Laurent Series and z-Transform - Geometric Series Properties Ⓑ

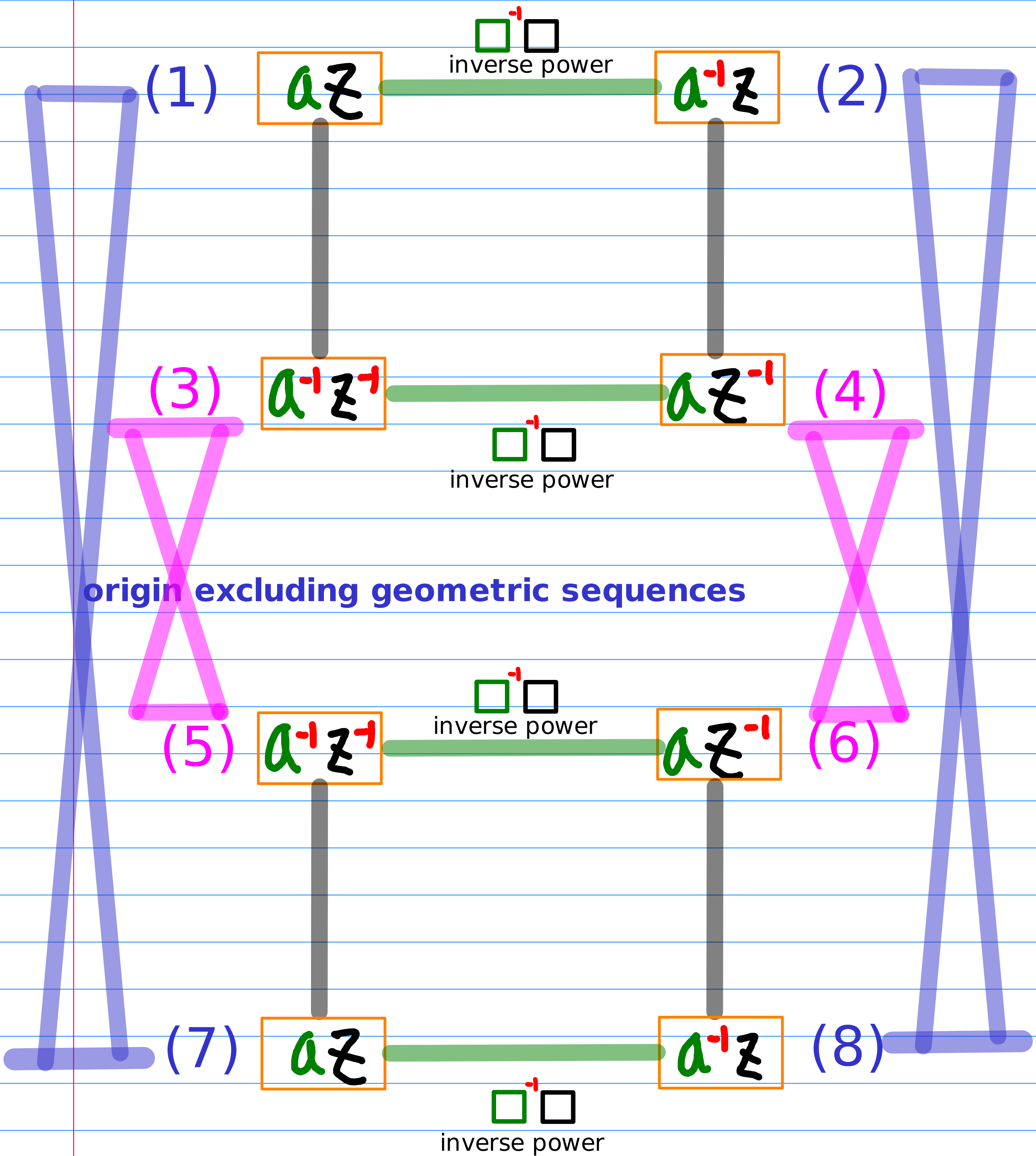
20220126 Wed

Geometric sequence cases - (1) CR view

Shifting relations

origin including geometric sequences

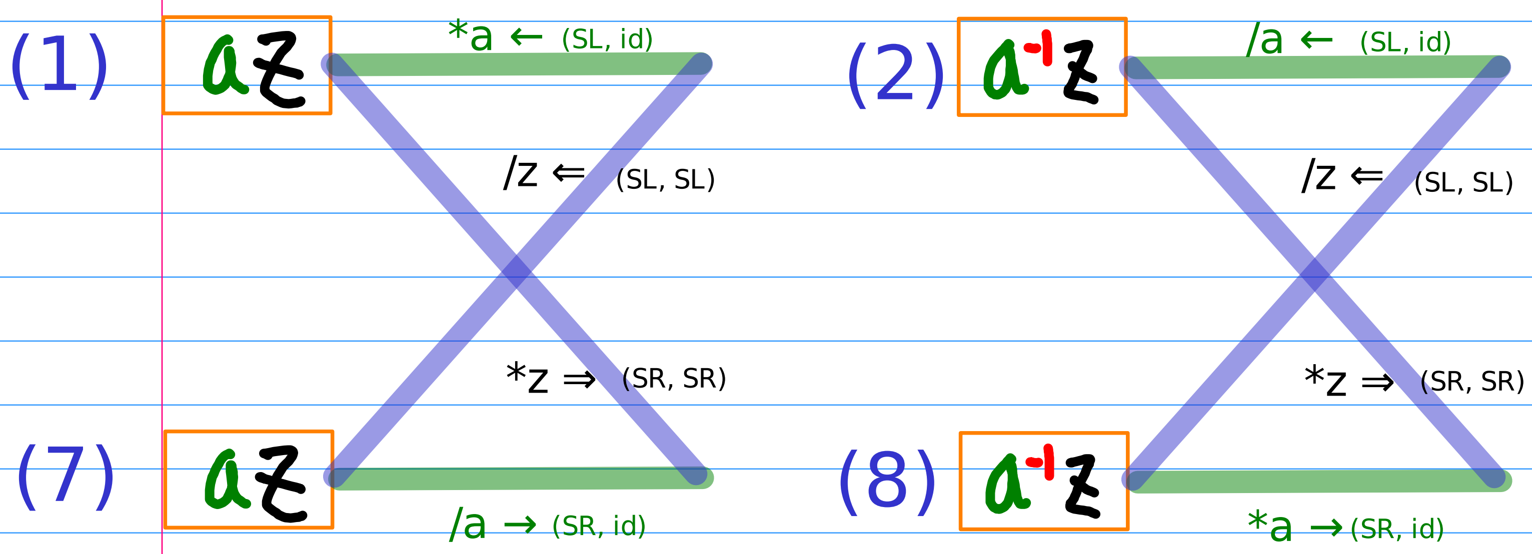
origin excluding geometric sequences



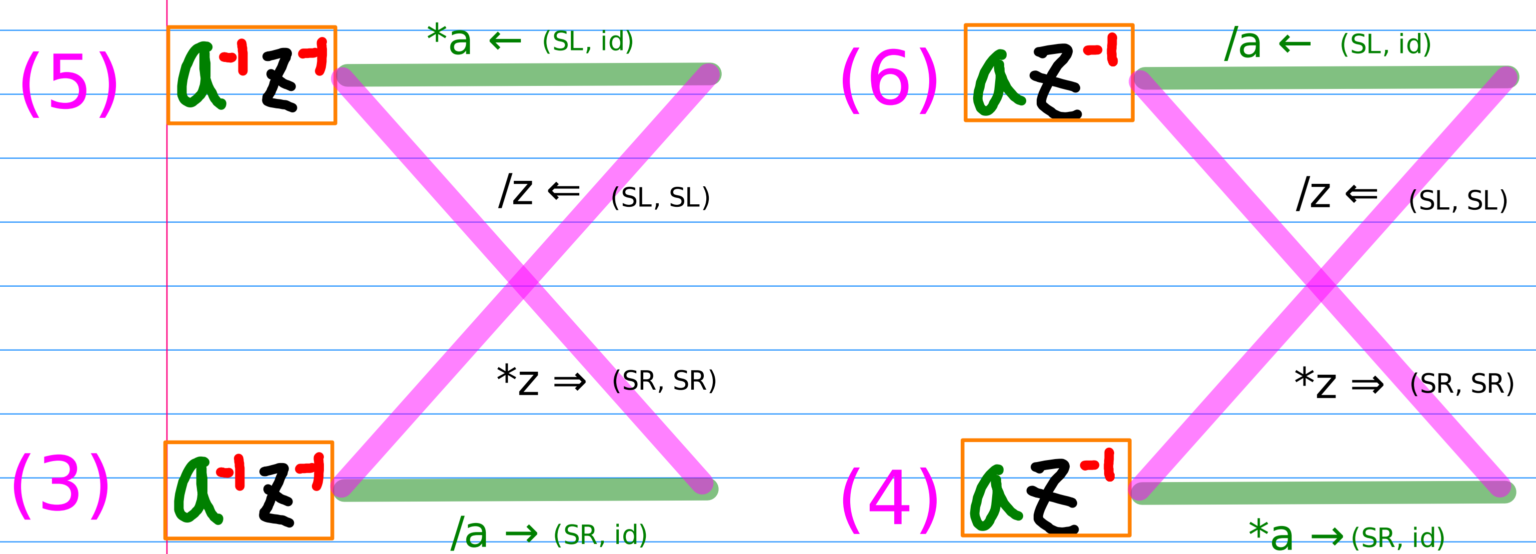
Geometric sequence cases - (1) CR view

Shifting relations

origin including geometric sequences

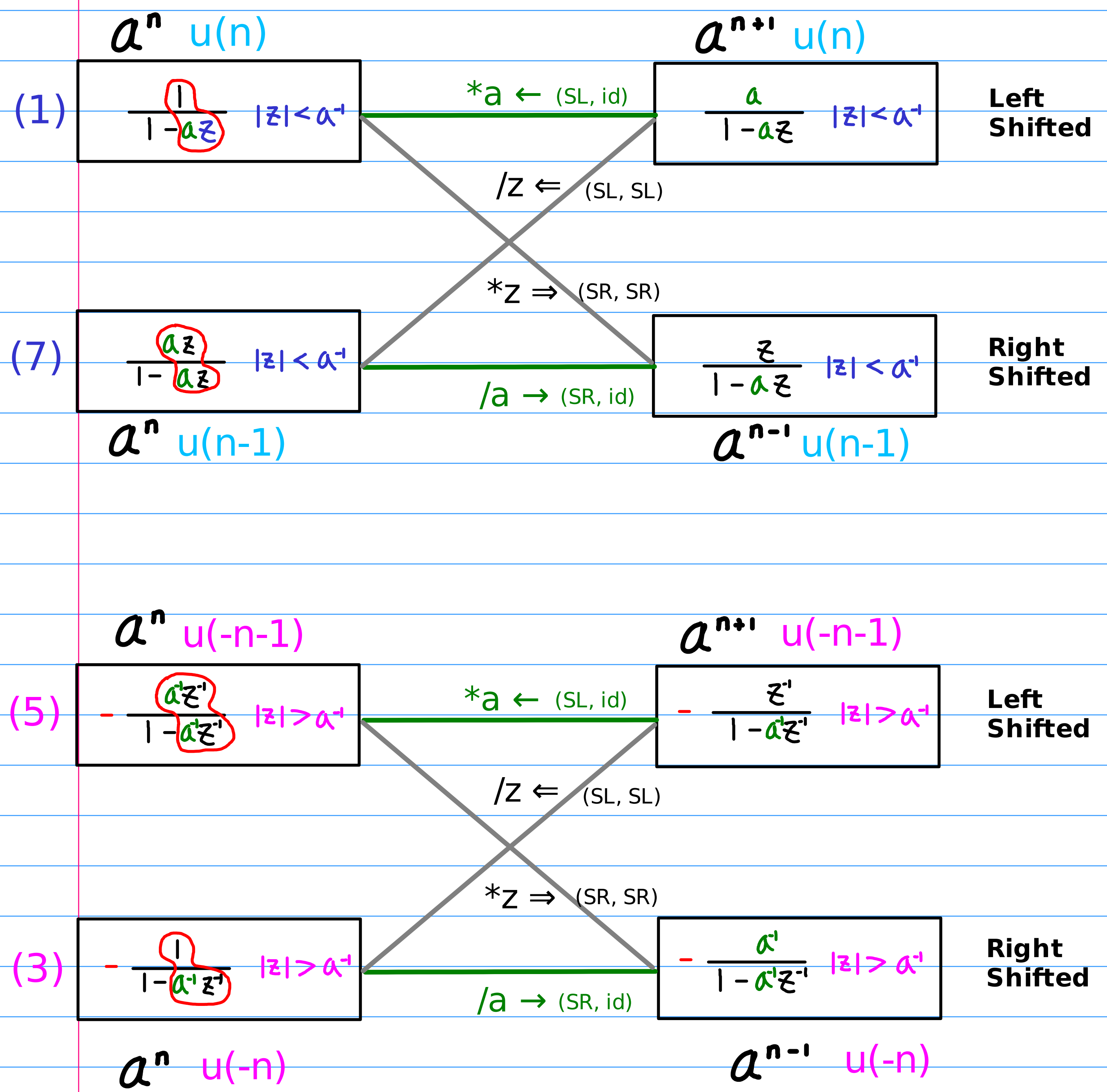


origin excluding geometric sequences



Shifting Geometric Series (1) positive exponent

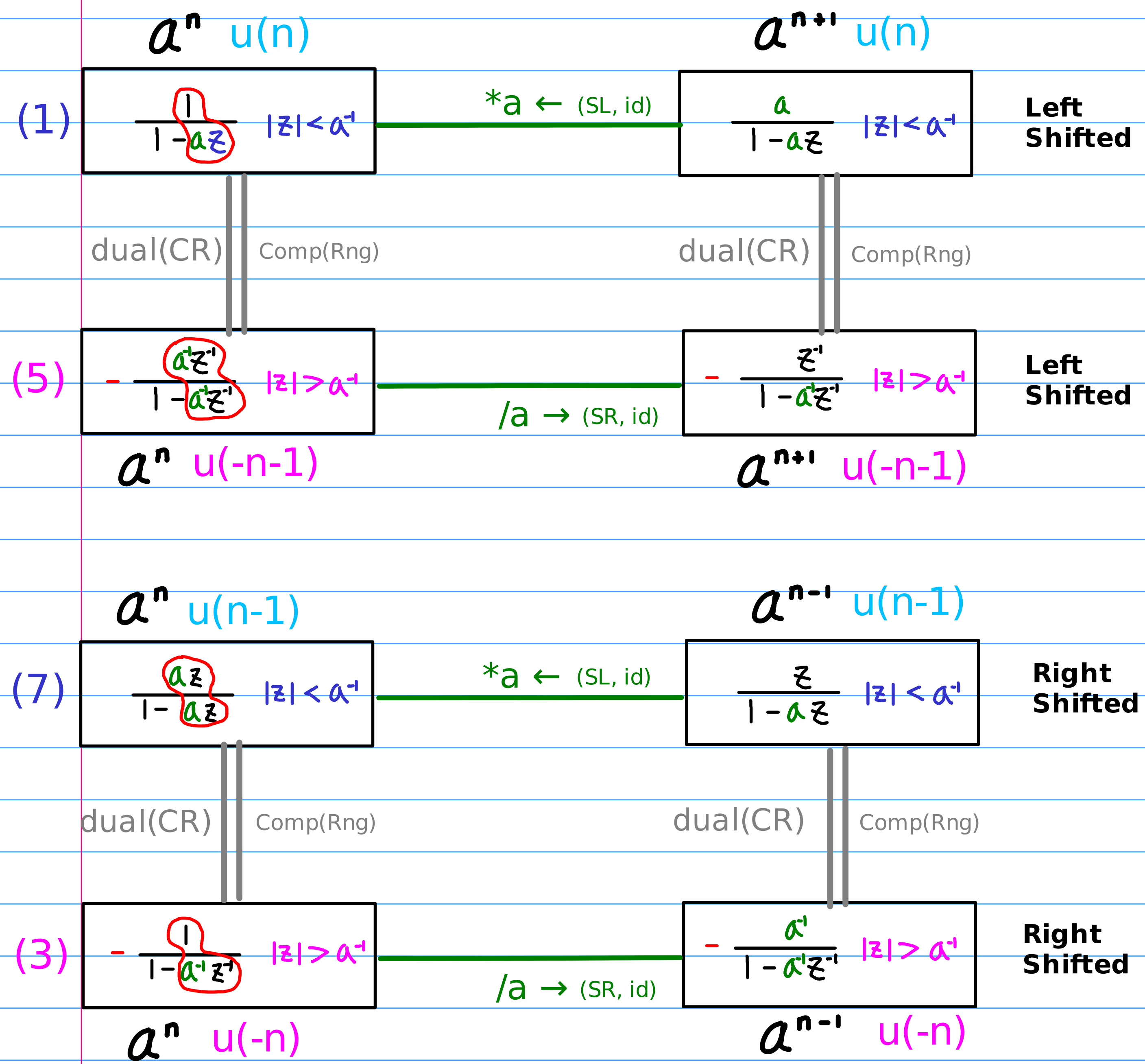
Positive Exponent a /z $n \leftarrow n+1$ *z $n \leftarrow n-1$



