(z)$

$Y(z)$	$X(z)$
$Y(z)$	$X(z)$

a_n	a_{-n}
$-a_n$	$-a_{-n}$

x_{-n}	x_n
$-x_{-n}$	$-x_n$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$[0,\infty)$	$(-\infty,0]$
$(-\infty,-1]$	$[1,\infty)$

$(-\infty,0]$	$[0,\infty)$
$[1,\infty)$	$(-\infty,-1]$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$(p^{0}, p^{1}, p^{2}, \cdots)$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$(p^{0}, p^{1}, p^{2}, \cdots)$

$f(z)$	$g(z)$
$f(z)$	$g(z)$

$Y(z)$	$X(z)$
$Y(z)$	$X(z)$

a_n	a_{-n}
$-a_n$	$-a_{-n}$

x_{-n}	x_n
$-x_{-n}$	$-x_n$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$[0,\infty)$	$(-\infty, 0]$
$(-\infty,-1]$	$[1,\infty)$

$(-\infty, 0]$	$[0,\infty)$
$[1,\infty)$	$(-\infty,-1]$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$(p^0, p^1, p^2, \cdots)$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$(p^0, p^1, p^2, \cdots)$

$$\boxed{\begin{matrix} a_n & a_{-n} \\ b_{-n} & b_n \end{matrix}}$$

$$\boxed{\begin{matrix} x_{-n} & x_n \\ y_n & y_{-n} \end{matrix}}$$

(1)

$f(z)$ $(|z| < p)$ $\leftrightarrow$ a_n $(n \geq 0)$

$\Updownarrow$ the same ROC $\qquad \|$ symmetric ranges

$X(z^{-1})$ $(|z| < p)$ $\leftrightarrow$ x_{-n} $(n < 1)$

$|z| < p$ $\qquad$ $n \geq 0$

$f(z)$ $\boxed{-\dfrac{p^{-1}}{1 - p^{-1} z}}$ $\qquad$ $\boxed{\text{causal } f(z) \quad -(p^{-1})^{n+1}}$ a_n

$|z| < p$ $\qquad$ $n < 1$

$X(z)$ $\boxed{-\dfrac{p^{-1}}{1 - p^{-1} z}}$ $\qquad$ $\boxed{\text{anti-causal } X(z) \quad -(p^{-1})^{-n+1}}$ x_{-n}

②

$f(z^{-1})\ (|z| > p^{-1}) \longleftrightarrow a_{-n}\ (n < 1)$

the same ROC

symmetric ranges

$X(z)\ (|z| > p^{-1}) \longleftrightarrow x_n\ (n \geq 0)$

$|z| > p^{-1}$

$f(z^{-1})\ \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$

$n < 1$

anti-causal f(z)

$-(p^{-1})^{-n+1}$ a_{-n}

$|z| > p^{-1}$

$X(z)\ \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$

$n \geq 0$

causal X(z)

$-(p^{-1})^{n+1}$ x_n

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$(z^{0}, z^{1}, z^{2}, \cdots) \qquad f(z), \ X(z^{-1})$

$(z^{0}, z^{-1}, z^{-2}, \cdots) \qquad f(z^{-1}), \ X(z)$

$(z^{-1}, z^{-2}, z^{-3}, \cdots) \qquad g(z^{-1}), \ Y(z)$

$(p^{0}, p^{1}, p^{2}, \cdots)$

$(z^{1}, z^{2}, z^{3}, \cdots) \qquad g(z), \ Y(z^{-1})$

a_n, x_{-n}	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$ —— $(z^0, z^1, z^2, \cdots)$	$f(z)$,	$X(z^{-1})$
b_n, x_n	$(p^0, p^1, p^2, \cdots)$ —— $(z^{-1}, z^{-2}, z^{-3}, \cdots)$	$g(z^{-1})$,	$Y(z)$
a_{-n}, y_n	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$ —— $(z^0, z^{-1}, z^{-2}, \cdots)$	$f(z^{-1})$,	$X(z)$
b_{-n}, y_{-n}	$(p^0, p^1, p^2, \cdots)$ —— $(z^1, z^2, z^3, \cdots)$	$g(z)$,	$Y(z^{-1})$

$$\boxed{\begin{array}{ll} f(z) & (|z| < p) \\ a_n & (n \geq 0) \\ \multicolumn{2}{l}{-(p^{-1}, p^{-2}, p^{-3}, \cdots)} \end{array}} \qquad \boxed{\begin{array}{ll} f(z^{-1}) & (|z| > p^{-1}) \\ a_{-n} & (n < 1) \\ \multicolumn{2}{l}{-(p^{-1}, p^{-2}, p^{-3}, \cdots)} \end{array}}$$

