Difference Equation Higher Order (H.3)

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Based on
Complex Analysis for Mathematics and Engineering
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a p-th order Linear Constant Coefficient Difference Equation

$$y[n] + a_1 y[n-1] + \cdots + a_p y[n-p]$$
$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_q x[n-q]$$

$\{a_i\}_{i=1}^{p}$ $\{b_j\}_{j=0}^{q}$

$\{x_n = x[n]\}_{n=0}^{\infty}$ input (given)

$\{y_n = y[n]\}_{n=0}^{\infty}$ output

p: the order of the difference equation

$$y[n] + \sum_{i=1}^{p} a_i y[n-i] = \sum_{j=0}^{q} b_j x[n-j]$$

$$y[n] = \sum_{j=0}^{q} b_j x[n-j] - \sum_{i=1}^{p} a_i y[n-i]$$

function of the part output values $y[n-i]$ and
the present input value $x[n]$ and
the previous input values $x[n-j]$

$$y[n] + \sum_{i=1}^{p} a_i\, y[n-i] = \sum_{j=0}^{q} b_j\, x[n-j]$$

$$Y(z) + \sum_{i=1}^{p} a_i\, Y(z)\, z^{-i} = \sum_{j=1}^{q} b_j\, X(z)\, z^{-j}$$

$$Y(z)\left(1 + \sum_{i=1}^{p} a_i z^{-i}\right) = X(z) \sum_{j=0}^{q} b_j z^{-j}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\sum_{j=0}^{q} b_j z^{-j}\right)}{\left(1 + \sum_{i=1}^{p} a_i z^{-i}\right)}$$

$$h[n] = \mathcal{Z}^{-1}[H(z)]$$

$$y_p[n] = h[n] * x[n] = \mathcal{Z}^{-1}[H(z)X(z)]$$

$$y_p[n] = h[n] * x[n] = \sum_{i=0}^{n} h[n-i]\, x[i]$$

Difference Equations with Initial Conditions

Only the present value of the input

$$y[n] + a_1 y[n-1] + \cdots + a_p y[n-p] = x[n]$$

$$y[n+i] + a_1 y[n-1+i] + \cdots + a_p y[n-p+i] = x[n+i]$$

$$y[n+p] + a_1 y[n-1+p] + \cdots + a_p y[n] = x[n+p]$$

$$y[n+2] - 2a\, y[n+1] + b\, y[n] = x[n+2] \qquad y[0] = y_0, \quad y[1] = y_1$$
$$x[0] = x_0, \quad x[1] = x_1$$

$$\mathcal{Z}[y[n+1]] = z\,(Y(z) - y_0)$$
$$\mathcal{Z}[y[n+2]] = z\Big(z\,(Y(z) - y_0) - y_1\Big) = z^2\,(Y(z) - y_0 - y_1 z^{-1})$$
$$\mathcal{Z}[x[n+2]] = z\Big(z\,(X(z) - x_0) - x_1\Big) = z^2\,(X(z) - x_0 - x_1 z^{-1})$$

$$z^2\,(Y(z) - y_0 - y_1 z^{-1}) - 2az\,(Y(z) - y_0) + bY(z) = z^2\,(X(z) - x_0 - x_1 z^{-1})$$
$$(z^2 - 2az + b)\,Y(z) - (y_0 z^2 + y_1 z - 2a y_0 z) = z^2 X(z) \qquad x_0 = x_1 = 0$$

$$Y(z) = \frac{z^2 X(z)}{(z^2 - 2az + b)} + \frac{y_0 z^2 + (y_1 - 2a y_0)\,z}{(z^2 - 2az + b)}$$

① $$y[n] = \mathcal{Z}^{-1}[Y(z)] = \mathcal{Z}^{-1}\left[\frac{z^2 X(z)}{(z^2 - 2az + b)}\right] + \mathcal{Z}^{-1}\left[\frac{y_0 z^2 + (y_1 - 2a y_0)\,z}{(z^2 - 2az + b)}\right]$$