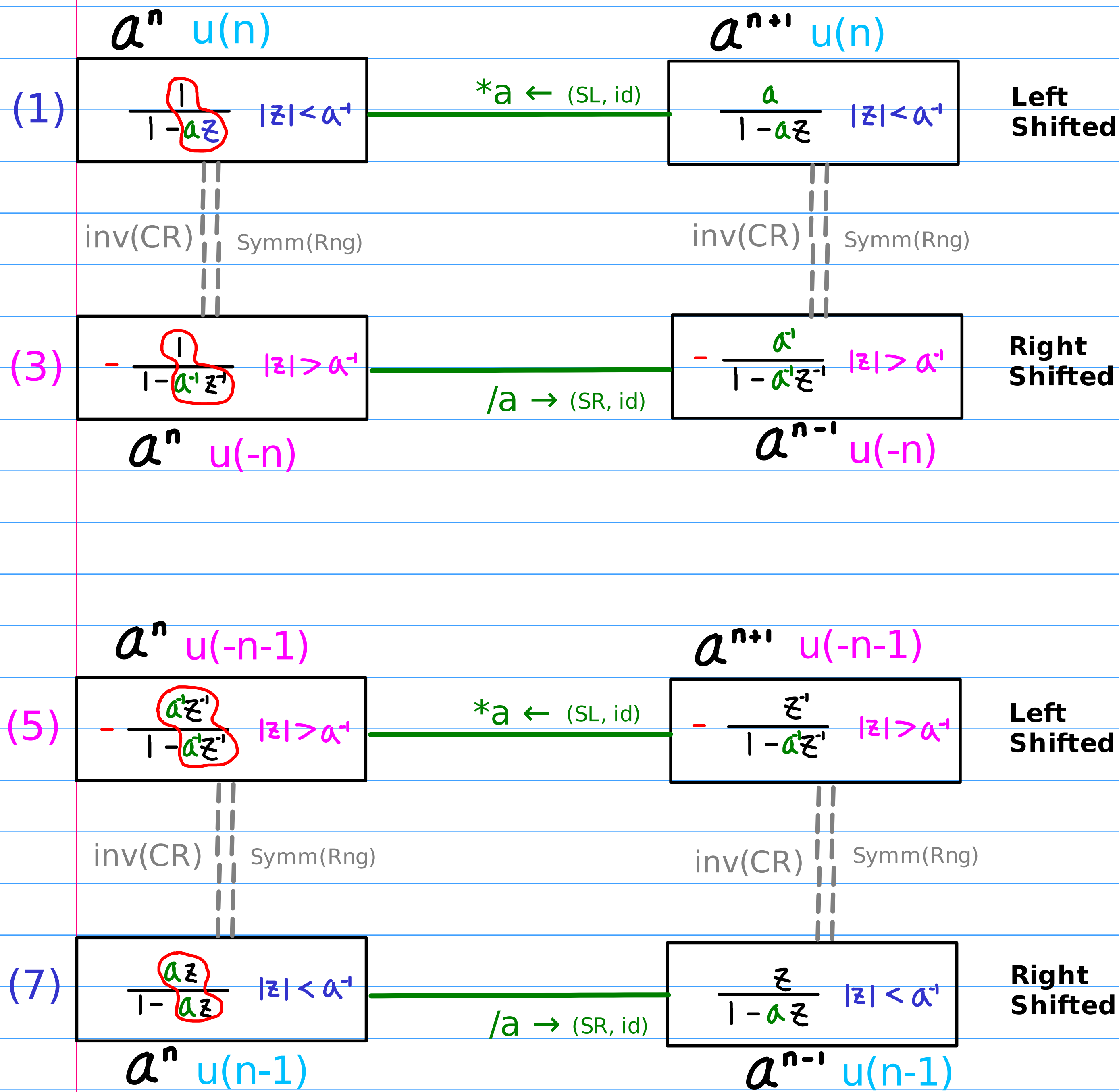
Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Dual expression w.r.t a common ratio

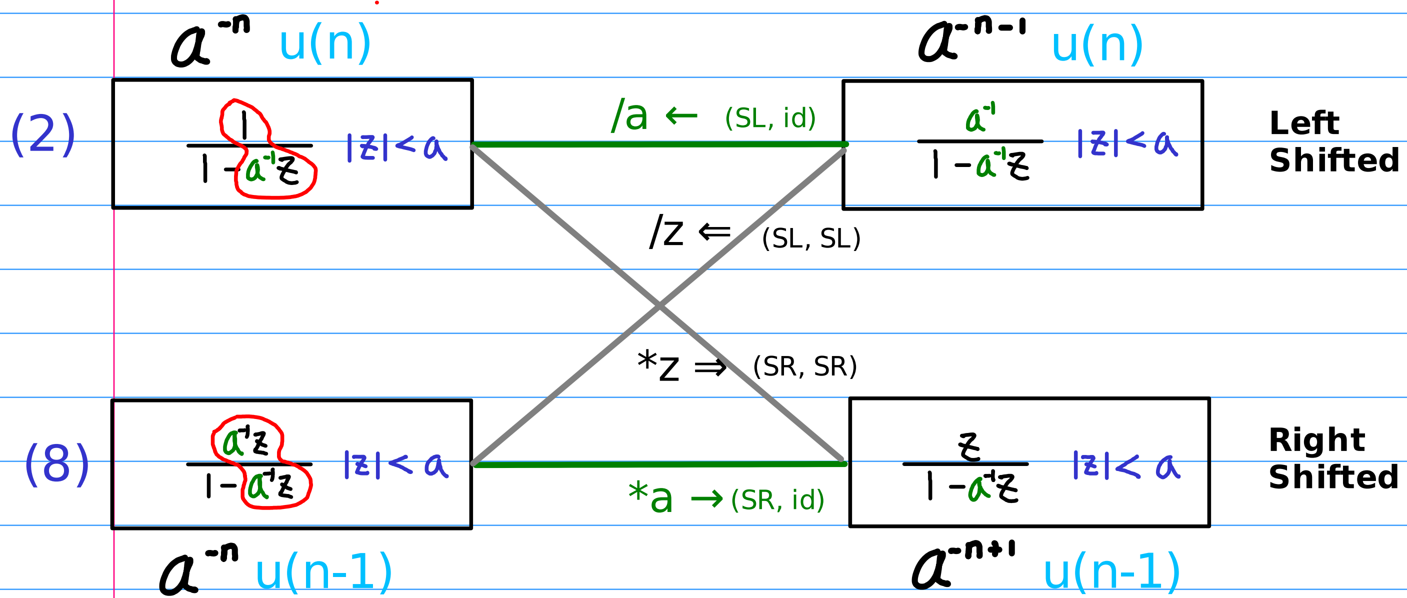


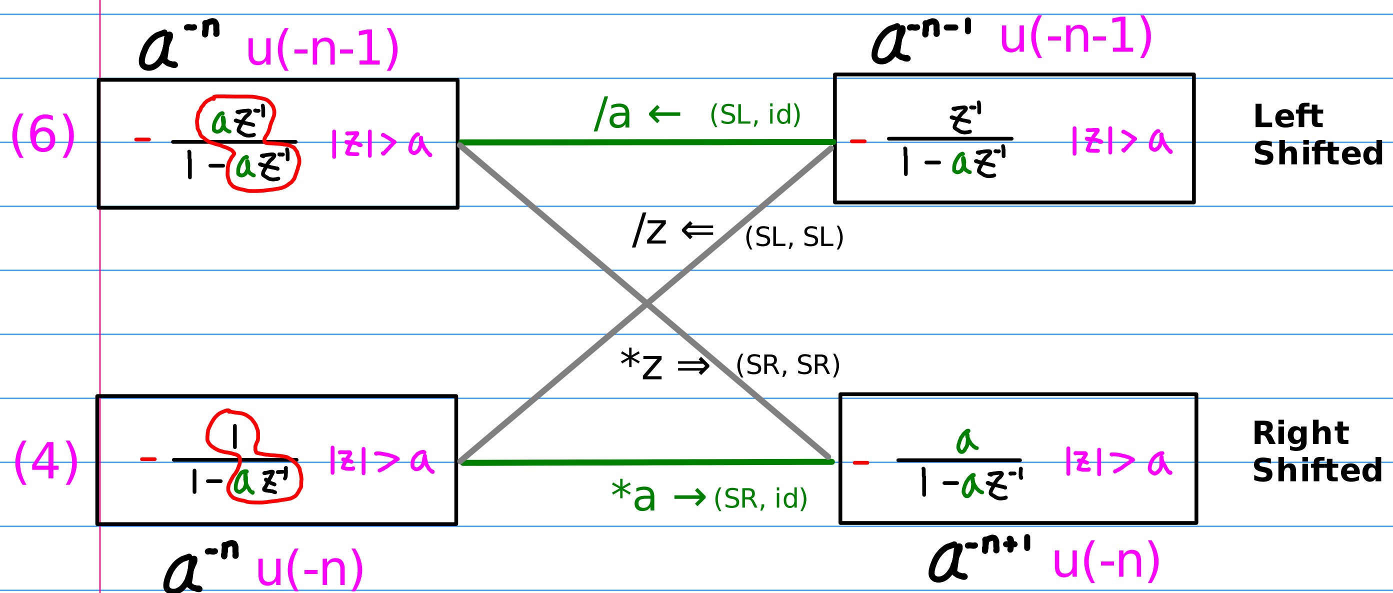
Reciprocal common ratio



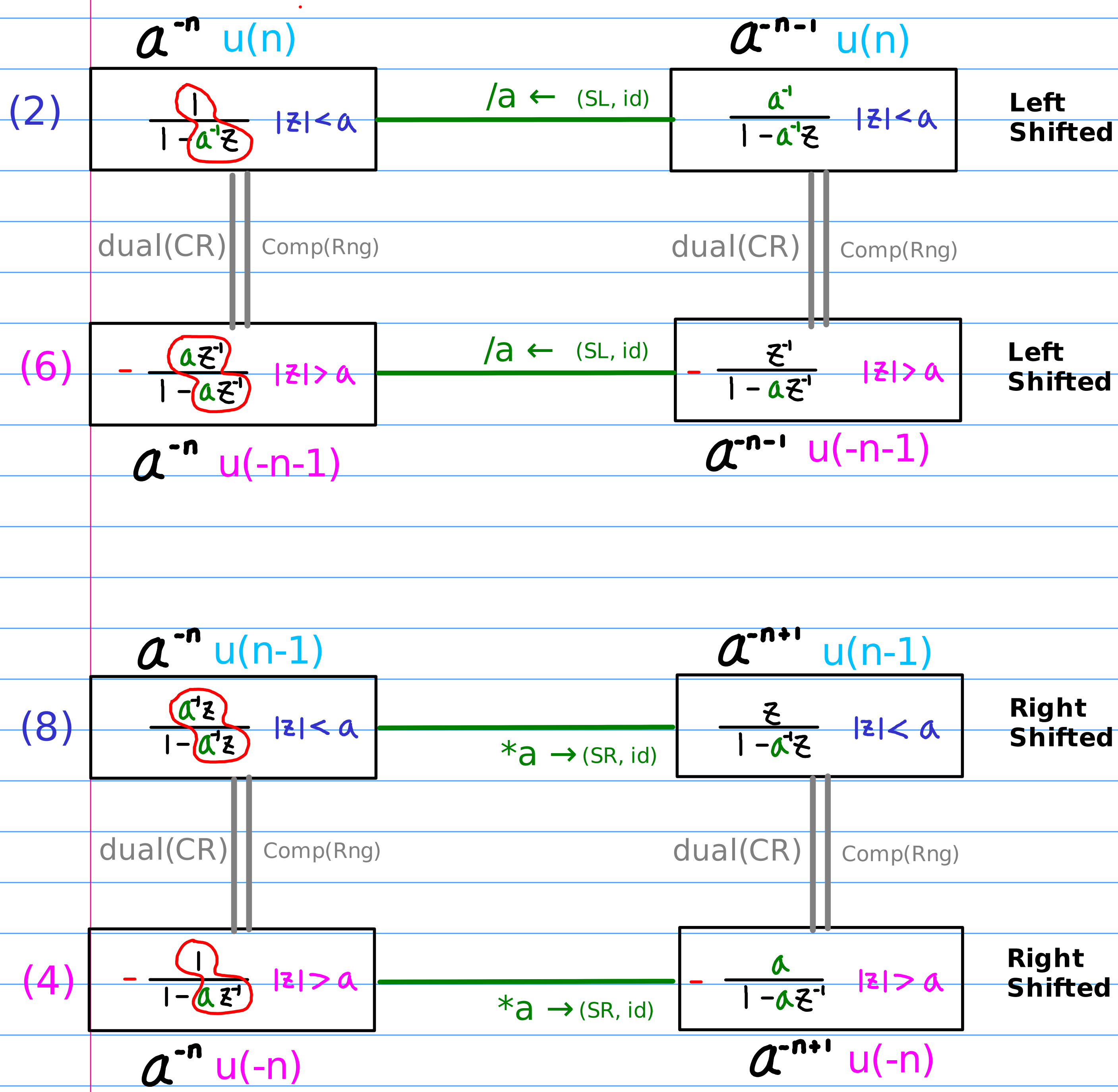
Shifting Geometric Series (2) negative exponent

Negative Exponent a^{-1} /z n ← n+1 *z n ← n-1

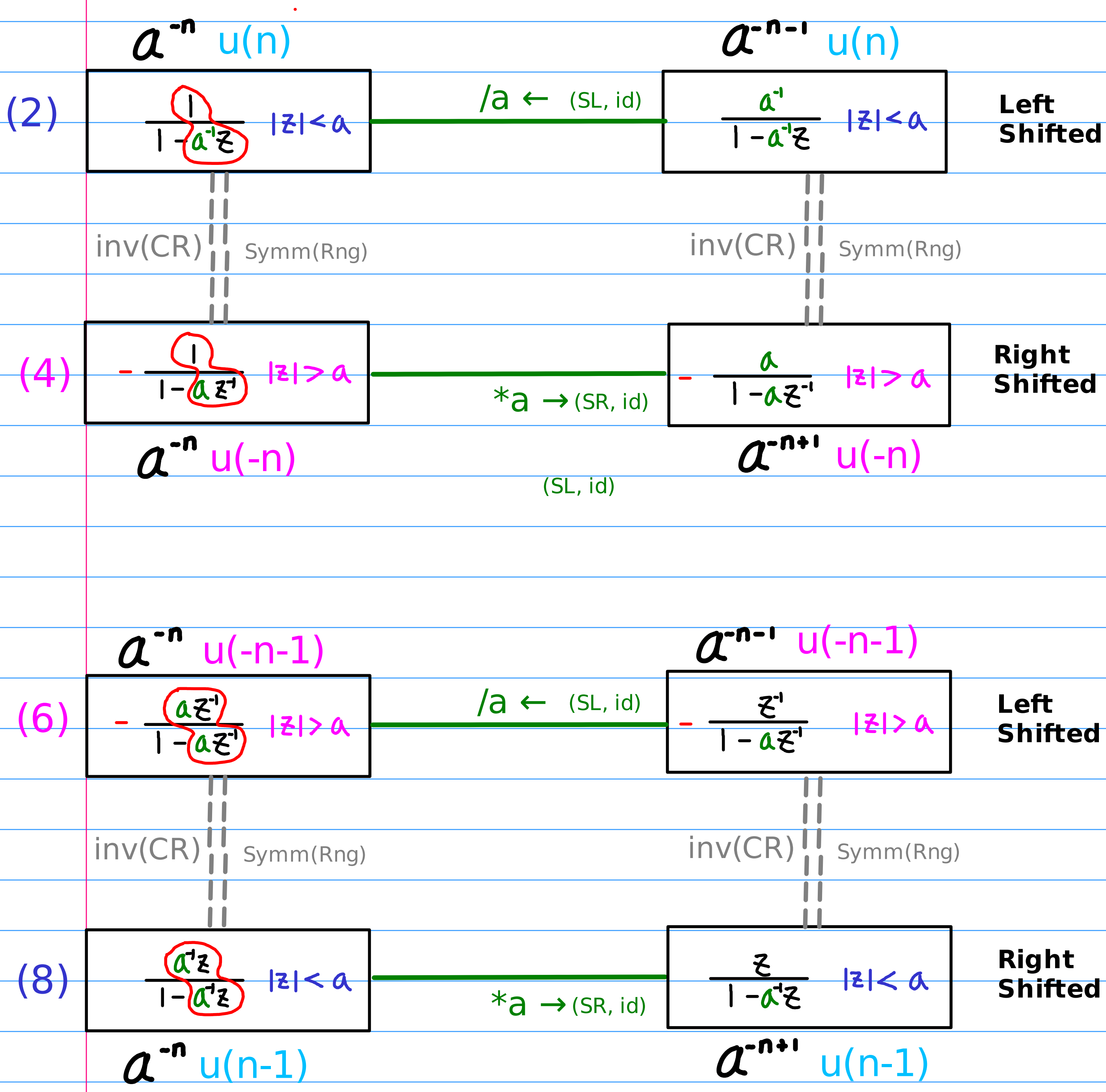




Dual expression w.r.t a common ratio



Reciprocal common ratio

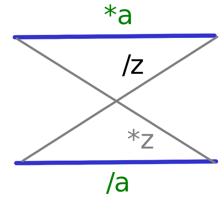


Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Shifted Geometric Series (1)

by multiplying a or a^{-1}



Positive Exponent

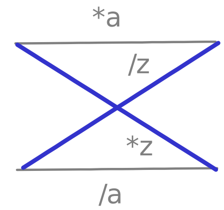
(1) ←	$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$a^n u(n)$	$\times a$	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$	$a^{n+1} u(n)$
(7) →	$\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$a^n u(n-1)$	$\times a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$	$a^{n-1} u(n-1)$
(5) ←	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$a^n u(-n-1)$	$\times a$	$\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$a^{n+1} u(-n-1)$
(3) →	$\frac{1}{1 \;\; a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$a^n u(-n)$	$\times a^{-1}$	$\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$a^{n-1} u(-n)$

Negative Exponent

(2) ←	$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$a^{-n} u(n)$	$\times a^{-1}$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$	$a^{-n-1} u(n)$
(8) →	$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$a^{-n} u(n-1)$	$\times a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$a^{-n+1} u(n-1)$
(6) ←	$\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$a^{-n} u(-n-1)$	$\times a^{-1}$	$\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$a^{-n-1} u(-n-1)$
(4) →	$\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$a^{-n} u(-n)$	$\times a$	$\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$	$a^{-n+1} u(-n)$

Shifted Geometric Series (2)

by multiplying z or z^{-1}



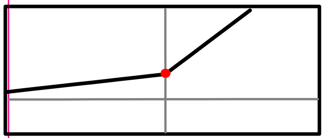
Positive Exponent

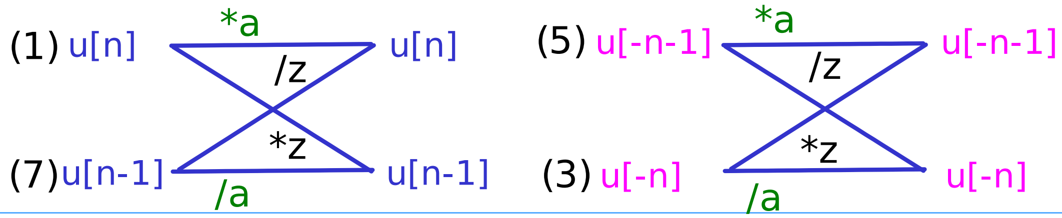
(1) ⇐	$\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$n \leftarrow n+1$ $a^n u(n-1)$ z^{-1}	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$	$a^{n+1} u(n)$
(7) ⇒	$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$n \leftarrow n-1$ $a^n u(n)$ $\times z$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$	$a^{n-1} u(n-1)$
(5) ⇐	$\frac{1}{1\ a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$n \leftarrow n+1$ $a^n u(-n)$ $\times z^{-1}$	$\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$a^{n+1} u(-n-1)$
(3) ⇒	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$n \leftarrow n-1$ $a^n u(-n-1)$ $\times z$	$\frac{a^{-1}}{1-a^{-1}z^{-}}$ $\lvert z\rvert > a^{-1}$	$a^{n-1} u(-n)$