$$\boxed{\begin{array}{ll} g(z^{-1}) & (|z| > p) \\ b_{-n} & (n < 0) \\ \multicolumn{2}{l}{(p^0, p^1, p^2, \cdots)} \end{array}} \qquad \boxed{\begin{array}{ll} g(z) & (|z| < p^{-1}) \\ b_n & (n \geq 1) \\ \multicolumn{2}{l}{(p^0, p^1, p^2, \cdots)} \end{array}}$$

$$\boxed{\begin{array}{ll} X(z^{-1}) & (|z| < p) \\ x_{-n} & (n < 1) \\ \multicolumn{2}{l}{-(p^{-1}, p^{-2}, p^{-3}, \cdots)} \end{array}} \qquad \boxed{\begin{array}{ll} X(z) & (|z| > p^{-1}) \\ x_n & (n \geq 0) \\ \multicolumn{2}{l}{-(p^{-1}, p^{-2}, p^{-3}, \cdots)} \end{array}}$$

$$\boxed{\begin{array}{ll} Y(z) & (|z| > p) \\ y_n & (n \geq 1) \\ \multicolumn{2}{l}{(p^0, p^1, p^2, \cdots)} \end{array}} \qquad \boxed{\begin{array}{ll} Y(z^{-1}) & (|z| < p^{-1}) \\ y_{-n} & (n < 0) \\ \multicolumn{2}{l}{(p^0, p^1, p^2, \cdots)} \end{array}}$$

$$a_n = -b_{-n} \qquad x_n = -y_{-n}$$

$$b_n = -a_{-n} \qquad y_n = -x_{-n}$$

$$a_n = x_{-n} \qquad b_n = y_{-n}$$

$$x_n = a_{-n} \qquad y_n = b_{-n}$$

$$a_n = -y_n \qquad b_n = -x_n$$

$$x_n = -b_n \qquad y_n = -a_n$$

getting causal sequence

$$
\begin{tikzcd}
\boxed{\dfrac{1}{z-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift right, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \arrow[l, shift left] & = f(z) \leftrightarrow \boxed{?} \\
\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \arrow[u, shift right] & &
\end{tikzcd}
$$

$$\parallel$$

$$Y(z) \leftrightarrow \boxed{?}$$

$$
\begin{tikzcd}
\boxed{\dfrac{1}{z^{-1}-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift right, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \arrow[l, shift left] & = X(z) \leftrightarrow \boxed{?} \\
\boxed{\dfrac{z}{1-pz}} \arrow[u, shift right] & &
\end{tikzcd}
$$

$$\parallel$$

$$g(z) \leftrightarrow \boxed{?}$$

Getting causal sequence w/o memorizing

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z}}$$

$$= f(z) \longleftrightarrow \boxed{-(p^{-1})^{n+1}}$$

$$\boxed{\frac{z}{1-pz}}$$ Left shift $(n \leftarrow n-1)$

$$= g(z) \longleftrightarrow \boxed{(p)^{n-1}}$$

$$\boxed{\frac{z^{-1}}{1-pz^{-1}}}$$ Left shift $(n \leftarrow n-1)$

$$= Y(z) \longleftrightarrow \boxed{(p)^{n-1}}$$

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}$$

$$= X(z) \longleftrightarrow \boxed{-(p^{-1})^{n+1}}$$

getting anti-causal sequence

$$
\begin{tikzcd}
\boxed{\dfrac{1}{z-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift left] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \arrow[l, shift left] \\
\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \arrow[u, shift left, "\frac{z^{-1}}{z^{-1}}"] &
\end{tikzcd}
$$

$$= f(z) \leftrightarrow -(p^{-1})^{n+1}$$

$$X(z^{-1}) \leftrightarrow \boxed{?}$$

$$\dfrac{z^{-1}}{1-pz^{-1}} = g(z^{-1}) \leftrightarrow \boxed{?}$$

$$Y(z) \leftrightarrow (p)^{n-1}$$

$$
\begin{tikzcd}
\boxed{\dfrac{1}{z^{-1}-p}} \arrow[r, shift left, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, shift left] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \arrow[l, shift left] \\
\boxed{\dfrac{z}{1-pz}} \arrow[u, shift left, "\frac{z^{-1}}{z^{-1}}"] &
\end{tikzcd}
$$

$$= f(z^{-1}) \leftrightarrow \boxed{?}$$

$$X(z) \leftrightarrow -(p^{-1})^{n+1}$$

$$\dfrac{z}{1-pz} = g(z) \leftrightarrow (p)^{n-1}$$

$$Y(z^{-1}) \leftrightarrow \boxed{?}$$

① $z \leftarrow z^{-1}$ ② $a_n \leftarrow a_{-n}$

$$\boxed{\frac{1}{z-p}} \xleftrightarrow{\frac{p^{-1}}{p^{-1}}} \boxed{-\frac{p^{-1}}{1-p^{-1}z}} = f(z) \leftrightarrow -(p^{-1})^{n+1} \qquad -p^{-n-1}$$