② $$y[n] = \mathcal{Z}^{-1}[Y(z)] = \sum_{i=1}^{k} \mathrm{Res}\left(\underline{Y(z)\,z^{n-1}},\ z_i\right)$$

z_i : poles of $\underline{Y(z)\,z^{n-1}}$

real coefficien function $f(z) = Y(z)\, z^{n-1}$

z_j pole
$\bar{z}_j$ pole

$$\operatorname{Res}\left[Y(z)\, z^{n-1}, \overline{z_j} \right] = \overline{\operatorname{Res}\left[Y(z)\, z^{n-1}, z_j \right]}$$

$$\operatorname{Res}\left[f(z), \overline{z_j} \right] = \overline{\operatorname{Res}\left[f(z), z_j \right]}$$

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] =$$
$$b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

	a_1		a_{N-1}	a_N
$y[n+N]$	$y[n+N-1]$	$\cdots$	$y[n+1]$	$y[n]$
$x[n+M]$	$x[n+M-1]$	$\cdots$	$x[n+1]$	$x[n]$
b_{N-M}	b_{N-M+1}		b_{N-1}	b_N

$N \geqslant M$ Causal condition

$N = M$

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] =$$
$$b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

	a_1		a_{N-1}	a_N
$y[n+N]$	$y[n+N-1]$	$\cdots$	$y[n+1]$	$y[n]$
$x[n+M]$	$x[n+M-1]$	$\cdots$	$x[n+1]$	$x[n]$
b_0	b_1		b_{N-1}	b_N

$$E\,x[n] = x[n+1]$$
$$E^2 x[n] = x[n+2]$$
$$\vdots \qquad\qquad \vdots$$
$$E^N x[n] = x[n+N]$$

$$y[n+1] - a\,y[n] = x[n+1]$$
$$E\,y[n] - a\,y[n] = E\,x[n]$$
$$(E - a)\,y[n] = E\,x[n]$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N)\,y[n] =$$
$$(b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N)\,x[n]$$

$$Q[E]\,y[n] = P[E]\,x[n]$$

$$Q[E] = (E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N)$$
$$P[E] = (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N)$$

ZIR (Zero Input Response)

$y_{ZI}[n] \leftarrow x[n] = 0$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N)\, y_{ZI}[n] = 0$$

$$y_{ZI}[n+N] + a_1 y_{ZI}[n+N-1] + \cdots + a_{N-1} y_{ZI}[n+1] + a_N y_{ZI}[n] = 0$$

the linear combination of $y_{ZI}[n]$ and advanced $y_{ZI}[n]$

$= 0$ always zero for all n

$y_{ZI}[n]$ and advanced $y_{ZI}[n]$ have the same form

$\Rightarrow y_{ZI}[n] = c\lambda^n$

$$E^k y_{ZI}[n] = y_{ZI}[n+k] = c\lambda^{n+k}$$

$$y_{ZI}[n+N] + a_1 y_{ZI}[n+N-1] + \cdots + a_{N-1} y_{ZI}[n+1] + a_N y_{ZI}[n] = 0$$

$$c\lambda^{n+N} + a_1 c\lambda^{n+N-1} + \quad + a_{N-1} c\lambda^{n+1} + a_N c\lambda^n = 0$$

$$c(\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N)\lambda^n = 0$$

$$Q[\lambda] = (\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

Characteristic Polynomial

Characteristic Equation

$y_{zi}[n] \leftarrow x[n]=0$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N)\, y_{zi}[n] = 0$$

$$y_{zi}[n+N] + a_1 y_{zi}[n+N-1] + \cdots + a_{N-1} y_{zi}[n+1] + a_N y_{zi}[n] = 0$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1}E + a_N)$$

$$Q[\lambda] = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

N-th order polynomial $\rightarrow$ N roots

characteristic roots
characteristic values
characteristic modes
natural modes

① N distinct real roots

$$Q[\lambda] = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_N) = 0$$

② some repeated roots (N−r+1) distinct roots

$$Q[\lambda] = (\lambda-\lambda_1)^r(\lambda-\lambda_{r+1})\cdots(\lambda-\lambda_N) = 0$$