Negative Exponent

(2) ⇐	$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$n \leftarrow n+1$ $a^{-n} u(n-1)$ $\times z^{-1}$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$	$a^{-n-1} u(n)$
(8) ⇒	$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$n \leftarrow n-1$ $a^{-n} u(n)$ $\times z$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$a^{-n+1} u(n-1)$
(6) ⇐	$\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$n \leftarrow n+1$ $a^{-n} u(-n)$ $\times z^{-1}$	$\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$a^{-n-1} u(-n-1)$
(4) ⇒	$\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$n \leftarrow n-1$ $a^{-n} u(-n-1)$ $\times z$	$\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$	$a^{-n+1} u(-n)$

a^n





(1) ***a**

$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$
$\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$

(7) **/a**

(1) ***a**

a^n u(n) $(a^0, a^1, a^2, \cdots)$	a^{n+1} u(n) $(a^1, a^2, a^3, \cdots)$
a^n u(n-1) $(a^1, a^2, a^3, \cdots)$	a^{n-1} u(n-1) $(a^0, a^1, a^2, \cdots)$

(7) **/a**

(1) ***z**

$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$
$\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$

(7) **/z**

(1) ***z**

a^n u(n) $(a^0, a^1, a^2, \cdots)$	a^{n-1} u(n-1) $(a^0, a^1, a^2, \cdots)$
a^n u(n-1) $(a^1, a^2, a^3, \cdots)$	a^{n+1} u(n) $(a^1, a^2, a^3, \cdots)$

(7) **/z**

(5) ***a**

$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$
$\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$

(3) **/a**

(5) ***a**

a^n u(-n-1) $(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$	a^{n+1} u(-n-1) $(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$
a^n u(-n) $(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	a^{n-1} u(-n) $(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$

(3) **/a**

(5) ***z**

$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$
$\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$

(3) **/z**

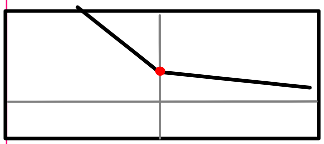
(5) ***z**

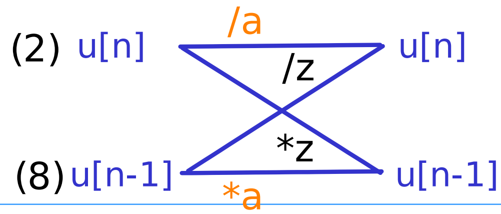
a^n u(-n-1) $(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$	a^{n-1} u(-n) $(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$
a^n u(-n) $(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	a^{n+1} u(-n-1) $(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$

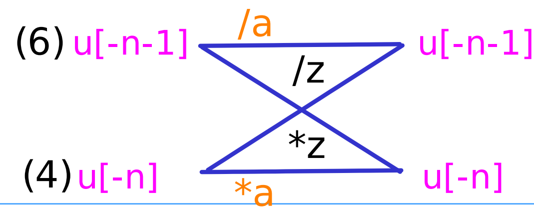
(3) **/z**

butterfly pair ordering

a^{-n}







(2) /a

$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$
$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$

(8) *a

(2) /a

$(\frac{1}{a})^n$ u(n) $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots)$	$(\frac{1}{a})^{n+1}$ u(n) $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots)$
$(\frac{1}{a})^n$ u(n-1) $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots)$	$(\frac{1}{a})^{n-1}$ u(n-1) $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots)$

(8) *a

(2) *z

$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$
$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$

(8) /z

(2) *z

$(\frac{1}{a})^n$ u(n) $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots)$	$(\frac{1}{a})^{n-1}$ u(n-1) $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots)$
$(\frac{1}{a})^n$ u(n-1) $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots)$	$(\frac{1}{a})^{n+1}$ u(n) $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots)$

(8) /z

(6) /a

$\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$
$\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$

(4) *a

(6) /a

$(\frac{1}{a})^n$ u(-n-1) $(\cdots, a^3, a^2, a^1)$	$(\frac{1}{a})^{n+1}$ u(-n-1) $(\cdots, a^2, a^1, a^0)$
$(\frac{1}{a})^n$ u(-n) $(\cdots, a^2, a^1, a^0)$	$(\frac{1}{a})^{n-1}$ u(-n) $(\cdots, a^3, a^2, a^1)$

(4) *a

(6) *z

$\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$
$\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$

(4) /z

(6) *z

$(\frac{1}{a})^n$ u(-n-1) $(\cdots, a^3, a^2, a^1)$	$(\frac{1}{a})^{n-1}$ u(-n) $(\cdots, a^3, a^2, a^1)$
$(\frac{1}{a})^n$ u(-n) $(\cdots, a^2, a^1, a^0)$	$(\frac{1}{a})^{n+1}$ u(-n-1) $(\cdots, a^2, a^1, a^0)$

(4) /z

butterfly pair ordering

Shifted Sequences of **f(z)**

(1)

$f(z) = \dfrac{1}{1 - az}$	$\|z\| < a^{-1}$
$a^n \, u(n)$	$(n \geq 0)$

(1')

$f_2(z) = \dfrac{a}{1 - az}$	$\|z\| < a^{-1}$
$a^{n+1} \, u(n)$	$(n \geq 0)$

(7)

$f_1(z) = \dfrac{az}{1 - az}$	$\|z\| < a^{-1}$
$a^n \, u(n-1)$	$(n \geq 1)$

(7')

$f_3(z) = \dfrac{z}{1 - az}$	$\|z\| < a^{-1}$
$a^{n-1} u(n-1)$	$(n \geq 1)$

(5)

$\bar{f}(z) = \dfrac{a^{-1} z^{-1}}{1 - a^{-1} z^{-1}}$	$\|z\| > a^{-1}$
$a^n \, u(-n-1)$	$(n < 0)$

(5')

$\bar{f}_2(z) = \dfrac{z^{-1}}{1 - a^{-1} z^{-1}}$	$\|z\| > a^{-1}$
$a^{n+1} \, u(-n-1)$	$(n < 0)$

(3)

$\bar{f}_1(z) = \dfrac{1}{1 - a^{-1} z^{-1}}$	$\|z\| > a^{-1}$
$a^n \, u(-n)$	$(n < 1)$

(3')

$\bar{f}_3(z) = \dfrac{a^{-1}}{1 - a^{-1} z^{-1}}$	$\|z\| > a^{-1}$
$a^{n-1} u(-n)$	$(n < 1)$

Shifted Sequences of **g(z)**

(2)

$g(z) = \dfrac{1}{1-a^{-1}z}$	$\lvert z \rvert < a^{-1}$
$\left(\frac{1}{a}\right)^{n} u(n)$	$(n \geq 0)$

(2')

$g_2(z) = \dfrac{a^{-1}}{1-a^{-1}z}$	$\lvert z \rvert < a^{-1}$
$\left(\frac{1}{a}\right)^{n+1} u(n)$	$(n \geq 0)$

(8)

$g_1(z) = \dfrac{a^{-1}z}{1-a^{-1}z}$	$\lvert z \rvert < a^{-1}$
$\left(\frac{1}{a}\right)^{n} u(n-1)$	$(n \geq 1)$

(8')

$g_3(z) = \dfrac{z}{1-a^{-1}z}$	$\lvert z \rvert < a^{-1}$
$\left(\frac{1}{a}\right)^{n-1} u(n-1)$	$(n \geq 1)$

(6)

$\bar{g}(z) = \dfrac{a z^{-1}}{1-a z^{-1}}$	$\lvert z \rvert > a^{-1}$
$\left(\frac{1}{a}\right)^{n} u(-n-1)$	$(n < 0)$

(6')

$\bar{g}_2(z) = \dfrac{z^{-1}}{1-a z^{-1}}$	$\lvert z \rvert > a^{-1}$
$\left(\frac{1}{a}\right)^{n+1} u(-n-1)$	$(n < 0)$

(4)

$\bar{g}_1(z) = \dfrac{1}{1-a z^{-1}}$	$\lvert z \rvert > a^{-1}$
$\left(\frac{1}{a}\right)^{n} u(-n)$	$(n < 1)$

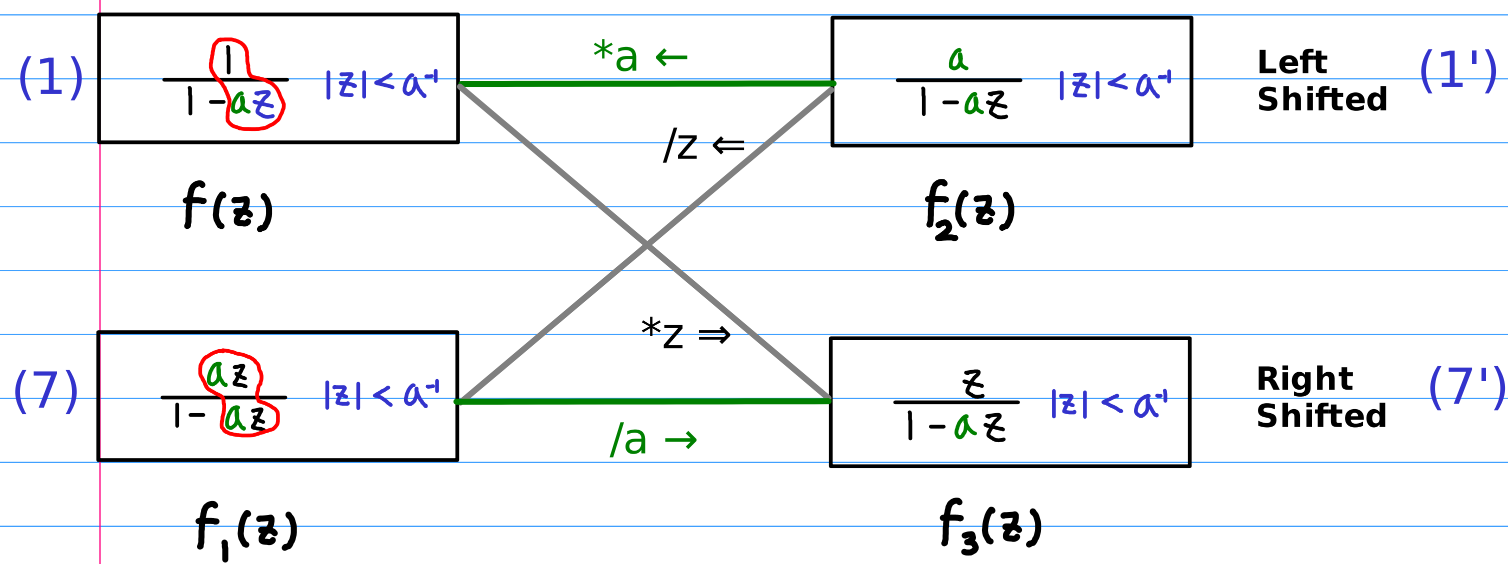
(4')

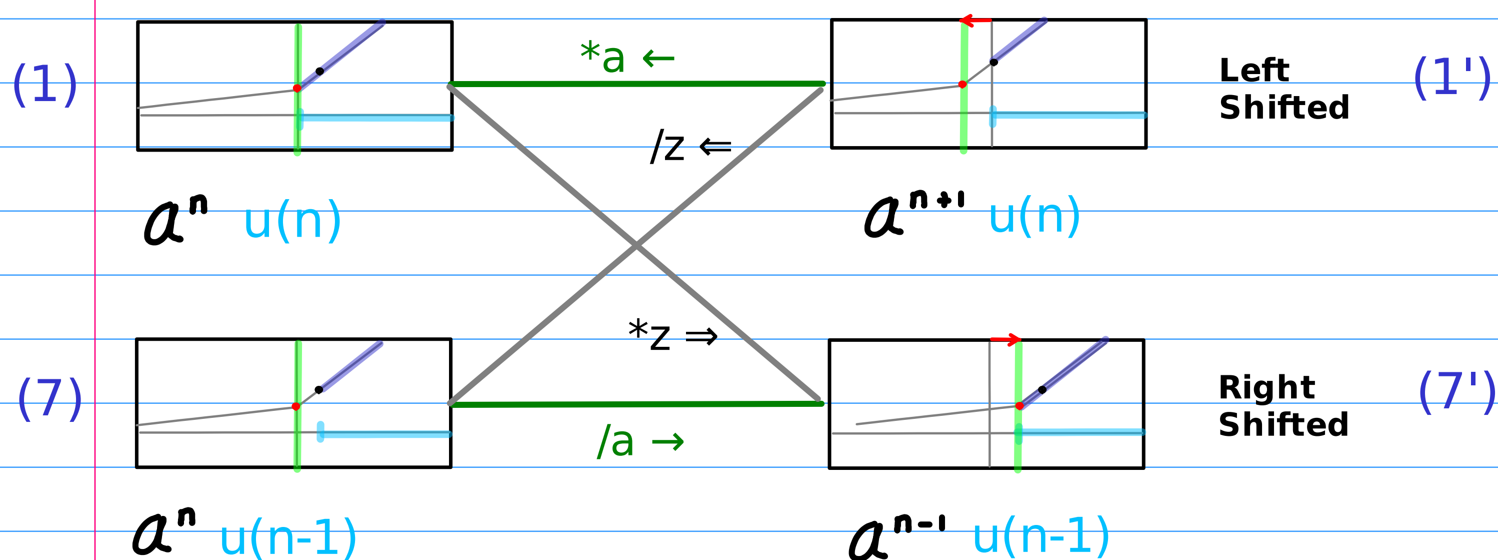
$\bar{g}_3(z) = \dfrac{a}{1-a z^{-1}}$	$\lvert z \rvert > a^{-1}$
$\left(\frac{1}{a}\right)^{n-1} u(-n)$	$(n < 1)$