$$X(z^{-1}) \leftrightarrow \boxed{-(p^{-1})^{-n+1}} \qquad -p^{n-1}$$

$$\Big\updownarrow \tfrac{z^{-1}}{z^{-1}}$$

$$\boxed{\frac{z^{-1}}{1-pz^{-1}}}$$

$$\parallel$$

$$g(z^{-1}) \leftrightarrow \boxed{(p)^{-n-1}} \qquad z^{-1} \qquad p^{-n-1}$$

$$Y(z) \leftrightarrow (p)^{n-1} \qquad p^{n-1}$$

z^{-1}

$$\boxed{\frac{1}{z^{-1}-p}} \xleftrightarrow{\frac{p^{-1}}{p^{-1}}} \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} = f(z^{-1}) \leftrightarrow \boxed{-(p^{-1})^{-n+1}} \qquad -p^{n-1}$$

$$X(z) \leftrightarrow -(p^{-1})^{n+1} \qquad -p^{-n-1}$$

$$\Big\updownarrow \tfrac{z^{-1}}{z^{-1}}$$

$$\boxed{\frac{z}{1-pz}}$$

$$\parallel$$

$$g(z) \leftrightarrow (p)^{n-1} \qquad p^{n-1}$$

$$Y(z^{-1}) \leftrightarrow \boxed{(p)^{-n-1}} \qquad p^{-n-1}$$

$f(z^{-1})$ $\quad g(z^{-1})$

① $\boxed{z^{-1} \to z}$ $\quad f(z),\ g(z)$

② $f(z) \leftrightarrow a_n$ $\quad g(z) \leftrightarrow b_n$

③ $\boxed{n \to -n}$ $\quad a_{-n},\ b_{-n}$

$X(z^{-1})$ $\quad Y(z^{-1})$

① $\boxed{z^{-1} \to z}$ $\quad X(z),\ Y(z)$

② $X(z) \leftrightarrow x_n$ $\quad Y(z) \leftrightarrow y_n$

③ $\boxed{n \to -n}$ $\quad x_{-n},\ x_{-n}$

$$f(z^{-1}) = \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \qquad g(z^{-1}) = \boxed{\frac{z^{-1}}{1-pz^{-1}}}$$

anti-causal

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$f(z) = \boxed{-\frac{p^{-1}}{1-p^{-1}z}} \qquad g(z) = \boxed{\frac{z}{1-pz}}$$

$$Y(z^{-1}) = \boxed{\frac{z}{1-pz}} \qquad X(z^{-1}) = \boxed{-\frac{p^{-1}}{1-p^{-1}z}}$$

anti-causal

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$Y(z) = \boxed{\frac{z^{-1}}{1-pz^{-1}}} \qquad X(z) = \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}$$

$f(z^{-1})$ $g(z^{-1})$

① $\boxed{z^{-1} \to z}$ $f(z)$, $g(z)$

② $f(z) \leftrightarrow a_n$ $g(z) \leftrightarrow b_n$

③ $\boxed{n \to -n}$ a_{-n}, b_{-n}

$$f(z^{-1}) = \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \qquad g(z^{-1}) = \boxed{\frac{z^{-1}}{1-pz^{-1}}}$$

$$\downarrow \qquad \downarrow$$

① $$f(z) = \boxed{-\frac{p^{-1}}{1-p^{-1}z}} \qquad g(z) = \boxed{\frac{z}{1-pz}}$$

② $$a_n = \boxed{-(p^{-1})^{n+1}} \qquad b_n = \boxed{(p)^{n-1}}$$

$$\downarrow \qquad \downarrow$$

③ $$a_{-n} = \boxed{-(p^{-1})^{-n+1}} \qquad b_{-n} = \boxed{(p)^{-n-1}}$$

$X(z^{-1})$ $\quad Y(z^{-1})$

① $\boxed{z^{-1} \to z}$ $\qquad X(z),\ Y(z)$

② $X(z) \leftrightarrow x_n$ $\qquad Y(z) \leftrightarrow y_n$

③ $\boxed{n \to -n}$ $\qquad x_{-n},\ x_{-n}$

$$Y(z^{-1}) = \boxed{\frac{z}{1 - pz}} \qquad X(z^{-1}) = \boxed{-\frac{p^{-1}}{1 - p^{-1}z}}$$

① $$Y(z) = \boxed{\frac{z^{-1}}{1 - pz^{-1}}} \qquad X(z) = \boxed{-\frac{p^{-1}}{1 - p^{-1}z^{-1}}}$$

② $$y_n = \boxed{(p)^{n-1}} \qquad x_n = \boxed{-(p^{-1})^{n+1}}$$

③ $$y_{-n} = \boxed{-(p^{-1})^{-n+1}} \qquad x_{-n} = \boxed{(p)^{-n-1}}$$