③ complex roots $(\frac{N}{2})$ complex conjugate roots

$$Q[\lambda] = (\lambda-\lambda_1)(\lambda-\bar{\lambda}_1)\cdots(\lambda-\lambda_{N/2})(\lambda-\bar{\lambda}_{N/2}) = 0$$

$$Q[\lambda] = (\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

$y_{zI}[n]$ zero input response

a linear combination of the characteristic modes

① N distinct real roots

$$Q[\lambda] = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_N) = 0$$

$$y_{zI}[n] = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_N\lambda_N^n$$

② some repeated roots $(N-r+1)$ distinct roots

$$Q[\lambda] = (\lambda - \lambda_1)^r(\lambda - \lambda_{r+1})\cdots(\lambda - \lambda_N) = 0$$

$$y_{zI}[n] = (c_1 + c_2 n + \cdots + c_r n^{r-1})\lambda_1^n + c_{r+1}\lambda_2^n + \cdots + c_N\lambda_N^n$$

③ complex roots $(\frac{N}{2})$ complex conjugate roots

$$Q[\lambda] = (\lambda - \lambda_1)(\lambda - \overline{\lambda}_1)\cdots(\lambda - \lambda_{N/2})(\lambda - \overline{\lambda_{N/2}}) = 0$$

$$y_{zI}[n] = c_1\lambda_1^n + c_2\overline{\lambda}_1^n + \cdots + c_{N-1}\lambda_{N/2}^n + c_N\overline{\lambda}_{N/2}^n$$

$$y_{zI}[n] = |\lambda_1|^n(c_1\cos(\beta_1 n) + c_2\sin(\beta_1 n)) + \cdots + |\lambda_{\frac{N}{2}}|^n(c_{N-1}\cos(\beta_{\frac{N}{2}} n) + c_N\sin(\beta_{\frac{N}{2}} n))$$

Impulse Response

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] =$$
$$b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y[n]$$
$$= (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N) x[n]$$

$$Q[E]\, y[n] = P[E]\, x[n]$$

$$Q[E]\, h[n] = P[E]\, \delta[n]$$

$$h[-1] = h[-2] = \cdots = h[-N] = 0$$

$h[n]$ the system response to input $\delta[n]$, which is initially at rest.

$\delta[n] = 0 \quad t < 0, \quad t > 0$

Zero input response

$\Rightarrow x[n] = 0 \Rightarrow y_{ZI}[n] \sim$ only mode terms $y_c[n]$

$h[n]$: zero input response for $\boxed{n > 0}$ $(\because x[n] = \delta[n] = 0)$

only mode terms for $n > 0$

$y_c[n]$ = linear combination of the char modes

* some non-zero value $A_0 \quad t=0$

$h[n] = A_0 \delta[n] + y_c[n] u[n]$

$$Q[E]h[n] = P[E]\delta[n]$$

$$h[-1] = h[-2] = \cdots = h[-N] = 0$$

$$h[n] = A_0\delta[n] + y_c[n]u[n]$$

$$Q[E]\left(A_0\delta[n] + y_c[n]u[n]\right) = P[E]\delta[n]$$

$$Q[E]y_c[n]u[n] = 0 \quad \text{lin comb of char modes}$$

$$A_0 Q[E]\delta[n] = P[E]\delta[n]$$

$$A_0(E^N + a_1E^{N-1} + \cdots + a_{N-1}E + a_N)\delta[n]$$
$$= (b_0E^N + b_1E^{N-1} + \cdots + b_{N-1}E + b_N)\delta[n]$$

$$A_0(\delta[n+N] + a_1\delta[n+N-1] + \cdots + a_N\delta[n])$$
$$= b_0\delta[n+N] + b_1\delta[n+N-1] + \cdots + b_N\delta[n])$$

$$A_0(\delta[0+N] + a_1\delta[0+N-1] + \cdots + a_N\delta[0]) \leftarrow n=0$$
$$= b_0\delta[0+N] + b_1\delta[0+N-1] + \cdots + b_N\delta[0])$$

$$\begin{cases} \delta[m] = 0 & m \neq 0 \\ \delta[0] = 1 & m = 0 \end{cases}$$

$$A_0 = \frac{b_N}{a_N} \Rightarrow h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n]$$