Shifted Sequences of **f(z)**

(1) $f(z)$ (1') $f_2(z)$

(7) $f_1(z)$ (7') $f_3(z)$

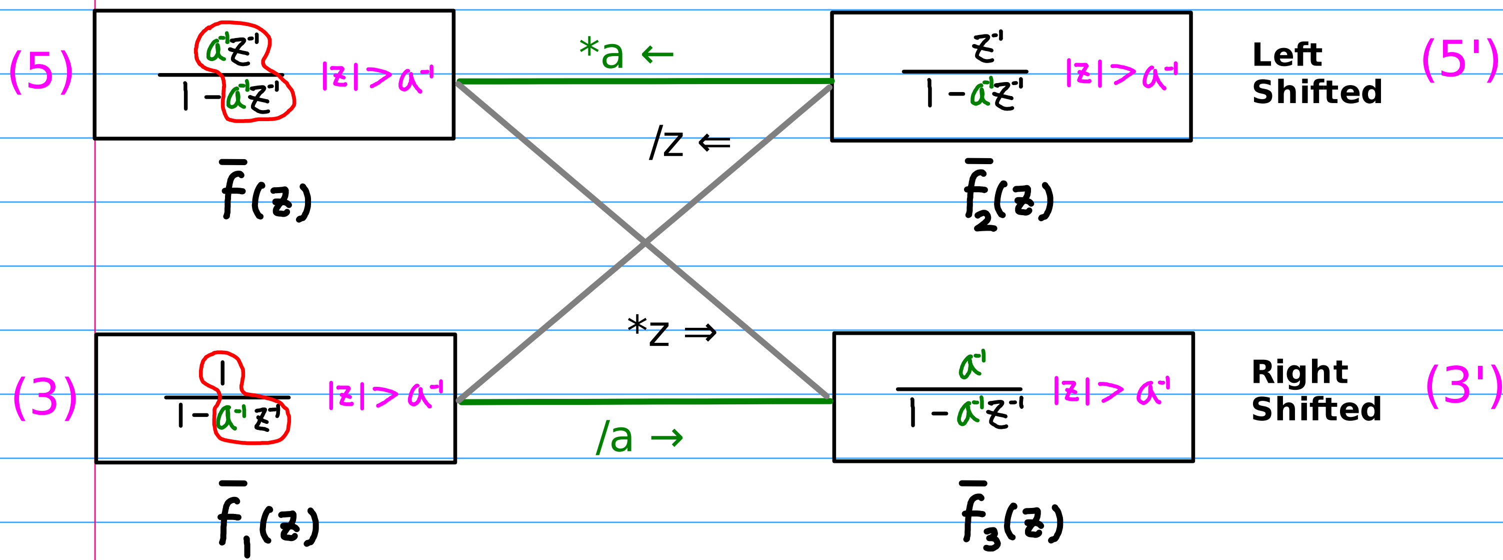


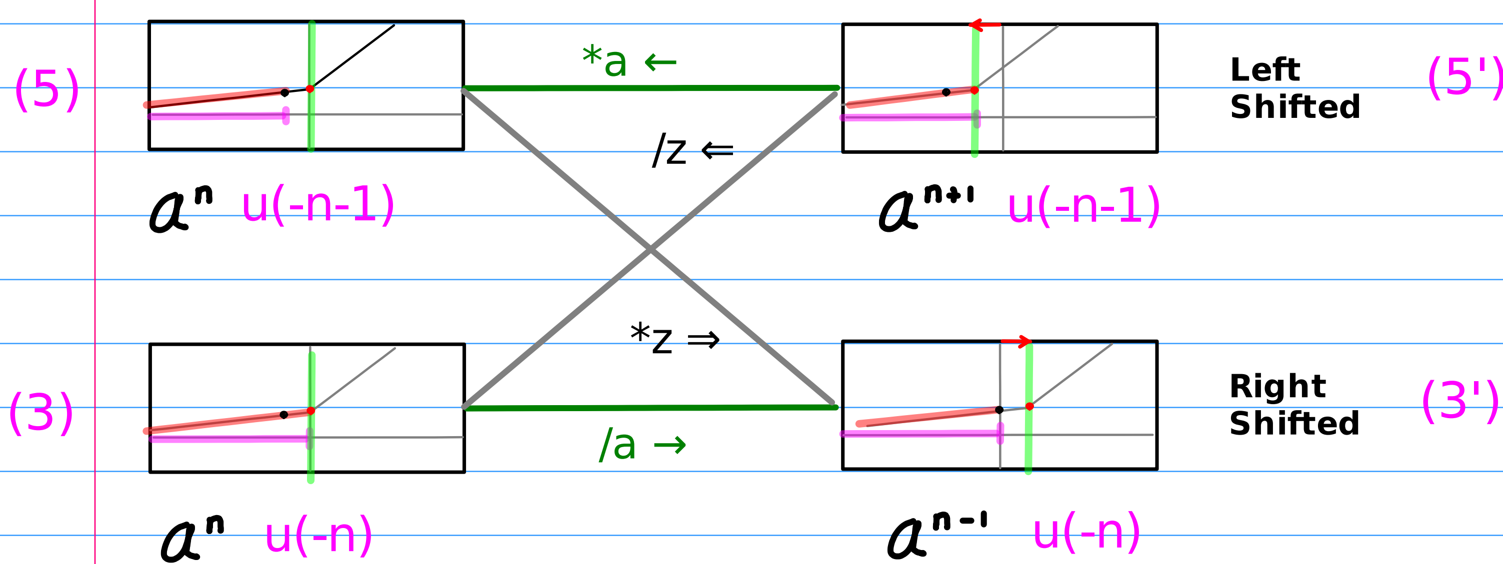


Shifted Sequences of **f(z)**

(5) $\bar{f}(z)$ (5') $\bar{f}_2(z)$

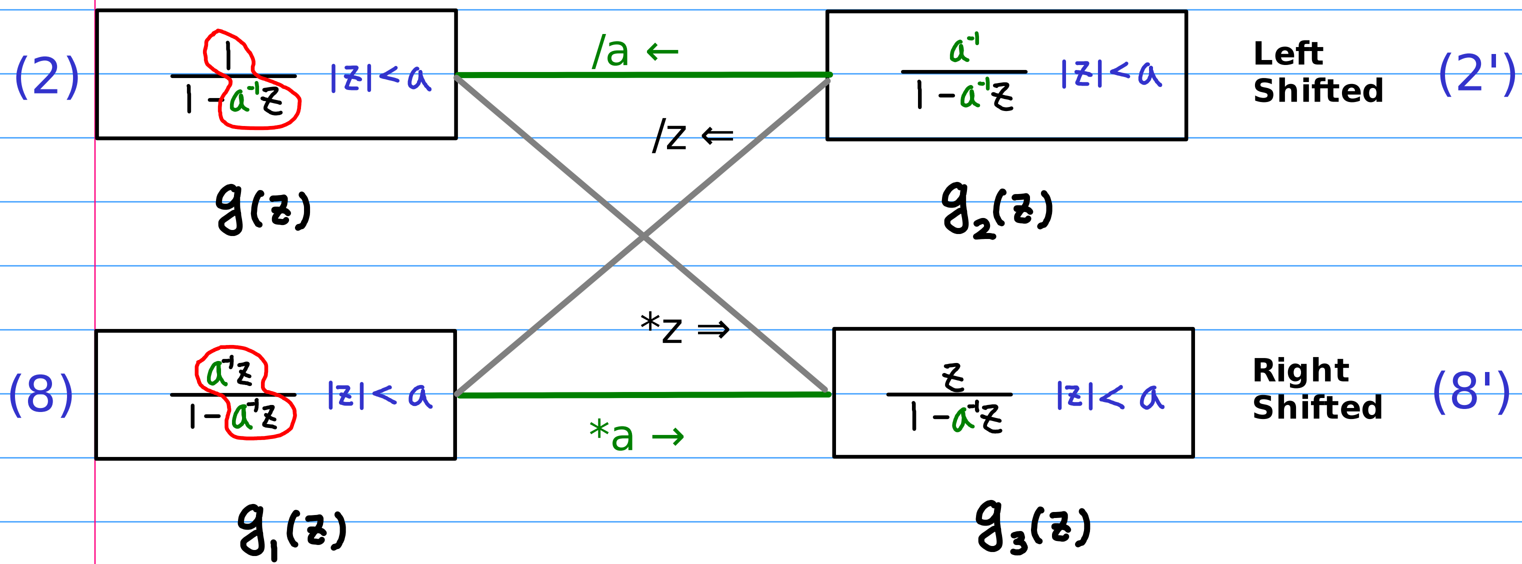
(3) $\bar{f}_1(z)$ (3') $\bar{f}_3(z)$

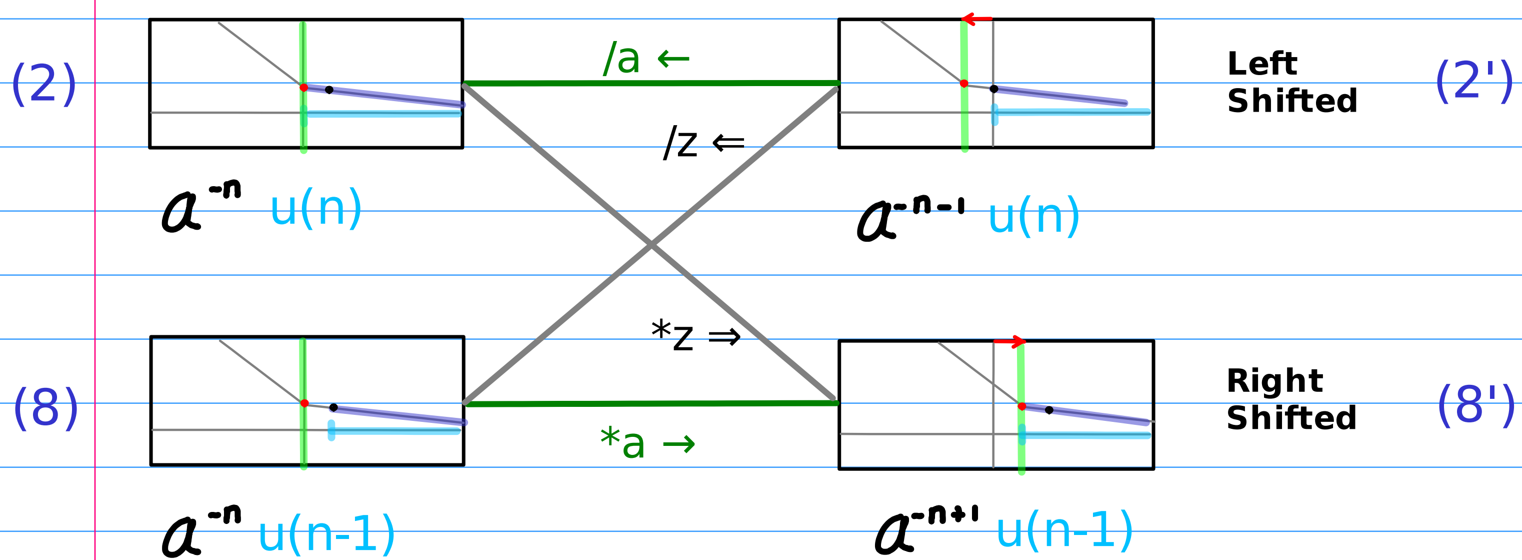




Shifted Sequences of **g(z)**

(2) $g(z)$ (2') $g_2(z)$
(8) $g_1(z)$ (8') $g_3(z)$

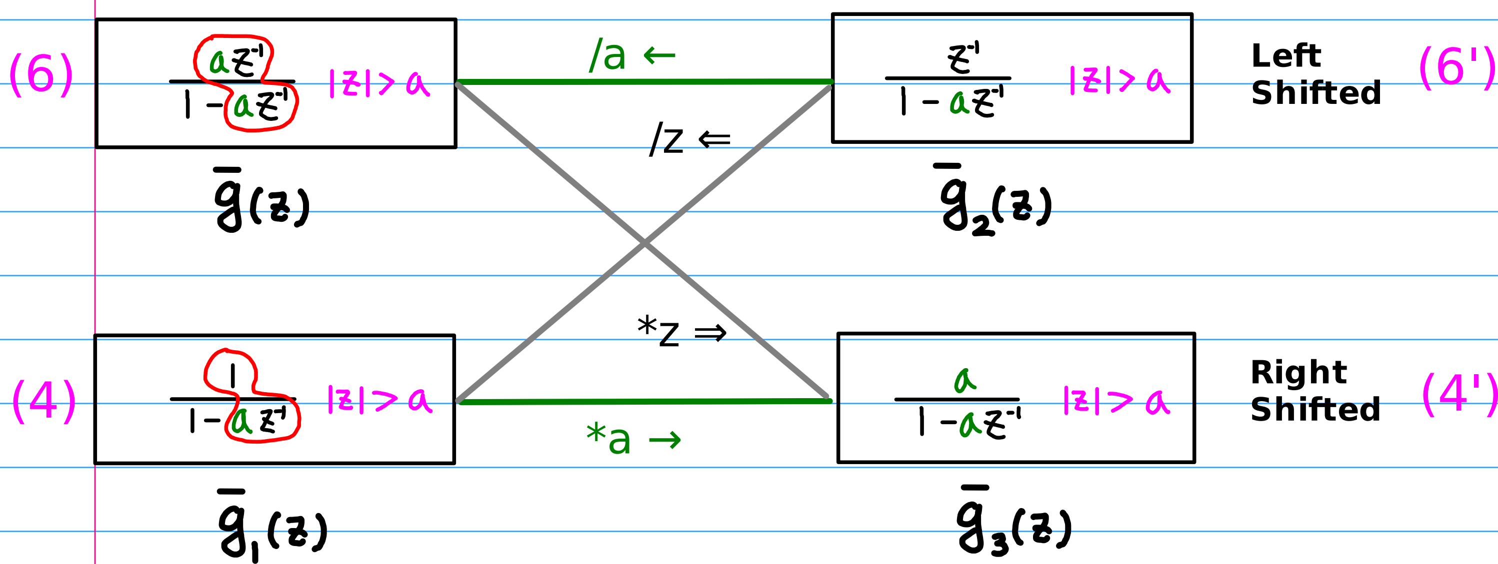


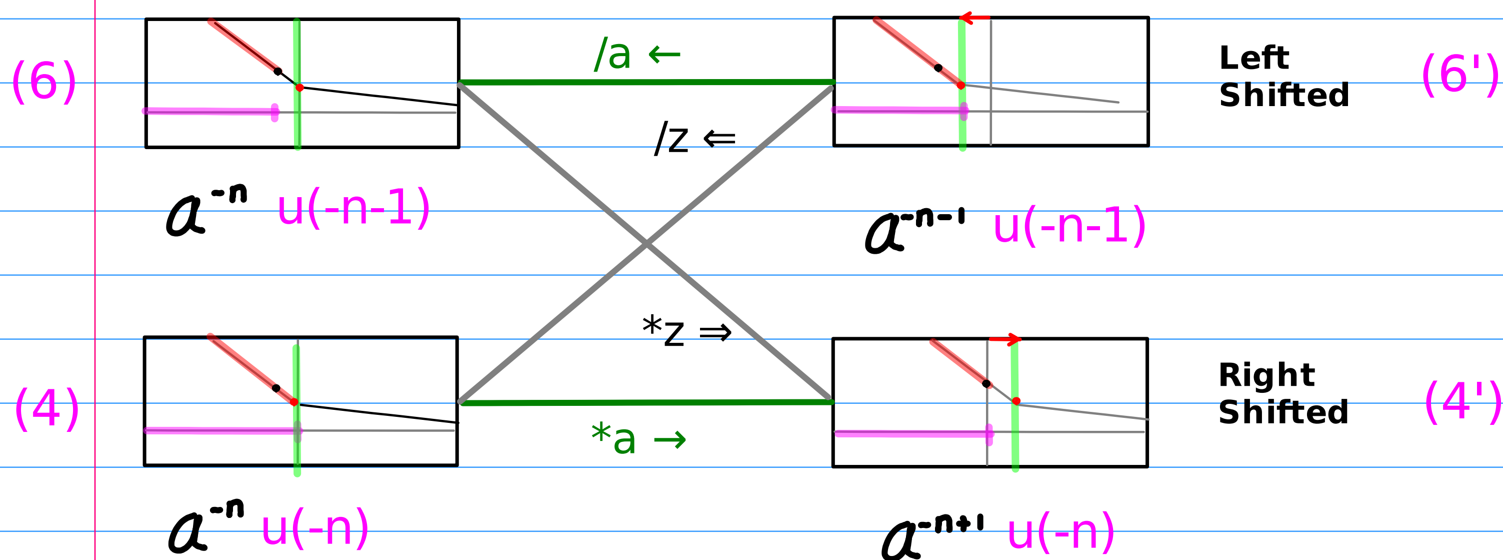


Shifted Sequences of **g(z)**

(6) $\bar{g}(z)$ (6') $\bar{g}_2(z)$

(4) $\bar{g}_1(z)$ (4') $\bar{g}_3(z)$





Assume all positive series $f(z)$ $\bar{f}(z)$ $f_1(z)$ $\bar{f}_1(z)$

$g(z)$ $g(z)$ $g_1(z)$ $\bar{g}_1(z)$

	pole $\left(\frac{1}{a}\right)$	pole a
causal Laurent	$f(z)$ a^n $u(n)$	$g(z)$ $\left(\frac{1}{a}\right)^n$ $u(n)$
anti-causal Laurent	$\bar{f}(z)$ a^n $u(-n-1)$	$\bar{g}(z)$ $\left(\frac{1}{a}\right)^n$ $u(-n-1)$
causal z-transform	$f_1(z)$ a^n $u(n-1)$	$g_1(z)$ $\left(\frac{1}{a}\right)^n$ $u(n-1)$
anti-causal z-transform	$\bar{f}_1(z)$ a^n $u(-n)$	$\bar{g}_1(z)$ $\left(\frac{1}{a}\right)^n$ $u(-n-1)$

Unshifted Geometric Series

Expressions

$f(z)$ $f_1(z)$ $g(z)$ $g_1(z)$

$\bar{f}(z)$ $\bar{f}_1(z)$ $\bar{g}(z)$ $\bar{g}_1(z)$

(1)

$f(z) = \dfrac{1}{1 - a z}$	$\lvert z \rvert < a^{-1}$
$a^n \; u(n)$	$(n \geq 0)$

(2)

$g(z) = \dfrac{1}{1 - a^{-1} z}$	$\lvert z \rvert < a$
$\left(\frac{1}{a}\right)^n \; u(n)$	$(n \geq 0)$

(3)

$\bar{f}_1(z) = \dfrac{1}{1 - a^{-1} z^{-1}}$	$\lvert z \rvert > a^{-1}$
$a^n \; u(-n)$	$(n < 1)$

(4)

$\bar{g}_1(z) = \dfrac{1}{1 - a z^{-1}}$	$\lvert z \rvert > a$
$\left(\frac{1}{a}\right)^n \; u(-n)$	$(n < 1)$

(5)

$\bar{f}(z) = \dfrac{a^{-1} z^{-1}}{1 - a^{-1} z^{-1}}$	$\lvert z \rvert > a^{-1}$
$a^n \; u(-n-1)$	$(n < 0)$

(6)

$\bar{g}(z) = \dfrac{a z^{-1}}{1 - a z^{-1}}$	$\lvert z \rvert > a$
$\left(\frac{1}{a}\right)^n \; u(-n-1)$	$(n < 0)$

(7)

$f_1(z) = \dfrac{a z}{1 - a z}$	$\lvert z \rvert < a^{-1}$
$a^n \; u(n-1)$	$(n \geq 1)$

(8)

$g_1(z) = \dfrac{a^{-1} z}{1 - a^{-1} z}$	$\lvert z \rvert < a$
$\left(\frac{1}{a}\right)^n \; u(n-1)$	$(n \geq 1)$

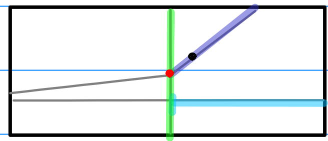
Unshifted Geometric Series

Graphs

$f(z)$ $f_1(z)$ $g(z)$ $g_1(z)$

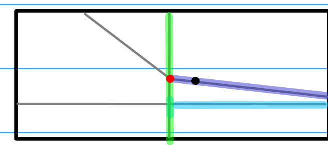
$\bar{f}(z)$ $\bar{f}_1(z)$ $\bar{g}(z)$ $\bar{g}_1(z)$

(1) $f(z)$



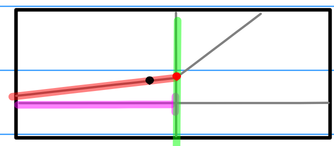
a^n u(n)

(2) $g(z)$



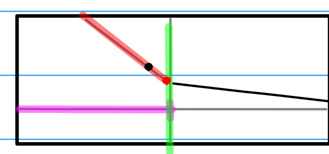
a^{-n} u(n)

(3) $\bar{f}_1(z)$



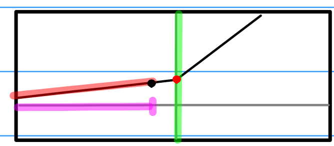
a^n u(-n)

(4) $\bar{g}_1(z)$



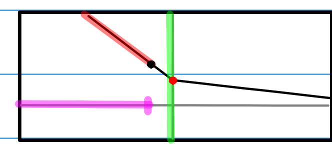
a^{-n} u(-n)

(5) $\bar{f}(z)$



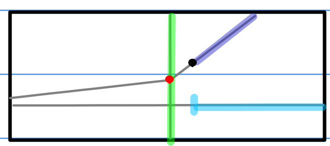
a^n u(-n-1)

(6) $\bar{g}(z)$



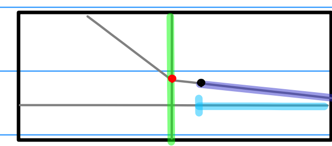
a^{-n} u(-n-1)

(7) $f_1(z)$



a^n u(n-1)

(8) $g_1(z)$



a^{-n} u(n-1)

Unshifted Geometric Series

$f(z)$ $f_1(z)$ $g(z)$ $g_1(z)$

$\bar{f}(z)$ $\bar{f}_1(z)$ $\bar{g}(z)$ $\bar{g}_1(z)$

Relationships (1)

Shifted Ranges (I)

$f(z)$	$f_1(z)$
$g(z)$	$g_1(z)$
u(n)	u(n-1)

Shifted Ranges (II)

$\bar{f}(z)$	$\bar{f}_1(z)$
$\bar{g}(z)$	$\bar{g}_1(z)$
u(-n-1)	u(-n)

Complementary Ranges (I)

$f(z)$	$\bar{f}(z)$
$g(z)$	$\bar{g}(z)$
u(n)	u(-n-1)

Complementary Ranges (II)

$f_1(z)$	$\bar{f}_1(z)$
$g_1(z)$	$\bar{g}_1(z)$
u(n-1)	u(-n)

Symmetric Ranges (I)

$f(z)$	$\bar{f}_1(z)$
$g(z)$	$\bar{g}_1(z)$
u(n)	u(-n)

Symmetric Ranges (II)

$\bar{f}(z)$	$f_1(z)$
$\bar{g}(z)$	$g_1(z)$
u(-n-1)	u(n-1)

Symmetric Sequences (I)

$f(z)$	$\bar{g}_1(z)$
$g(z)$	$\bar{f}_1(z)$
u(n)	u(-n)

Symmetric Sequences (II)

$\bar{f}(z)$	$g_1(z)$
$\bar{g}(z)$	$f_1(z)$
u(-n-1)	u(n-1)

Unshifted Geometric Series

Relationships (2)

$f(z)$ $f_1(z)$ $g(z)$ $g_1(z)$

$\bar{f}(z)$ $\bar{f}_1(z)$ $\bar{g}(z)$ $\bar{g}_1(z)$

$$f(z) * a\,z = f_1(z)$$
$$g(z) * a^{-1}z = g_1(z)$$

u(n) u(n-1) **Shifted Ranges (I)**

$$\bar{f}(z) * a\,z = \bar{f}_1(z)$$
$$\bar{g}(z) * a^{-1}z = \bar{g}_1(z)$$

u(-n) u(-n-1) **Shifted Ranges (II)**

$$-f(z) * \frac{(a\,z)^{-1}}{(a\,z)^{-1}} = \bar{f}(z)$$
$$-g(z) * \frac{(a^{-1}z)^{-1}}{(a^{-1}z)^{-1}} = \bar{g}(z)$$

u(n) u(-n-1) **Complementary Ranges (I)**

$$-f_1(z) * \frac{(a\,z)^{-1}}{(a\,z)^{-1}} = \bar{f}_1(z)$$
$$-g_1(z) * \frac{(a^{-1}z)^{-1}}{(a^{-1}z)^{-1}} = \bar{g}_1(z)$$

u(n-1) u(-n) **Complementary Ranges (II)**

common ratios

(1)	$f(z)$	$a\,z$	(5)	$\bar{f}(z)$	$a^{-1}z^{-1}$
(2)	$g(z)$	$a^{-1}z$	(6)	$\bar{g}(z)$	$a\,z^{-1}$
(3)	$\bar{f}_1(z)$	$a^{-1}z^{-1}$	(7)	$f_1(z)$	$a\,z$
(4)	$\bar{g}_1(z)$	$a\,z^{-1}$	(8)	$g_1(z)$	$a^{-1}z$

Unshifted Geometric Series

Relationships (3)

$f(z)$ $f_1(z)$ $g(z)$ $g_1(z)$

$\bar{f}(z)$ $\bar{f}_1(z)$ $\bar{g}(z)$ $\bar{g}_1(z)$

$$-f(z) * a z * \frac{(a z)^{-1}}{(a z)^{-1}} = \bar{f}_1(z)$$

$$-g(z) * a^{-1} z * \frac{(a^{-1} z)^{-1}}{(a^{-1} z)^{-1}} = \bar{g}_1(z)$$

u(n) u(-n) **Symmetric Ranges (I)**

$$-\bar{f}(z) * a z * \frac{(a z)^{-1}}{(a z)^{-1}} = f_1(z)$$

$$-\bar{g}(z) * a^{-1} z * \frac{(a^{-1} z)^{-1}}{(a^{-1} z)^{-1}} = g_1(z)$$

u(-n-1) u(n-1) **Symmetric Ranges (II)**

$$f(z^{-1}) = \bar{g}_1(z)$$

$$g(z^{-1}) = \bar{f}_1(z)$$

u(n) u(-n) **Symmetric Sequences (I)**

$$\bar{f}(z^{-1}) = g_1(z)$$

$$\bar{g}(z^{-1}) = f_1(z)$$

u(-n-1) u(n-1) **Symmetric Sequences (II)**

common ratios					
(1)	$f(z)$	$a z$	(5)	$\bar{f}(z)$	$a^{-1} z^{-1}$
(2)	$g(z)$	$a^{-1} z$	(6)	$\bar{g}(z)$	$a z^{-1}$
(3)	$\bar{f}_1(z)$	$a^{-1} z^{-1}$	(7)	$f_1(z)$	$a z$
(4)	$\bar{g}_1(z)$	$a z^{-1}$	(8)	$g_1(z)$	$a^{-1} z$

Shifted Geometric Series

Expressions

$f_2(z)$ $f_3(z)$ $g_2(z)$ $g_3(z)$
$\bar{f}_2(z)$ $\bar{f}_3(z)$ $\bar{g}_2(z)$ $\bar{g}_3(z)$

(1') $f_2(z) = \dfrac{a}{1 - az}$ | $|z| < a^{-1}$
a^{n+1} u(n) | $(n \geq 0)$

(2') $g_2(z) = \dfrac{a^{-1}}{1 - a^{-1}z}$ | $|z| < a^{-1}$
$\left(\frac{1}{a}\right)^{n+1}$ u(n) | $(n \geq 0)$

(3') $\bar{f}_3(z) = \dfrac{a^{-1}}{1 - a^{-1}z^{-1}}$ | $|z| > a^{-1}$
a^{n-1} u(-n) | $(n < 1)$

(4') $\bar{g}_3(z) = \dfrac{a}{1 - az^{-1}}$ | $|z| > a^{-1}$
$\left(\frac{1}{a}\right)^{n-1}$ u(-n) | $(n < 1)$

(5') $\bar{f}_2(z) = \dfrac{z^{-1}}{1 - a^{-1}z^{-1}}$ | $|z| > a^{-1}$
a^{n+1} u(-n-1) | $(n < 0)$

(6') $\bar{g}_2(z) = \dfrac{z^{-1}}{1 - az^{-1}}$ | $|z| > a^{-1}$
$\left(\frac{1}{a}\right)^{n+1}$ u(-n-1) | $(n < 0)$

(7') $f_3(z) = \dfrac{z}{1 - az}$ | $|z| < a^{-1}$
a^{n-1} u(n-1) | $(n \geq 1)$

(8') $g_3(z) = \dfrac{z}{1 - a^{-1}z}$ | $|z| < a^{-1}$
$\left(\frac{1}{a}\right)^{n-1}$ u(n-1) | $(n \geq 1)$

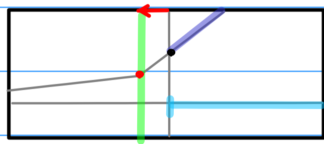
Shifted Geometric Series

Graphs

$f_2(z)$ $f_3(z)$ $g_2(z)$ $g_3(z)$

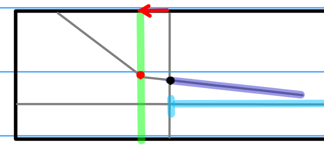
$\bar{f}_2(z)$ $\bar{f}_3(z)$ $\bar{g}_2(z)$ $\bar{g}_3(z)$

(1') $f_2(z)$



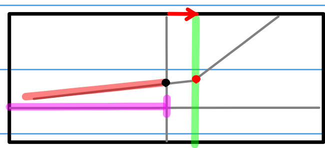
a^{n+1} u(n)

(2') $g_2(z)$



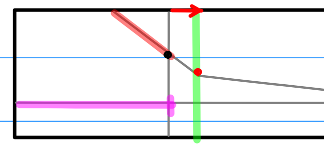
a^{-n-1} u(n)

(3') $\bar{f}_3(z)$



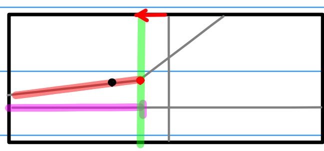
a^{n-1} u(-n)

(4') $\bar{g}_3(z)$



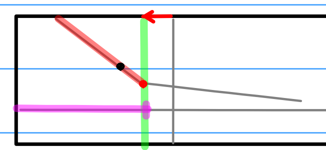
a^{-n+1} u(-n)

(5') $\bar{f}_2(z)$



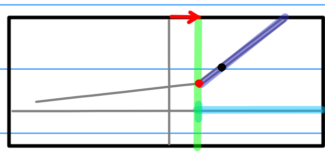
a^{n+1} u(-n-1)

(6') $\bar{g}_2(z)$



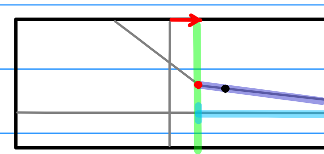
a^{-n-1} u(-n-1)

(7') $f_3(z)$



a^{n-1} u(n-1)

(8') $g_3(z)$



a^{-n+1} u(n-1)

Shifted Geometric Series

Relationships (1)

$f_2(z)$ $f_3(z)$ $g_2(z)$ $g_3(z)$

$\bar{f}_2(z)$ $\bar{f}_3(z)$ $\bar{g}_2(z)$ $\bar{g}_3(z)$

Shifted Ranges (I)

$f_2(z)$	$f_3(z)$
$g_2(z)$	$g_3(z)$
u(n)	u(n-1)

Shifted Ranges (II)

$\bar{f}_2(z)$	$\bar{f}_3(z)$
$\bar{g}_2(z)$	$\bar{g}_3(z)$
u(-n-1)	u(-n)

Complementary Ranges (I)

$f_2(z)$	$\bar{f}_2(z)$
$g_2(z)$	$\bar{g}_2(z)$
u(n)	u(-n-1)

Complementary Ranges (II)

$f_3(z)$	$\bar{f}_3(z)$
$g_3(z)$	$\bar{g}_3(z)$
u(n-1)	u(-n)

Symmetric Ranges (I)

$f_2(z)$	$\bar{f}_3(z)$
$g_2(z)$	$\bar{g}_3(z)$
u(n)	u(-n)

Symmetric Ranges (II)

$\bar{f}_2(z)$	$f_3(z)$
$\bar{g}_2(z)$	$g_3(z)$
u(-n-1)	u(n-1)

Symmetric Sequences (I)

$f_2(z)$	$\bar{g}_3(z)$
$g_2(z)$	$\bar{f}_3(z)$
u(n)	u(-n)

Symmetric Sequences (II)

$\bar{f}_2(z)$	$g_3(z)$
$\bar{g}_2(z)$	$f_3(z)$
u(-n-1)	u(n-1)

Shifted Geometric Series

Relationships (2)

$f_2(z)$ $f_3(z)$ $g_2(z)$ $g_3(z)$

$\bar{f}_2(z)$ $\bar{f}_3(z)$ $\bar{g}_2(z)$ $\bar{g}_3(z)$

$$f_2(z) * a^{-1}z = f_3(z)$$
$$g_2(z) * a\,z = g_3(z)$$

u(n) u(n-1)

Shifted Ranges (I)

$$\bar{f}_2(z) * a^{-1}z = \bar{f}_3(z)$$
$$\bar{g}_2(z) * a\,z = \bar{g}_3(z)$$

u(-n) u(-n-1)

Shifted Ranges (II)

$$-f_2(z) * \frac{(a\,z)^{-1}}{(a\,z)^{-1}} = \bar{f}_2(z)$$
$$-g_2(z) * \frac{(a^{-1}z)^{-1}}{(a^{-1}z)^{-1}} = \bar{g}_2(z)$$

u(n) u(-n-1)

Complementary Ranges (I)

$$-f_3(z) * \frac{(a\,z)^{-1}}{(a\,z)^{-1}} = \bar{f}_3(z)$$
$$-g_3(z) * \frac{(a^{-1}z)^{-1}}{(a^{-1}z)^{-1}} = \bar{g}_3(z)$$

u(n-1) u(-n)

Complementary Ranges (II)

common ratios

(1')	$f_2(z)$	$a\,z$	(5')	$\bar{f}_2(z)$	$a^{-1}z^{-1}$
(2')	$g_2(z)$	$a^{-1}z$	(6')	$\bar{g}_2(z)$	$a\,z^{-1}$
(3')	$\bar{f}_3(z)$	$a^{-1}z^{-1}$	(7')	$f_3(z)$	$a\,z$
(4')	$\bar{g}_3(z)$	$a\,z^{-1}$	(8')	$g_3(z)$	$a^{-1}z$

$$-f_2(z) * a z * \frac{(a z)^{-1}}{(a z)^{-1}} = \bar{f}_3(z)$$

$$-g_2(z) * a^{-1} z * \frac{(a^{-1} z)^{-1}}{(a^{-1} z)^{-1}} = \bar{g}_3(z)$$

u(n) u(-n) **Symmetric Ranges (I)**

$$-\bar{f}_2(z) * a z * \frac{(a z)^{-1}}{(a z)^{-1}} = f_3(z)$$

$$-\bar{g}_2(z) * a^{-1} z * \frac{(a^{-1} z)^{-1}}{(a^{-1} z)^{-1}} = g_3(z)$$

u(-n-1) u(n-1) **Symmetric Ranges (II)**

$$f_2(z^{-1}) = \bar{g}_3(z)$$

$$g_2(z^{-1}) = \bar{f}_3(z)$$

u(n) u(-n)

Symmetric Sequences (I)

$$\bar{f}_2(z^{-1}) = g_3(z)$$

$$\bar{g}_2(z^{-1}) = f_3(z)$$

u(-n-1) u(n-1)

Symmetric Sequences (II)

common ratios

(1')	$f_2(z)$	$a\,z$	(5')	$\bar{f}_2(z)$	$a^{-1} z^{-1}$
(2')	$g_2(z)$	$a^{-1} z$	(6')	$\bar{g}_2(z)$	$a\,z^{-1}$
(3')	$\bar{f}_3(z)$	$a^{-1} z^{-1}$	(7')	$f_3(z)$	$a\,z$
(4')	$\bar{g}_3(z)$	$a\,z^{-1}$	(8')	$g_3(z)$	$a^{-1} z$

Shifted Geometric Series

Relationships (3)

$f_2(z)\quad f_3(z)\quad g_2(z)\quad g_3(z)$

$\overline{f}_2(z)\quad \overline{f}_3(z)\quad \overline{g}_2(z)\quad \overline{g}_3(z)$

$$f_2(z) * a^{-1} * z = f(z) * z = f_3(z)$$
$$g_2(z) * a * z = g(z) * z = g_3(z)$$

u(n) u(n) u(n-1) **Shifted Ranges**

$$\overline{f}_2(z) * a^{-1} * z = \overline{f}(z) * z = \overline{f}_3(z)$$
$$\overline{g}_2(z) * a * z = \overline{g}(z) * z = \overline{g}_3(z)$$

u(-n-1) u(-n-1) u(-n) **Shifted Ranges**

$$f_3(z) * a * z^{-1} = f_1(z) * z^{-1} = f_2(z)$$
$$g_3(z) * a^{-1} * z^{-1} = g_1(z) * z^{-1} = g_2(z)$$

u(n-1) u(n-1) u(n) **Shifted Ranges**

$$\overline{f}_3(z) * a * z^{-1} = \overline{f}_1(z) * z^{-1} = \overline{f}_2(z)$$
$$\overline{g}_3(z) * a^{-1} * z^{-1} = \overline{g}_1(z) * z^{-1} = \overline{g}_2(z)$$

u(-n) u(-n) u(-n-1) **Shifted Ranges**

	common ratios				
(1')	$f_2(z)$	$a\,z$	(5')	$\overline{f}_2(z)$	$a^{-1} z^{-1}$
(2')	$g_2(z)$	$a^{-1} z$	(6')	$\overline{g}_2(z)$	$a\,z^{-1}$
(3')	$\overline{f}_3(z)$	$a^{-1} z^{-1}$	(7')	$f_3(z)$	$a\,z$
(4')	$\overline{g}_3(z)$	$a\,z^{-1}$	(8')	$g_3(z)$	$a^{-1} z$

Shifted Geometric Series

Relationships (4)

$f_2(z)$ $f_3(z)$ $g_2(z)$ $g_3(z)$

$\overline{f}_2(z)$ $\overline{f}_3(z)$ $\overline{g}_2(z)$ $\overline{g}_3(z)$

$$f(z) * a z = f_1(z)$$
$$g(z) * a^{-1} z = g_1(z)$$

u(n) u(n-1)

$$\overline{f}_1(z) * a^{-1} z^{-1} = \overline{f}(z)$$
$$\overline{g}_1(z) * a z^{-1} = \overline{g}(z)$$

u(-n-1) u(-n)

$$f_2(z) * a^{-1} z = f_3(z)$$
$$g_2(z) * a z = g_3(z)$$

u(n) u(n-1)

$$\overline{f}_3(z) * a z^{-1} = \overline{f}_2(z)$$
$$\overline{g}_3(z) * a^{-1} z^{-1} = \overline{g}_2(z)$$

u(-n-1) u(-n)

common ratios					
(1')	$f_2(z)$	$a z$	(5')	$\overline{f}_2(z)$	$a^{-1} z^{-1}$
(2')	$g_2(z)$	$a^{-1} z$	(6')	$\overline{g}_2(z)$	$a z^{-1}$
(3')	$\overline{f}_3(z)$	$a^{-1} z^{-1}$	(7')	$f_3(z)$	$a z$
(4')	$\overline{g}_3(z)$	$a z^{-1}$	(8')	$g_3(z)$	$a^{-1} z$

(1) $f(z) = \frac{1}{1-az}$ $\qquad$ $f_2(z) = \frac{a}{1-az}$ (1')

(7) $f_1(z) = \frac{az}{1-az}$ $\qquad$ $f_3(z) = \frac{z}{1-az}$ (7')

(5) $\bar{f}(z) = \frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\qquad$ $\bar{f}_2(z) = \frac{z^{-1}}{1-a^{-1}z^{-1}}$ (5')

(3) $\bar{f}_1(z) = \frac{1}{1-a^{-1}z^{-1}}$ $\qquad$ $\bar{f}_3(z) = \frac{a^{-1}}{1-a^{-1}z^{-1}}$ (3')

(2) $g(z) = \frac{1}{1-a^{-1}z}$ $\qquad$ $g_2(z) = \frac{a^{-1}}{1-a^{-1}z}$ (2')

(8) $g_1(z) = \frac{a^{-1}z}{1-a^{-1}z}$ $\qquad$ $g_3(z) = \frac{z}{1-a^{-1}z}$ (8')

(6) $\bar{g}(z) = \frac{az^{-1}}{1-az^{-1}}$ $\qquad$ $\bar{g}_2(z) = \frac{z^{-1}}{1-a^{-1}z^{-1}}$ (6')

(4) $\bar{g}_1(z) = \frac{1}{1-az^{-1}}$ $\qquad$ $\bar{g}_3(z) = \frac{a^{-1}}{1-a^{-1}z^{-1}}$ (4')

$\boxed{a}\ \boxed{z}$

(1) $f(z) = \dfrac{1}{1 - \square\square}$ $\qquad f_2(z) = \dfrac{\square}{1 - \square\square}$ (1')

(7) $f_1(z) = \dfrac{\square\square}{1 - \square\square}$ $\qquad f_3(z) = \dfrac{\square}{1 - \square\square}$ (7')

$\boxed{a^{-1}}\ \boxed{z^{-1}}$

(5) $\bar{f}(z) = \dfrac{\square\square}{1 - \square\square}$ $\qquad \bar{f}_2(z) = \dfrac{\square}{1 - \square\square}$ (5')

(3) $\bar{f}_1(z) = \dfrac{1}{1 - \square\square}$ $\qquad \bar{f}_3(z) = \dfrac{\square}{1 - \square\square}$ (3')

$\boxed{a^{-1}}\ \boxed{z}$

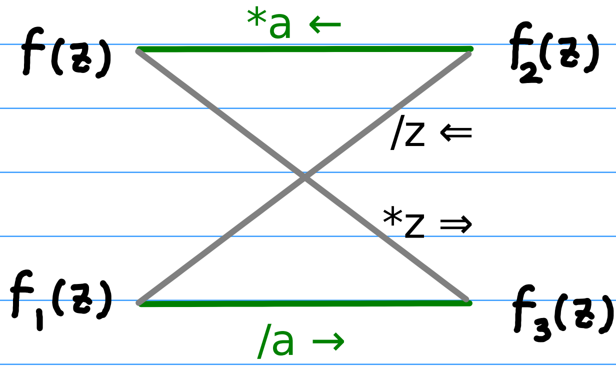
(2) $g(z) = \dfrac{1}{1 - \square\square}$ $\qquad g_2(z) = \dfrac{\square}{1 - \square\square}$ (2')