$y_c[n]$: Linear Comb. of Char Modes

$$h[n] = \frac{b_N}{a_N}\delta[n] + \boxed{y_c[n]}u[n]$$

$$Q[\lambda] = (\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

$y_{zI}[n]$ zero input response

$= y_c[n]$ a linear combination of the characteristic modes

① N distinct real roots

$$Q[\lambda] = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_N) = 0$$

$$y_{zI}[n] = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_N\lambda_N^n$$

$$y_c[n] = k_1\lambda_1^n + k_2\lambda_2^n + \cdots + k_N\lambda_N^n$$

② some repeated roots $(N-r+1)$ distinct roots

$$Q[\lambda] = (\lambda - \lambda_1)^r(\lambda - \lambda_{r+1})\cdots(\lambda - \lambda_N) = 0$$

$$y_{zI}[n] = (c_1 + c_2 n + \cdots + c_r n^{r-1})\lambda_1^n + c_{r+1}\lambda_2^n + \cdots + c_N\lambda_N^n$$

$$y_c[n] = (k_1 + k_2 n + \cdots + k_r n^{r-1})\lambda_1^n + k_{r+1}\lambda_2^n + \cdots + k_N\lambda_N^n$$

③ complex roots $\left(\frac{N}{2}\right)$ complex conjugate roots

$$Q[\lambda] = (\lambda - \lambda_1)(\lambda - \overline{\lambda_1})\cdots(\lambda - \lambda_{N/2})(\lambda - \overline{\lambda_{N/2}}) = 0$$

$$y_{zI}[n] = c_1\lambda_1^n + c_2\overline{\lambda_1}^n + \cdots + c_{N-1}\lambda_{N/2}^n + c_N\overline{\lambda_{N/2}}^n$$

$$y_{zI}[n] = |\lambda_1|^n(c_1\cos(\beta_1 n) + c_2\sin(\beta_1 n)) + \cdots + |\lambda_{\frac{N}{2}}|^n(c_{N-1}\cos(\beta_{\frac{N}{2}} n) + c_N\sin(\beta_{\frac{N}{2}} n))$$

$$y_{zI}[n] = |\lambda_1|^n(k_1\cos(\beta_1 n) + k_2\sin(\beta_1 n)) + \cdots + |\lambda_{\frac{N}{2}}|^n(k_{N-1}\cos(\beta_{\frac{N}{2}} n) + k_N\sin(\beta_{\frac{N}{2}} n))$$

$$h[n] = \frac{b_N}{a_N}\delta[n] + \boxed{y_c[n]}\,u[n]$$

N coefficients

Compute $h[0], h[1], \cdots, h[N-1]$

$$h[-1] = h[-2] = \cdots = h[-N] = 0$$

Zero State Response

$$y[n] = \cdots \quad \boxed{\begin{matrix} x[-2] \\ \cdot \\ \delta[n+2] \end{matrix}} \quad \boxed{\begin{matrix} x[-1] \\ \cdot \\ \delta[n+1] \end{matrix}} \quad \boxed{\begin{matrix} x[0] \\ \cdot \\ \delta[n] \end{matrix}} \quad \boxed{\begin{matrix} x[1] \\ \cdot \\ \delta[n-1] \end{matrix}} \quad \boxed{\begin{matrix} x[2] \\ \cdot \\ \delta[n-2] \end{matrix}} \quad \boxed{\begin{matrix} x[3] \\ \cdot \\ \delta[n-3] \end{matrix}} \cdots$$

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,\delta[n-m]$$

$$x[n] \Longrightarrow y[n]$$

$$\delta[n] \qquad h[n]$$

$$\delta[n-m] \qquad h[n-m]$$

$$x[m]\,\delta[n-m] \qquad x[m]\,h[n-m]$$

$$\sum_{m=-\infty}^{+\infty} x[m]\,\delta[n-m] \qquad \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m]$$

$$x[n] \qquad y[n]$$

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m]$$

Causality and ZSR

causal input $\rightarrow$ $x[n]=0 \quad n<0$

$\rightarrow$ $x[m]=0 \quad m<0$

$\boxed{0 \leqslant m \leqslant n}$

causal system $\rightarrow$ $h[n]=0 \quad n<0$

$\rightarrow$ $h[n-m]=0 \quad n<m$

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{+\infty} x[m]h[n-m]$$

$$\left.\begin{array}{l}\text{causal input}\\ \text{causal system}\end{array}\right\} = \sum_{m=0}^{n} x[m]h[n-m]$$