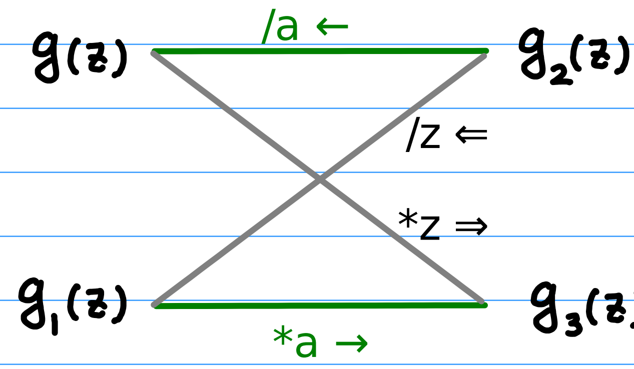
(8) $g_1(z) = \dfrac{\square\square}{1 - \square\square}$ $\qquad g_3(z) = \dfrac{\square}{1 - \square\square}$ (8')

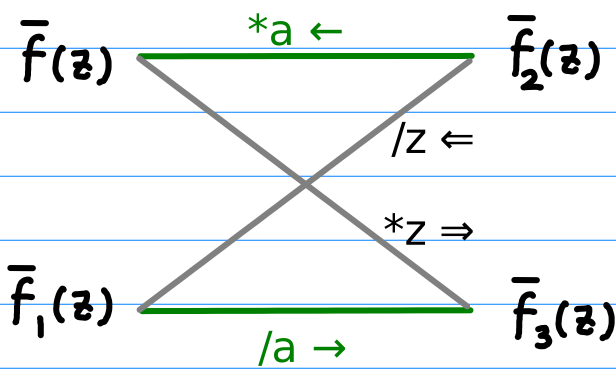
$\boxed{a}\ \boxed{z^{-1}}$

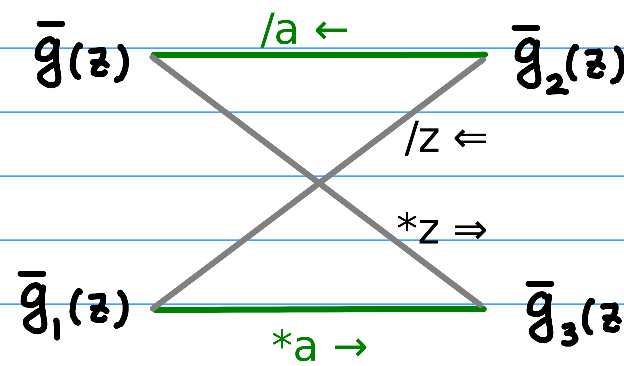
(6) $\bar{g}(z) = \dfrac{\square\square}{1 - \square\square}$ $\qquad \bar{g}_2(z) = \dfrac{\square}{1 - \square\square}$ (6')

(4) $\bar{g}_1(z) = \dfrac{1}{1 - \square\square}$ $\qquad \bar{g}_3(z) = \dfrac{\square}{1 - \square\square}$ (4')


$f(z)$
*a ←
$f_2(z)$
/z ⇐
*z ⇒
$f_1(z)$
/a →
$f_3(z)$


$g(z)$
/a ←
$g_2(z)$
/z ⇐
*z ⇒
$g_1(z)$
*a →


$\bar{f}(z)$
*a ←
$\bar{f}_2(z)$
/z ⇐
*z ⇒
$\bar{f}_1(z)$
/a →
$\bar{f}_3(z)$


$\bar{g}(z)$
/a ←
/z ⇐
*z ⇒
$\bar{g}_1(z)$
*a →

(1) $h_1(a,z) = f(z) = \dfrac{1}{1-az}$

(1') $h'_1(a,z) = f_2(z) = \dfrac{a}{1-az}$

(2) $h_2(a,z) = g(z) = \dfrac{1}{1-a^{-1}z}$

(2') $h'_2(a,z) = g_2(z) = \dfrac{a^{-1}}{1-a^{-1}z}$

(3) $h_3(a,z) = \bar{f}_1(z) = \dfrac{1}{1-a^{-1}z^{-1}}$

(3') $h'_3(a,z) = \bar{f}_3(z) = \dfrac{a^{-1}}{1-a^{-1}z^{-1}}$

(4) $h_4(a,z) = \bar{g}_1(z) = \dfrac{1}{1-az^{-1}}$

(4') $h'_4(a,z) = \bar{g}_3(z) = \dfrac{a^{-1}}{1-a^{-1}z^{-1}}$

(5) $h_5(a,z) = \bar{f}(z) = \dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$

(5') $h'_5(a,z) = \bar{f}_2(z) = \dfrac{z^{-1}}{1-a^{-1}z^{-1}}$

(6) $h_6(a,z) = \bar{g}(z) = \dfrac{a z^{-1}}{1-a z^{-1}}$

(6') $h'_6(a,z) = \bar{g}_2(z) = \dfrac{z^{-1}}{1-az^{-1}}$

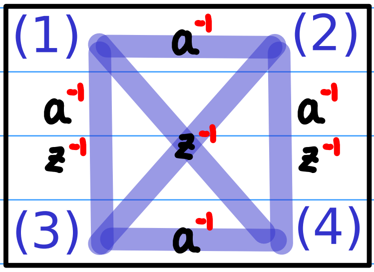
(7) $h_7(a,z) = f_1(z) = \dfrac{a z}{1-a z}$

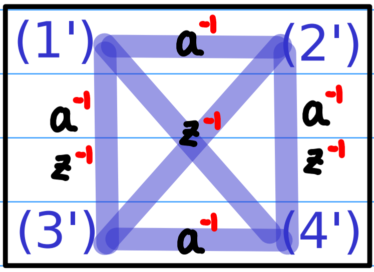
(7') $h'_7(a,z) = f_3(z) = \dfrac{z}{1-az}$

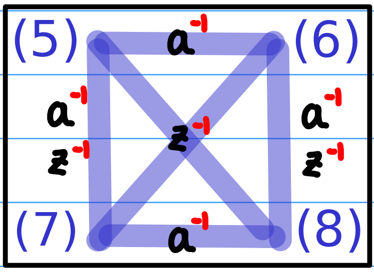
(8) $h_8(a,z) = g_1(z) = \dfrac{a^{-1}z}{1-a^{-1}z}$

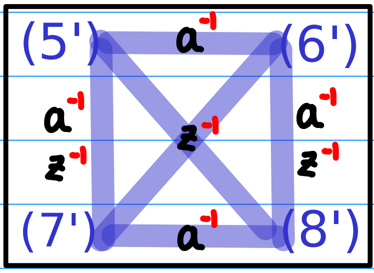
(8') $h'_8(a,z) = g_3(z) = \dfrac{z}{1-a^{-1}z}$

Substitute with a Multiplicative Inverse



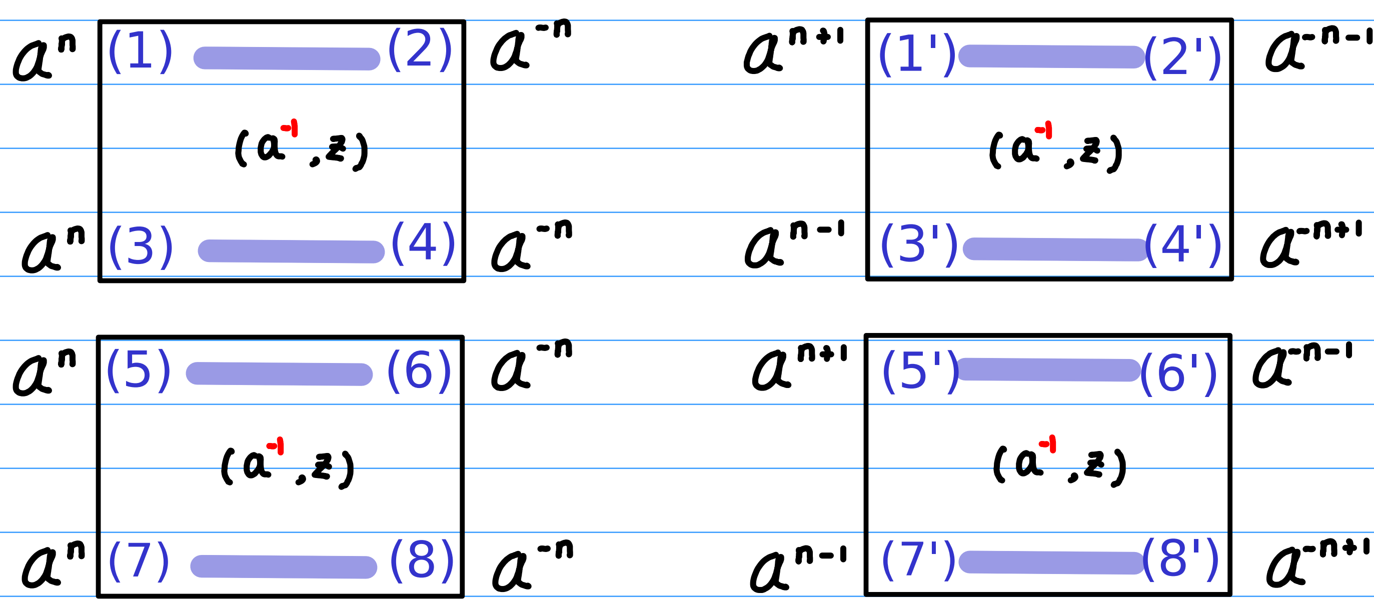






Substitute **a** with its Multiplicative Inverse

$h_1(a^{-1},z) = h_2(a,z)$	$h_1'(a^{-1},z) = h_2'(a,z)$
$h_2(a^{-1},z) = h_1(a,z)$	$h_2'(a^{-1},z) = h_1'(a,z)$
$h_3(a^{-1},z) = h_4(a,z)$	$h_3'(a^{-1},z) = h_4'(a,z)$
$h_4(a^{-1},z) = h_3(a,z)$	$h_4'(a^{-1},z) = h_3'(a,z)$
$h_5(a^{-1},z) = h_6(a,z)$	$h_5'(a^{-1},z) = h_6'(a,z)$
$h_6(a^{-1},z) = h_5(a,z)$	$h_6'(a^{-1},z) = h_5'(a,z)$
$h_7(a^{-1},z) = h_8(a,z)$	$h_7'(a^{-1},z) = h_8'(a,z)$
$h_8(a^{-1},z) = h_7(a,z)$	$h_8'(a^{-1},z) = h_7'(a,z)$



Substitute z with its Multiplicative Inverse

$h_1(a, z^{-1}) = h_4(a, z)$ $\quad$ $h'_1(a, z^{-1}) = h'_4(a, z)$

$h_2(a, z^{-1}) = h_3(a, z)$ $\quad$ $h'_2(a, z^{-1}) = h'_3(a, z)$

$h_3(a, z^{-1}) = h_2(a, z)$ $\quad$ $h'_3(a, z^{-1}) = h'_2(a, z)$

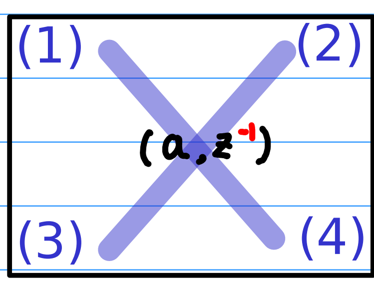
$h_4(a, z^{-1}) = h_1(a, z)$ $\quad$ $h'_4(a, z^{-1}) = h'_1(a, z)$

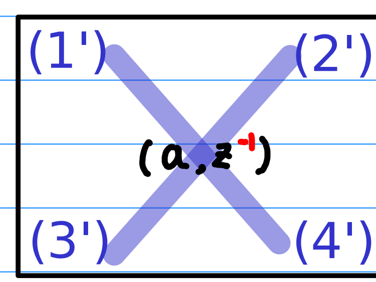
$h_5(a, z^{-1}) = h_8(a, z)$ $\quad$ $h'_5(a, z^{-1}) = h'_8(a, z)$

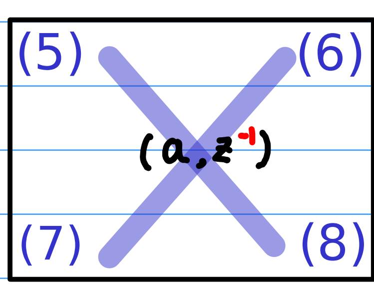
$h_6(a, z^{-1}) = h_7(a, z)$ $\quad$ $h'_6(a, z^{-1}) = h'_7(a, z)$

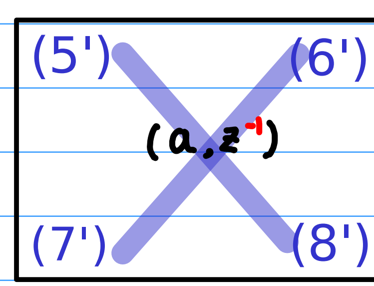
$h_7(a, z^{-1}) = h_6(a, z)$ $\quad$ $h'_7(a, z^{-1}) = h'_6(a, z)$

$h_8(a, z^{-1}) = h_5(a, z)$ $\quad$ $h'_8(a, z^{-1}) = h'_5(a, z)$









Substitute **a** with its Multiplicative Inverse

Substitute **z** with its Multiplicative Inverse

$$h_1(a^{-1}, z^{-1}) = h_3(a, z) \qquad h'_1(a^{-1}, z^{-1}) = h'_3(a, z)$$

$$h_2(a^{-1}, z^{-1}) = h_4(a, z) \qquad h'_2(a^{-1}, z^{-1}) = h'_4(a, z)$$

$$h_3(a^{-1}, z^{-1}) = h_1(a, z) \qquad h'_3(a^{-1}, z^{-1}) = h'_1(a, z)$$

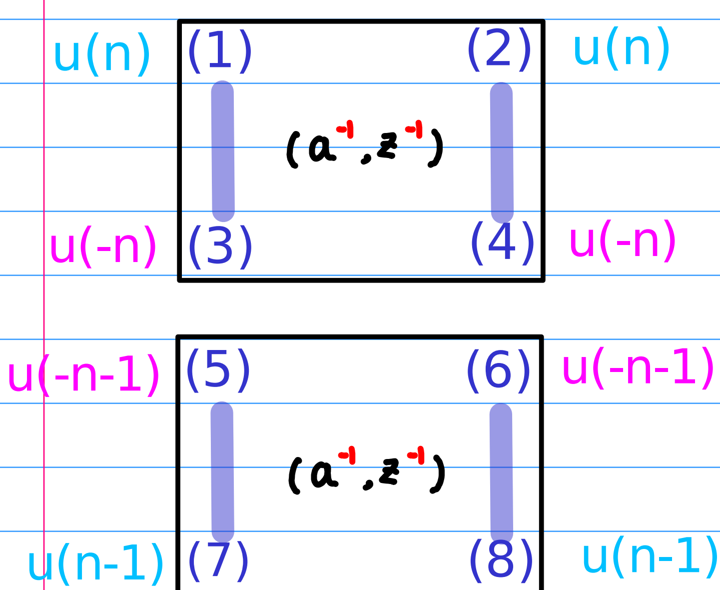
$$h_4(a^{-1}, z^{-1}) = h_2(a, z) \qquad h'_4(a^{-1}, z^{-1}) = h'_2(a, z)$$

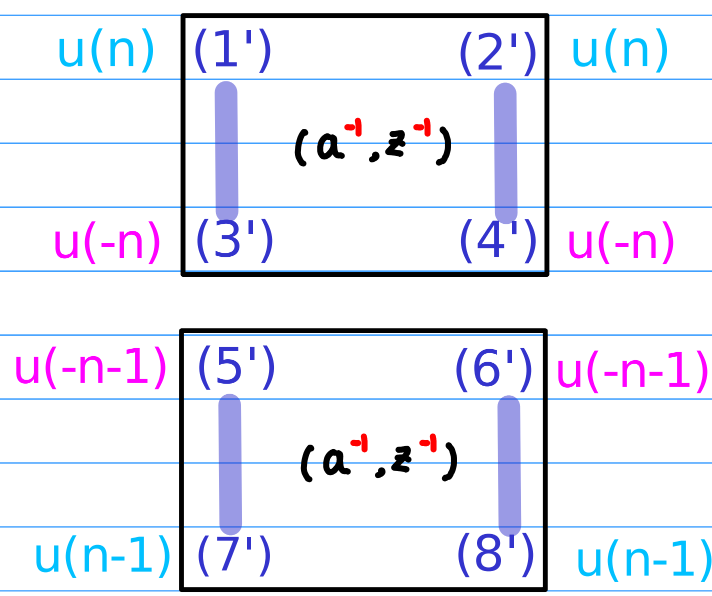
$$h_5(a^{-1}, z^{-1}) = h_7(a, z) \qquad h'_5(a^{-1}, z^{-1}) = h'_7(a, z)$$

$$h_6(a^{-1}, z^{-1}) = h_8(a, z) \qquad h'_6(a^{-1}, z^{-1}) = h'_8(a, z)$$

$$h_7(a^{-1}, z^{-1}) = h_5(a, z) \qquad h'_7(a^{-1}, z^{-1}) = h'_5(a, z)$$

$$h_8(a^{-1}, z^{-1}) = h_6(a, z) \qquad h'_8(a^{-1}, z^{-1}) = h'_6(a, z)$$





$$f(z) \;*\; a \;(\leftarrow) = f_2(z) \qquad g(z) \;/\; a \;(\leftarrow) = g_2(z)$$

$$f_1(z) \;/\; a \;(\rightarrow) = f_3(z) \qquad g_1(z) \;*\; a \;(\rightarrow) = g_3(z)$$

$$\bar{f}(z) \;*\; a \;(\leftarrow) = \bar{f}_2(z) \qquad \bar{g}(z) \;/\; a \;(\leftarrow) = \bar{g}_2(z)$$

$$\bar{f}_1(z) \;/\; a \;(\rightarrow) = \bar{f}_3(z) \qquad \bar{g}_1(z) \;*\; a \;(\rightarrow) = \bar{g}_3(z)$$

$$f(z) \;*\; z \;(\Rightarrow) = f_3(z) \qquad g(z) \;*\; z \;(\Rightarrow) = g_3(z)$$

$$f_1(z) \;/\; z \;(\Leftarrow) = f_2(z) \qquad g_1(z) \;/\; z \;(\Leftarrow) = g_2(z)$$

$$\bar{f}(z) \;*\; z \;(\Rightarrow) = \bar{f}_3(z) \qquad \bar{g}(z) \;*\; z \;(\Rightarrow) = \bar{g}_3(z)$$

$$\bar{f}_1(z) \;/\; z \;(\Leftarrow) = \bar{f}_2(z) \qquad \bar{g}_1(z) \;/\; z \;(\Leftarrow) = \bar{g}_2(z)$$

(1) $f(z)$ (1') $f_2(z)$ (5) $\bar{f}(z)$ (5') $\bar{f}_2(z)$

(7) $f_1(z)$ (7') $f_3(z)$ (3) $\bar{f}_1(z)$ (3') $\bar{f}_3(z)$

(2) $g(z)$ (2') $g_2(z)$ (6) $\bar{g}(z)$ (6') $\bar{g}_2(z)$

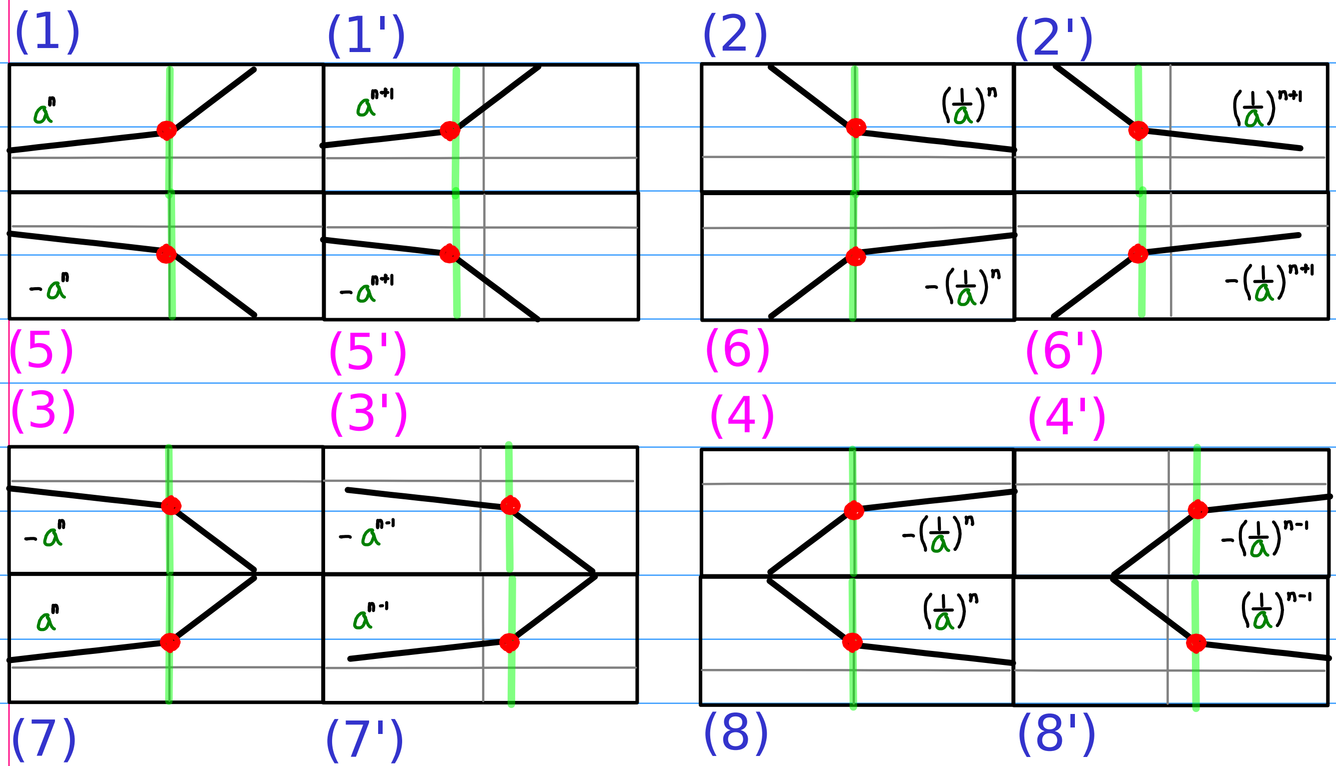
(8) $g_1(z)$ (8') $g_3(z)$ (4) $\bar{g}_1(z)$ (4') $\bar{g}_3(z)$

(1) $f(z)$	(2) $g(z)$	u(n)
(5) $\bar{f}(z)$	(6) $\bar{g}(z)$	u(-n-1)
a^n	a^{-n}	

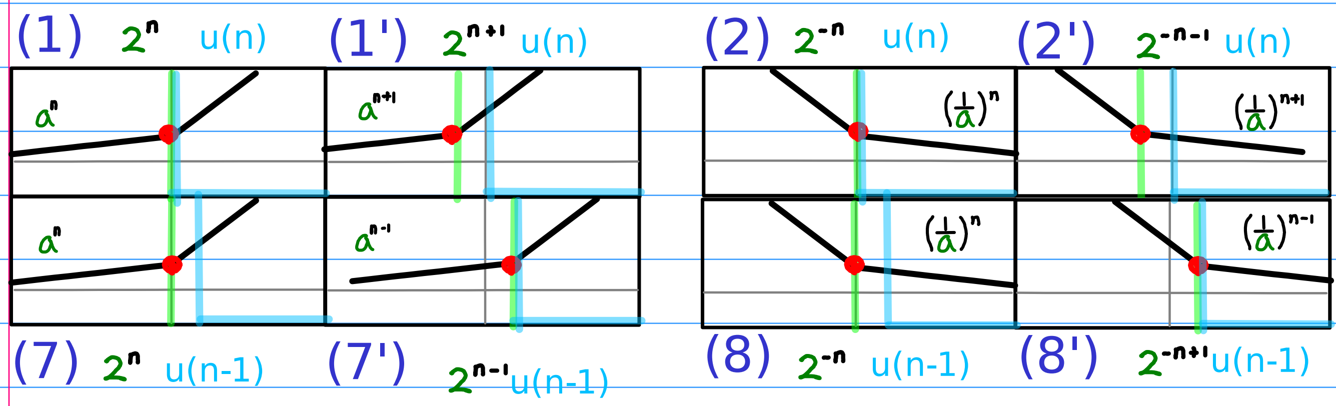
(7) $f_1(z)$	(8) $g_1(z)$	u(n-1)
(3) $\bar{f}_1(z)$	(4) $\bar{g}_1(z)$	u(-n)
a^n	a^{-n}	

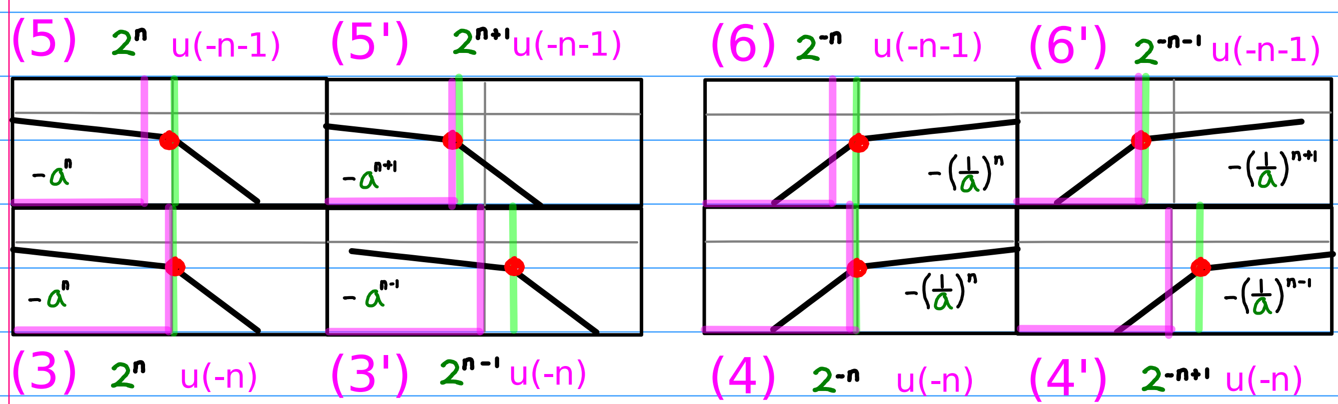
(1') $f_2(z)$	(2') $g_2(z)$	u(n)
(5') $\bar{f}_2(z)$	(6') $\bar{g}_2(z)$	u(-n-1)
a^{n+1}	a^{-n-1}	

(7') $f_3(z)$	(8') $g_3(z)$	u(n-1)
(3') $\bar{f}_3(z)$	(4') $\bar{g}_3(z)$	u(-n)
a^{n-1}	a^{-n+1}	



(1)
(1')
(2)
(2')
a^n
a^{n+1}
$(\frac{1}{a})^n$
$(\frac{1}{a})^{n+1}$
$-a^n$
$-a^{n+1}$
$-(\frac{1}{a})^n$
$-(\frac{1}{a})^{n+1}$
(5)
(5')
(6)
(6')
(3)
(3')
(4)
(4')
$-a^n$
$-a^{n-1}$
$-(\frac{1}{a})^n$
$-(\frac{1}{a})^{n-1}$
a^n
a^{n-1}
$(\frac{1}{a})^n$
$(\frac{1}{a})^{n-1}$
(7)
(7')
(8)
(8')


(1) 2^n u(n)
(1') 2^{n+1} u(n)
(2) 2^{-n} u(n)
(2') 2^{-n-1} u(n)
a^n
a^{n+1}
$(\frac{1}{a})^n$
$(\frac{1}{a})^{n+1}$
a^n
a^{n-1}
$(\frac{1}{a})^n$
$(\frac{1}{a})^{n-1}$
(7) 2^n u(n-1)
(7') 2^{n-1}u(n-1)
(8) 2^{-n} u(n-1)
(8') 2^{-n+1}u(n-1)

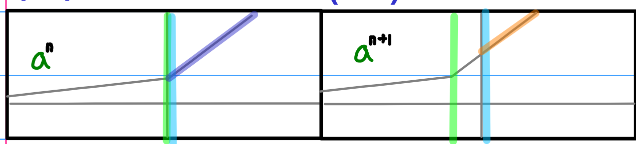

(5) 2^n u(-n-1)
(5') 2^{n+1}u(-n-1)
(6) 2^{-n} u(-n-1)
(6') 2^{-n-1} u(-n-1)
$-a^n$
$-a^{n+1}$
$-(\frac{1}{a})^n$
$-(\frac{1}{a})^{n+1}$
$-a^n$
$-a^{n-1}$
$-(\frac{1}{a})^n$
$-(\frac{1}{a})^{n-1}$
(3) 2^n u(-n)
(3') 2^{n-1} u(-n)
(4) 2^{-n} u(-n)
(4') 2^{-n+1} u(-n)

Exp: n → n+1 (SL) Rng: id (ID)

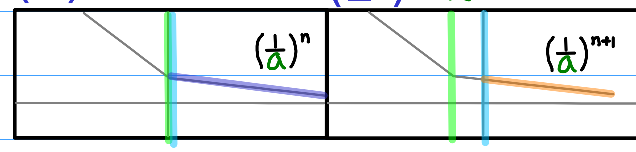
Exp: n → n-1 (SR) Rng: id (ID)

(1) 2^n u(n) (1') 2^{n+1} u(n)

(2) 2^{-n} u(n) (2') 2^{-n-1} u(n)



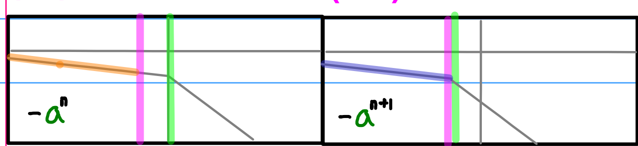
(SL, ID)



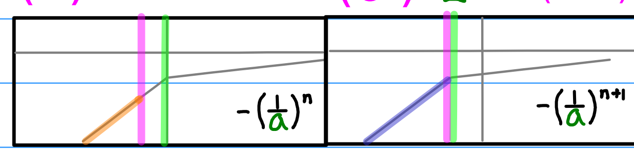
(SL, ID)

(5) 2^n u(-n-1) (5') 2^{n+1} u(-n-1)

(6) 2^{-n} u(-n-1) (6') 2^{-n-1} u(-n-1)

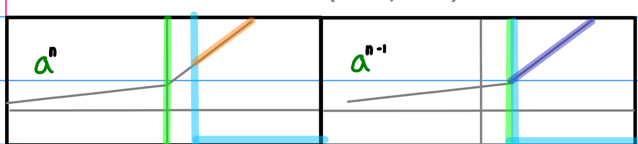


(SL, ID)



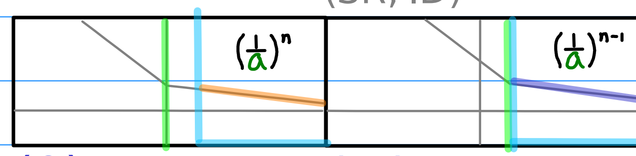
(SL, ID)

(SR, ID)



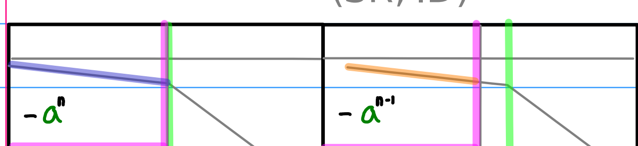
(7) 2^n u(n-1) (7') 2^{n-1} u(n-1)

(SR, ID)



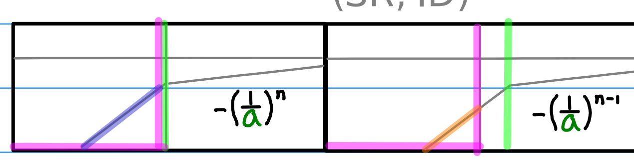
(8) 2^{-n} u(n-1) (8') 2^{-n+1} u(n-1)

(SR, ID)



(3) 2^n u(-n) (3') 2^{n-1} u(-n)

(SR, ID)



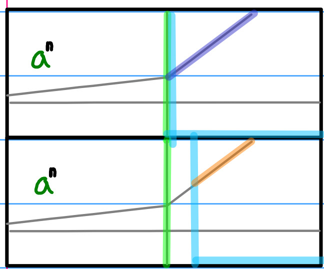
(4) 2^{-n} u(-n) (4') 2^{-n+1} u(-n)

Exp: id (ID) Rng: n → n-1 (SR)

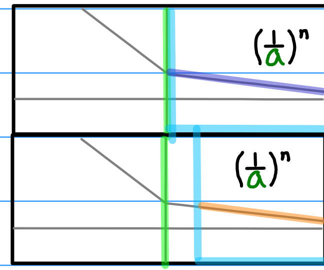
Exp: n → n-2 (SR2) Rng: n → n-1 (SR)

(1) 2^{n} u(n)

(2) 2^{-n} u(n)



(ID, SR)



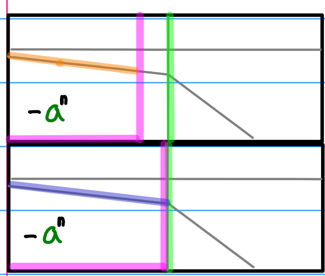
(ID, SR)

(7) 2^{n} u(n-1)

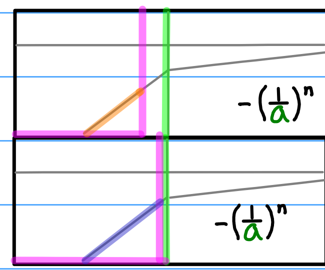
(8) 2^{-n} u(n-1)

(5) 2^{n} u(-n-1)

(6) 2^{-n} u(-n-1)



(ID, SR)



(ID, SR)

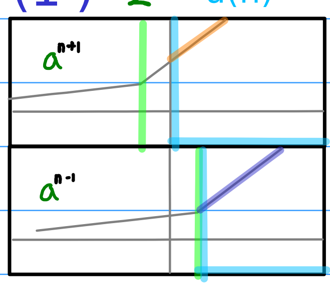
(3) 2^{n} u(-n)

(4) 2^{-n} u(-n)

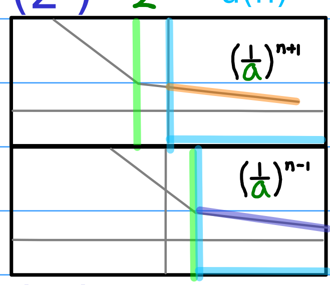
(1') 2^{n+1} u(n)

(2') 2^{-n-1} u(n)

(SR2, SR)



(SR2, SR)



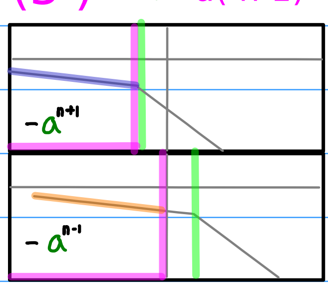
(7') 2^{n-1} u(n-1)

(8') 2^{-n+1} u(n-1)

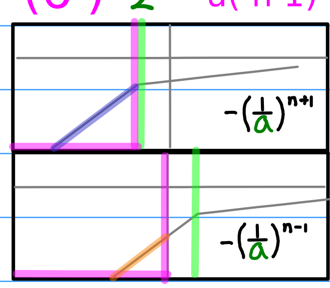
(5') 2^{n+1} u(-n-1)

(6') 2^{-n-1} u(-n-1)

(SR2, SR)



(SR2, SR)



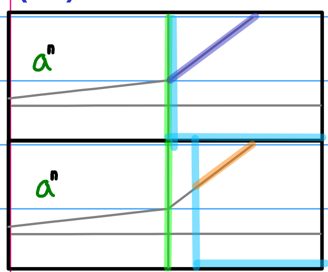
(3') 2^{n-1} u(-n)

(4') 2^{-n+1} u(-n)

Exp: id . (ID) Rng: $n \to n-1$ (SR)

Exp: $n \to n-2$ (SR2) Rng: $n \to n-1$ (SR)

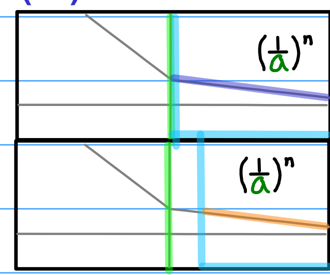
(1) 2^n $u(n)$



(ID, SR)

(7) 2^n $u(n-1)$

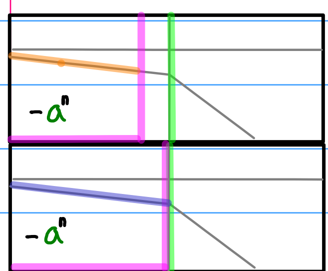
(2) 2^{-n} $u(n)$



(ID, SR)

(8) 2^{-n} $u(n-1)$

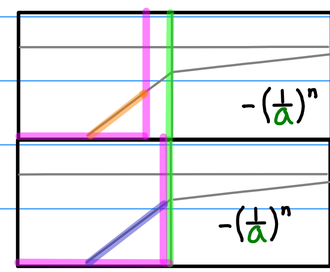
(5) 2^n $u(-n-1)$



(ID, SR)

(3) 2^n $u(-n)$

(6) 2^{-n} $u(-n-1)$

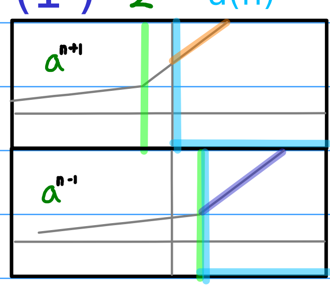


(ID, SR)