Convolution – Graphical Procedure

1. Invert $h[m]$ about the vertical axis $(m=0)$
   $\Rightarrow h[-m]$

2. Shift $h[-m]$ by n units
   $\Rightarrow h[n-m]$

   $+$ shift = right shift
   $-$ shift = left shift

3. multiply $x[m] \cdot h[n-m]$ for $0 \leq m \leq n$
   add all the products

$$y[n] = \sum_{m=0}^{n} x[m]h[n-m]$$

Everlasting Exponential z^n

$$e^{st} \longrightarrow \boxed{H(s)} \longrightarrow H(s)e^{st}$$

$$z^n \longrightarrow \boxed{H[z]} \longrightarrow H[z]\, z^n$$

$$\begin{aligned} y[n] &= h[n] * z^n \\ &= \sum_{m=-\infty}^{\infty} h[m]\, z^{n-m} \\ &= z^n \sum_{m=-\infty}^{\infty} h[m]\, z^{-m} \\ &= z^n H[z] \qquad H[z] = \sum_{m=-\infty}^{\infty} h[m]\, z^{-m} \end{aligned}$$

$$y[n] = H[z]\, z^n$$

Transfer Function

$$H[z] = \left. \frac{\text{Output Signal}}{\text{Input Signal}} \right|_{\text{input} = z^n \,:\, \text{everlasting exponential}}$$

LTID system only meaningful

everlasting exponential	z^n :	started at $n = -\infty$
	$z^n u[n]$:	started at $n = 0$

$\rightarrow$ initial conditions gives no contribution $\rightarrow$ ignore safely

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

$$x[n] = z^n \longrightarrow \boxed{H[z]} \longrightarrow H[z]\, z^n = y[n]$$

$$H[z]\left(z^{n+N} + a_1 z^{n+N-1} + \cdots + a_{N-1} z^{n+1} + a_N z^n\right) =$$
$$\left(b_{N-M} z^{n+M} + b_{N-M+1} z^{n+M-1} + \cdots + b_{N-1} z^{n+1} + b_N z^n\right)$$

$$\left(z^N + a_1 z^{N-1} + \cdots + a_{N-1} z^1 + a_N\right) z^n H[z] =$$
$$\left(b_{N-M} z^M + b_{N-M+1} z^{M-1} + \cdots + b_{N-1} z + b_N\right) z^n$$

$$\left(z^N + a_1 z^{N-1} + \cdots + a_{N-1} z^1 + a_N\right) H[z] =$$
$$\left(b_{N-M} z^M + b_{N-M+1} z^{M-1} + \cdots + b_{N-1} z + b_N\right)$$

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] =$$
$$b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

$$E^1 y(n) = y[n+1]$$
$$E^2 y[n] = y[n+2]$$
$$\vdots \qquad \vdots$$
$$E^N y(n) = y[n+N]$$

$$E^1 x(n) = x[n+1]$$
$$E^2 x[n] = x[n+2]$$
$$\vdots \qquad \vdots$$
$$E^N x(n) = x[n+N]$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E^1 + a_N)\, y[n] =$$
$$(b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E^1 + b_N)\, x[n]$$

$$Q[E]\, y[n] = P[E]\, x[n]$$

$$Q[E] = (E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N)$$
$$P[E] = (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N)$$

$$Q[z] = (z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N)$$
$$P[z] = (b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1} z + b_N)$$

$$(z^N + a_1 z^{N-1} + \cdots + a_{N-1} z^1 + a_N)\, H[z] =$$
$$(b_{N-M} z^M + b_{N-M+1} z^{M-1} + \cdots + b_{N-1} z + b_N)$$

$$Q[z]\, H[z] = P[z]$$

$$H[z] = \frac{P[z]}{Q[z]}$$

$$y[n] = H[z]\, z^n \qquad x[n] = z^n$$

$$\boxed{\begin{aligned} E^1 y[n] &= H[z]\, z^{n+1} \\ E^2 y[n] &= H[z]\, z^{n+2} \\ &\vdots \\ E^N y[n] &= H[z]\, z^{n+N} \end{aligned}} \qquad \boxed{\begin{aligned} E^1 x[n] &= z^{n+1} \\ E^2 x[n] &= z^{n+2} \\ &\vdots \\ E^N x[n] &= z^{n+N} \end{aligned}}$$

$$\boxed{\begin{aligned} &(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E^1 + a_N)\, y[n] = \\ &(b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E^1 + b_N)\, x[n] \end{aligned}}$$

$$H[z]\{Q[E]\, z^n\} = P[E]\, z^n$$

$$\Downarrow \qquad \Downarrow \qquad\qquad E^k z^n = z^{n+k} = z^k \cdot z^n$$

$$H[z]\{Q[z]\, z^n\} = P[z]\, z^n$$

$$(z^N + a_1 z^{N-1} + \cdots + a_{N-1} z^1 + a_N)\, H[z]\, z^n =$$
$$(b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1} z^1 + b_N)\, z^n$$

$$H[z] = \frac{P[z]}{Q[z]}$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N)\, y[n] =$$
$$(b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N)\ x[n]$$

$$Q[E]\, y[n] = P[E]\, x[n]$$

$$Q[E] = (E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N)$$
$$P[E] = (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N)$$

$$\mathcal{Z}\{Q[E]\, y[n]\} = \mathcal{Z}\{P[E]\, x[n]\}$$

$$P[z]\, Y[z] = Q[z]\, X[z]$$

$$H[z]\{Q[E]\, z^n\} = P[E]\, z^n$$

$$y[n] = H[z]\, z^n$$

$$H[z] = \frac{P[z]}{Q[z]}$$

$$H[z]\ Q[z] = P[z]$$

Total Response

$$\text{total response} = \boxed{ZIR} + \boxed{ZSR}$$

$$= \boxed{\text{lin comb of char modes}} + \boxed{\text{convolution of } x[n] \text{ and } h[n]}$$

$$= \boxed{\sum_{i=1}^{N} c_i \lambda_i} + \boxed{x[n] * h[n]}$$

$$\text{total response} = \text{Natural Response} + \text{Forced Response}$$

$$= \boxed{\text{all mode terms}} + \boxed{\text{all non-mode terms}}$$

$$= \underbrace{\boxed{y_n[n]}}_{\text{homogeneous}} + \underbrace{\boxed{y_f[n]}}_{\text{particular}}$$

$$Q[E]\,(y_n[n] + y_f[n]) = P[E]\,x[n]$$

$$Q[E]\,y_n[n] + Q[E]\,y_f[n] = P[E]\,x[n]$$

$$Q[E]\,y_f[n] = P[E]\,x[n]$$

$$x[n] = r^n \quad (r \neq r_i) \qquad y_f[n] = c\, r^n$$

$$x[n] = r^n \quad (r = r_i) \qquad y_f[n] = c\, n\, r^n$$

$$x[n] = \cos(\beta n + \theta) \qquad y_f[n] = c \cos(\beta n + \theta)$$

$$x[n] = \left(\sum_{i=0}^{m} \alpha_i n^i\right) r^n \qquad y_f[n] = \left(\sum_{i=0}^{m} c_i n^i\right) r^n$$

Initial Conditions

Classical Method requires

aux conditions $y[0], y[1], \cdots, y[N-1]$

Classical method does not separate modes components of ZIR & ZSR

I.C must be applied to the total response

given I.C. $y[-1], y[-2], \cdots, y[-N]$

compute $y[0], y[1], \cdots, y[N-1]$

Exponential Input

$$Q[E]\,y[n] = P[E]\,x[n]$$

$y_f[n] = H[r]\,r^n$ $\quad r \neq \gamma_i$ not char mode

$$H[r] = \frac{P[r]}{Q[r]}$$

$E^i x[n] = x[n+i] = r^{n+i} = r^i r^n$ $\qquad P[E]\,x[n] = P[r]\,r^n$

$E^i y_f[n] = y_f[n+i] = c\,r^{n+i} = c\,r^i r^n$ $\qquad Q[E]\,x[n] = c\,Q[r]\,r^n$

$$c\,Q[r]\,r^n = P[r]\,r^n$$

$$c = \frac{P[r]\,r^n}{Q[r]\,r^n} = \frac{P[r]}{Q[r]} = H[r]$$

A Constant Input $\quad x[n] = C \qquad c\,r^n \quad r = 1$

$$y_f[n] = C\,\frac{P[1]}{Q[1]} = C\,H[1]$$

A Sinusoidal Input $\quad x[n] = e^{j\Omega n} \qquad r^n \quad r = e^{j\Omega}$

$$x[n] = e^{j\Omega n}$$

$$y_f[n] = H[e^{j\Omega n}]\,e^{j\Omega n} = \frac{P[e^{j\Omega n}]}{Q[e^{j\Omega n}]}\,e^{j\Omega n}$$

$$x[n] = e^{-j\Omega n}$$

$$y_f[n] = H[e^{-j\Omega n}]\,e^{-j\Omega n} = \frac{P[e^{-j\Omega n}]}{Q[e^{-j\Omega n}]}\,e^{-j\Omega n}$$

$$x[n] = \cos \Omega n = \frac{1}{2}\left(e^{+j\Omega n} + e^{-j\Omega n}\right)$$

$$y_f[n] = \frac{1}{2}\left(H[e^{j\Omega n}]\,e^{j\Omega n} + H[e^{-j\Omega n}]\,e^{-j\Omega n}\right)$$

$$= \mathrm{Re}\left\{H[e^{j\Omega n}]\,e^{j\Omega n}\right\}$$

$$H[e^{j\Omega n}] = \left|H[e^{j\Omega n}]\right| e^{j\angle H[e^{j\Omega n}]}$$

$$y_f[n] = \mathrm{Re}\left\{\left|H[e^{j\Omega n}]\right| e^{j\angle H[e^{j\Omega n}]}\right\}$$

$$= \left|H[e^{j\Omega n}]\right| \cos\left(\Omega n + \angle H[e^{j\Omega}]\right)$$

$$x[n] = \cos(\Omega n + \theta)$$

$$y_f[n] = \left|H[e^{j\Omega n}]\right| \cos\left(\Omega n + \theta + \angle H[e^{j\Omega}]\right)$$

Impulse Response when $a_N = 0$

$$\boxed{a_N = 0 \Rightarrow \quad A_0 = \frac{b_N}{a_N} \text{ indeterminate}}$$

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] =$$
$$b_0 x[n+N] + b_1 x[n+N-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N)\, y[n]$$
$$= (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N)\, x[n]$$

$$Q[E]\, y[n] = p[E]\, x[n]$$
$$Q[E]\, h[n] = p[E]\, \delta[n]$$

$$h[-1] = h[-2] = \cdots = h[-N] = 0$$

$h[n]$ $\quad n > 0$

: ZIR ($\because$ $x[n] = \delta[n] = 0$) $\rightarrow$ only mode terms

$$h[n] = A_0 \delta[n] + y_c[n] u[n]$$

$$Q[E]\,(A_0 \delta[n] + y_c[n] u[n]) = p[E]\, \delta[n]$$

$$Q[E]\, y_c[n] u[n] = 0$$ lin comb of char modes

$$A_0 Q[E]\, \delta[n] = p[E]\, \delta[n]$$

$$A_0 (\delta[0+N] + a_1 \delta[0+N-1] + \cdots + a_N \delta[0])$$ $\leftarrow$ $n = 0$
$$= b_0 \delta[0+N] + b_1 \delta[0+N-1] + \cdots + b_N \delta[0])$$

$$\boxed{A_0 = \frac{b_N}{a_N}}$$

$$\boxed{a_N = 0 \Rightarrow \quad A_0 = \frac{b_N}{a_N} \text{ indeterminate}}$$

$$\boxed{\begin{aligned} &y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + \cancel{a_N} y[n] = \\ &b_0 x[n+N] + b_1 x[n+N-1] + \cdots + b_{N-1} x[n+1] + b_N x[n] \end{aligned}}$$

$$\left(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + \cancel{a_N}\right) y[n]$$
$$= \left(b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N\right) x[n]$$

$$Q[E]\, y[n] = p[E]\, x[n]$$
$$Q[E]\, h[n] = p[E]\, \delta[n]$$

$$h[-1] = h[-2] = \cdots = h[-N] = 0$$

$$\boxed{\begin{aligned} &y[n+N-1] + a_1 y[n+N-2] + \cdots + a_{N-1} y[n] = \\ &b_0 x[n+N-1] + b_1 x[n+N-2] + \cdots + b_{N-1} x[n] + b_N x[n-1] \end{aligned}}$$

$$\boxed{\begin{aligned} &\left(E^{N-1} + a_1 E^{N-2} + \cdots + a_{N-2} E + a_{N-1}\right) y[n] = \\ &\left(b_0 E^{N-1} + b_1 E^{N-2} + \cdots + b_{N-2} E + b_{N-1} + b_N E^{-1}\right) x[n] \end{aligned}}$$

$$\hat{Q}[E]\, y[n] = \hat{p}[E]\, x[n]$$

$$\boxed{\begin{aligned} &E\left(E^{N-1} + a_1 E^{N-2} + \cdots + a_{N-2} E + a_{N-1}\right) y[n] = \\ &\left(b_0 E^{N} + b_1 E^{N-1} + \cdots + b_{N-2} E^2 + b_{N-1} E + b_N\right) E^{-1} x[n] \end{aligned}}$$

$$E\hat{Q}[E]\, y[n] = p[E]\{E\, x[n]\}$$

$$\hat{Q}[E]\, y[n] = p[E]\, x[n-1]$$
$$\hat{Q}[E]\, h[n] = p[E]\, \delta[n-1]$$

$$\boxed{\begin{aligned} & y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + \cancelto{0}{a_N y[n]} = \\ & b_0 x[n+N] + b_1 x[n+N-1] + \cdots + b_{N-1} x[n+1] + b_N x[n] \end{aligned}}$$

$$\boxed{\begin{aligned} & y[n+N-1] + a_1 y[n+N-2] + \cdots + a_{N-1} y[n] = \\ & b_0 x[n+N-1] + b_1 x[n+N-2] + \cdots + b_{N-1} x[n] + b_N x[n-1] \end{aligned}}$$

$$E\{y[n+N-1] + a_1 y[n+N-2] + \cdots + a_{N-1} y[n]\}$$

$$= E\,\hat{Q}[E]\,y[n]$$

$$b_0 Ex[n+N-1] + b_1 Ex[n+N-2] + \cdots + b_{N-1} Ex[n] + b_N Ex[n-1]$$

$$= P[E]\{Ex[n-1]\}$$

$$\begin{aligned} E\,\hat{Q}[E]\,y[n] &= P[E]\{Ex[n-1]\} \\ &= E\,P[E]\,x[n-1] \end{aligned}$$

$$\hat{Q}[E]\,y[n] = P[E]\,x[n-1]$$

$$\hat{Q}[E]\, y[n] = P[E]\, x[n-1]$$

$$\hat{Q}[E]\, h[n] = P[E]\, \delta[n-1]$$

the input $P[E]\delta[n-1]$ becomes zero for $n \geq 2$

not for $n \geq 1$

the response consists of

$$\begin{cases} \text{the zero input term} \\ \text{an impulse } A_0\delta[n] \text{ at } n=0 \\ \text{an impulse } A_1\delta[n-1] \text{ at } n=1 \end{cases}$$

$$h[n] = A_0\delta[n] + A_1\delta[n-1] + y_c[n]u[n]$$

A_0 $\quad$ A_1 $\quad$ $\underline{N-1 \text{ coefficients}}$

$a_N = 0 \rightarrow \hat{Q}[\gamma] : (N-1)$ order polynomial

$N+1$ unknown coefficients

from $N+1$ initial values $h[0], h[1], \cdots, h[N]$

the iterative solution $\quad Q[E]\, h[n] = P[E]\, \delta[n]$

if $a_N = a_{N-1} = 0$

$h[n] = A_0 \delta[n] + A_1 \delta[n-1] + A_2 \delta[n-2] + y_c[n] u[n]$

$N+1$ unknown coefficients

from $N+1$ initial values $h[0], h[1], \cdots, h[N]$