(4) 2^{-n} $u(-n)$

(1') 2^{n+1} $u(n)$

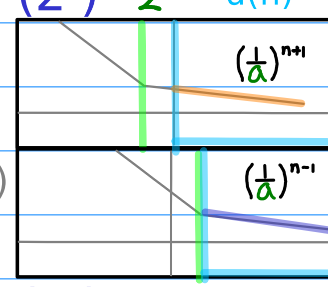
(SR2, SR)



(7') 2^{n-1} $u(n-1)$

(2') 2^{-n-1} $u(n)$

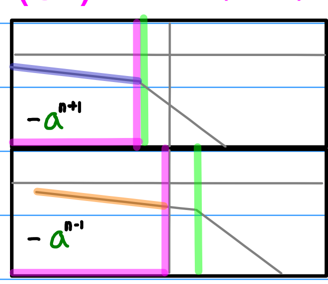
(SR2, SR)



(8') 2^{-n+1} $u(n-1)$

(5') 2^{n+1} $u(-n-1)$

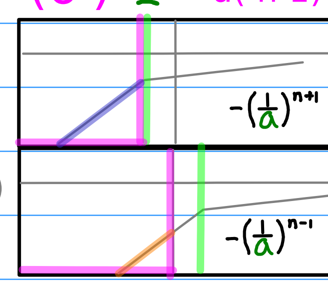
(SR2, SR)



(3') 2^{n-1} $u(-n)$

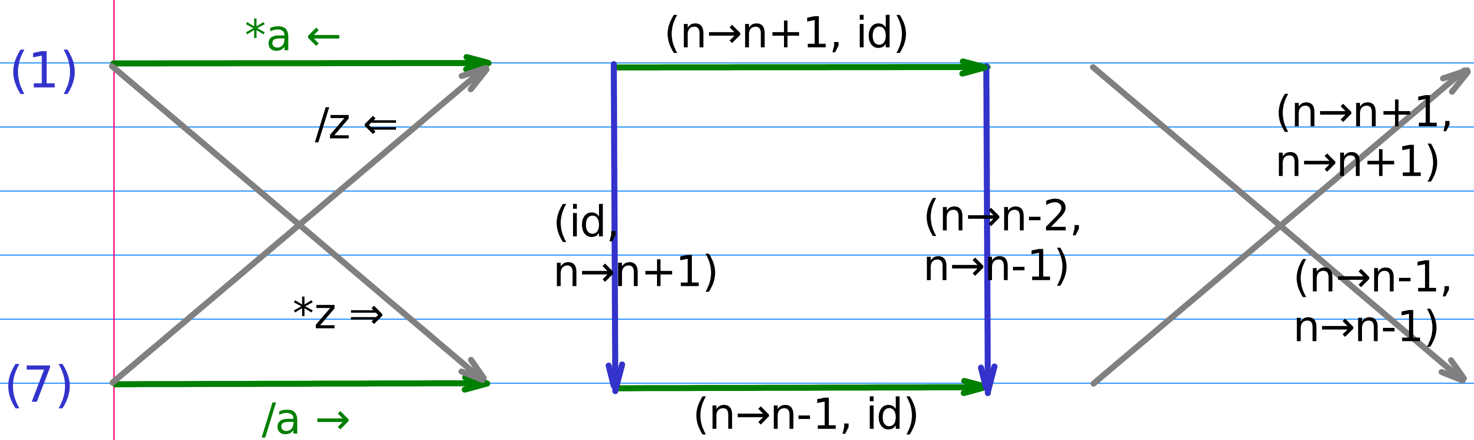
(6') 2^{-n-1} $u(-n-1)$

(SR2, SR)

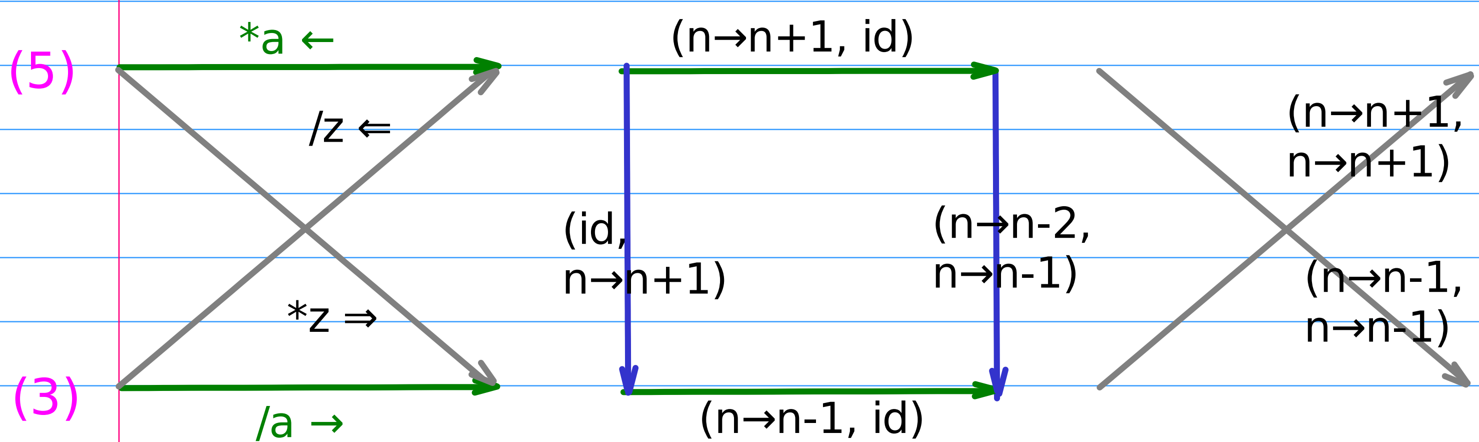


(4') 2^{-n+1} $u(-n)$

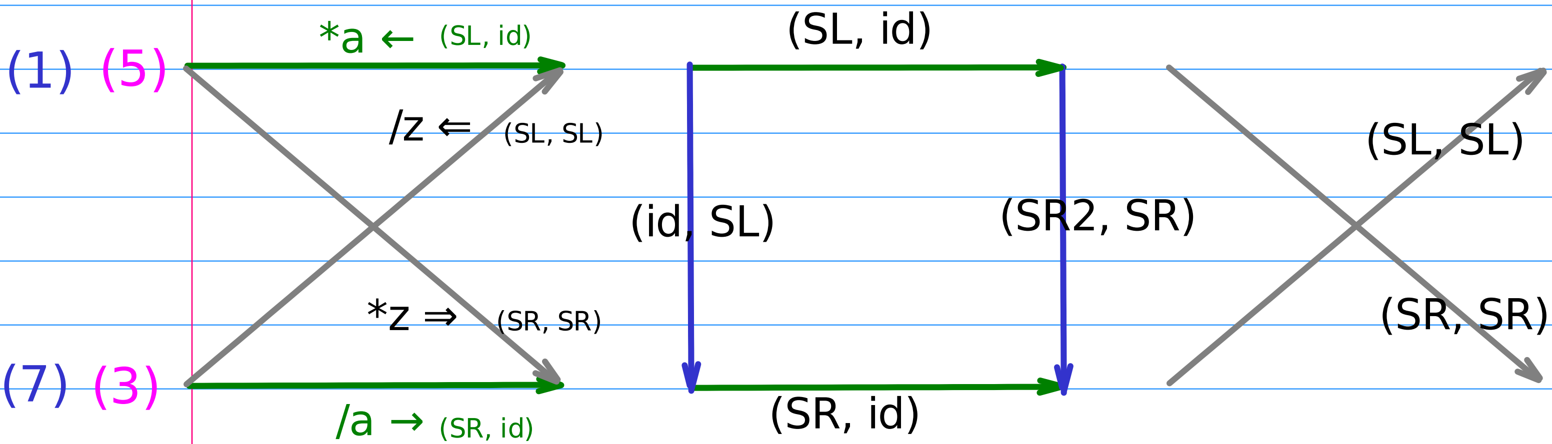
(Exp, Range)



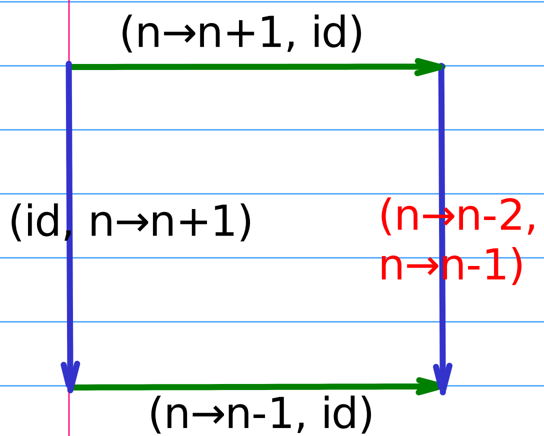
(1)
(7)
*a ←
/z ⇐
*z ⇒
/a →
(n→n+1, id)
(id, n→n+1)
(n→n-2, n→n-1)
(n→n-1, id)
(n→n+1, n→n+1)
(n→n-1, n→n-1)

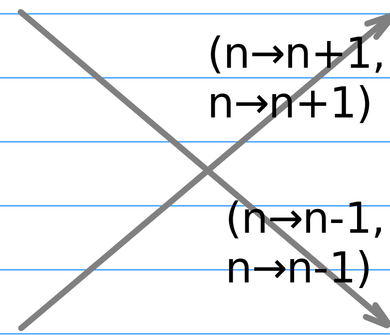


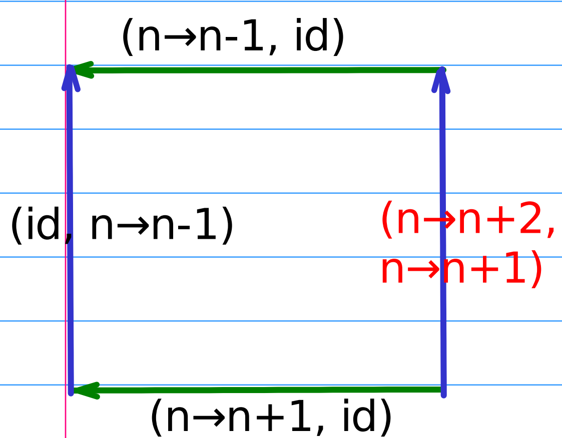
(5)
(3)
*a ←
/z ⇐
*z ⇒
/a →
(n→n+1, id)
(id, n→n+1)
(n→n-2, n→n-1)
(n→n-1, id)
(n→n+1, n→n+1)
(n→n-1, n→n-1)

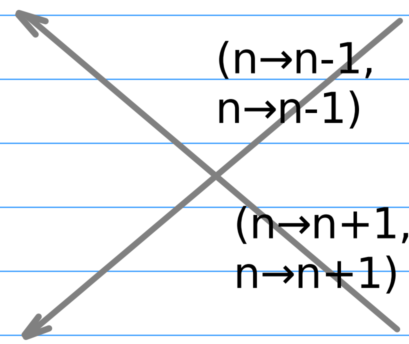


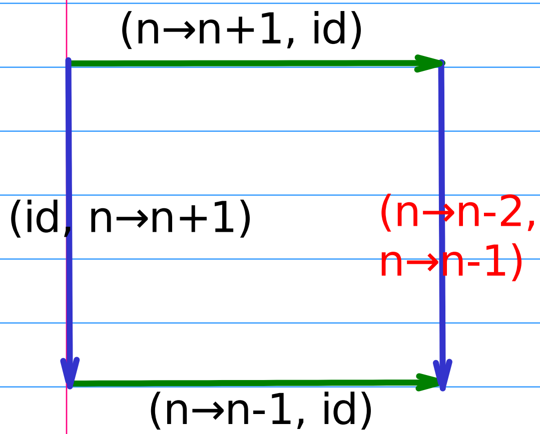
(1) (5)
(7) (3)
*a ← (SL, id)
/z ⇐ (SL, SL)
*z ⇒ (SR, SR)
/a → (SR, id)
(SL, id)
(id, SL)
(SR2, SR)
(SR, id)
(SL, SL)
(SR, SR)

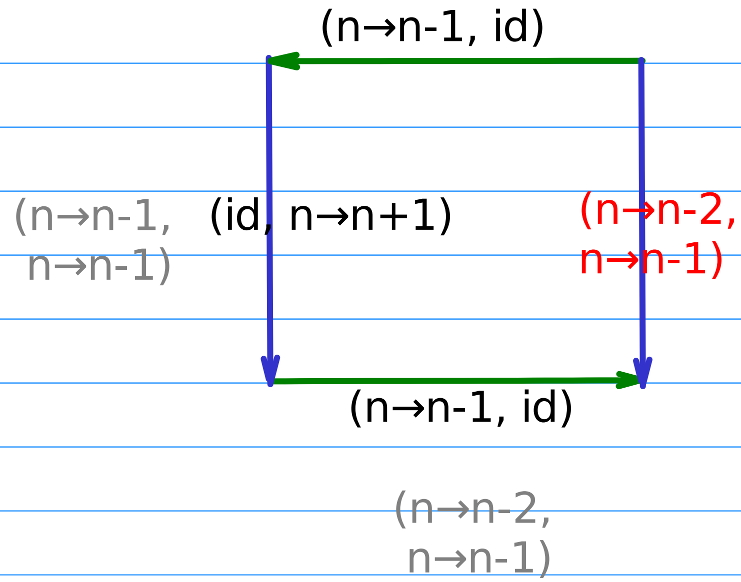

(n→n+1, id)
(id, n→n+1)
(n→n-2,
n→n-1)
(n→n-1, id)

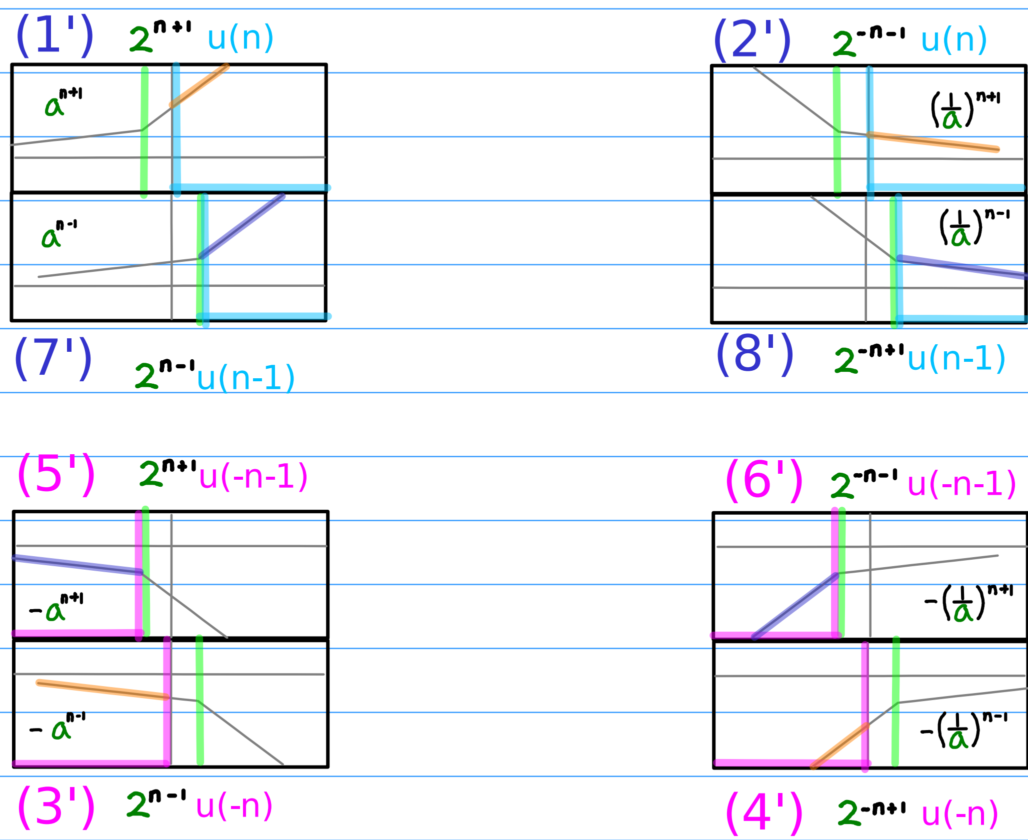

(n→n+1,
n→n+1)
(n→n-1,
n→n-1)


(n→n-1, id)
(id, n→n-1)
(n→n+2,
n→n+1)
(n→n+1, id)

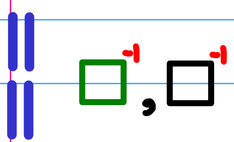

(n→n-1,
n→n-1)
(n→n+1,
n→n+1)


(n→n+1, id)
(id, n→n+1)
(n→n-2, n→n-1)
(n→n-1, id)


(n→n-1, id)
(n→n-1, n→n-1)
(id, n→n+1)
(n→n-2, n→n-1)
(n→n-1, id)
(n→n-2, n→n-1)


(1') 2^n+1 u(n)
a^n+1
a^n-1
(7') 2^n-1 u(n-1)
(2') 2^-n-1 u(n)
(1/a)^n+1
(1/a)^n-1
(8') 2^-n+1 u(n-1)
(5') 2^n+1 u(-n-1)
-a^n+1
- a^n-1
(3') 2^n-1 u(-n)
(6') 2^-n-1 u(-n-1)
-(1/a)^n+1
-(1/a)^n-1
(4') 2^-n+1 u(-n)

(ID, SM)



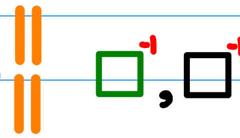
symmetric range

Inv(a, z)

Symm(Rng)

a ← 1 / a,
z ← 1 / z;

(ID, SM)



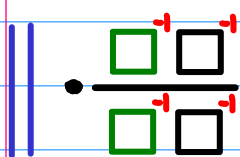
symmetric range

Inv(a, z)

Symm(Rng)

a ← 1 / a,
z ← 1 / z;

(ID, CP)



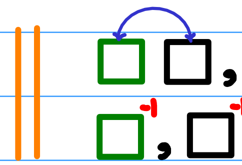
complementary range

dual(CR)

Comp(Rng)

numer ← numer / CR
denom ← denom / CR

(ID, CP)



complementary range

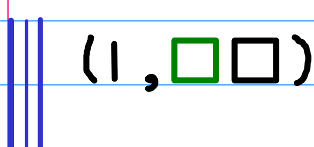
nSwap(a, z), Inv(a, z)

Comp(Rng)

a ↔ z;

a ← 1 / a,
z ← 1 / z;

(ID, SR)



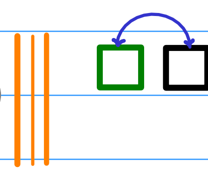
shifted range

nSel(1, CR)

Shift(Rng)

numer ← 1 or CR

(ID, SR)



shifted range

nSwap(a, z)

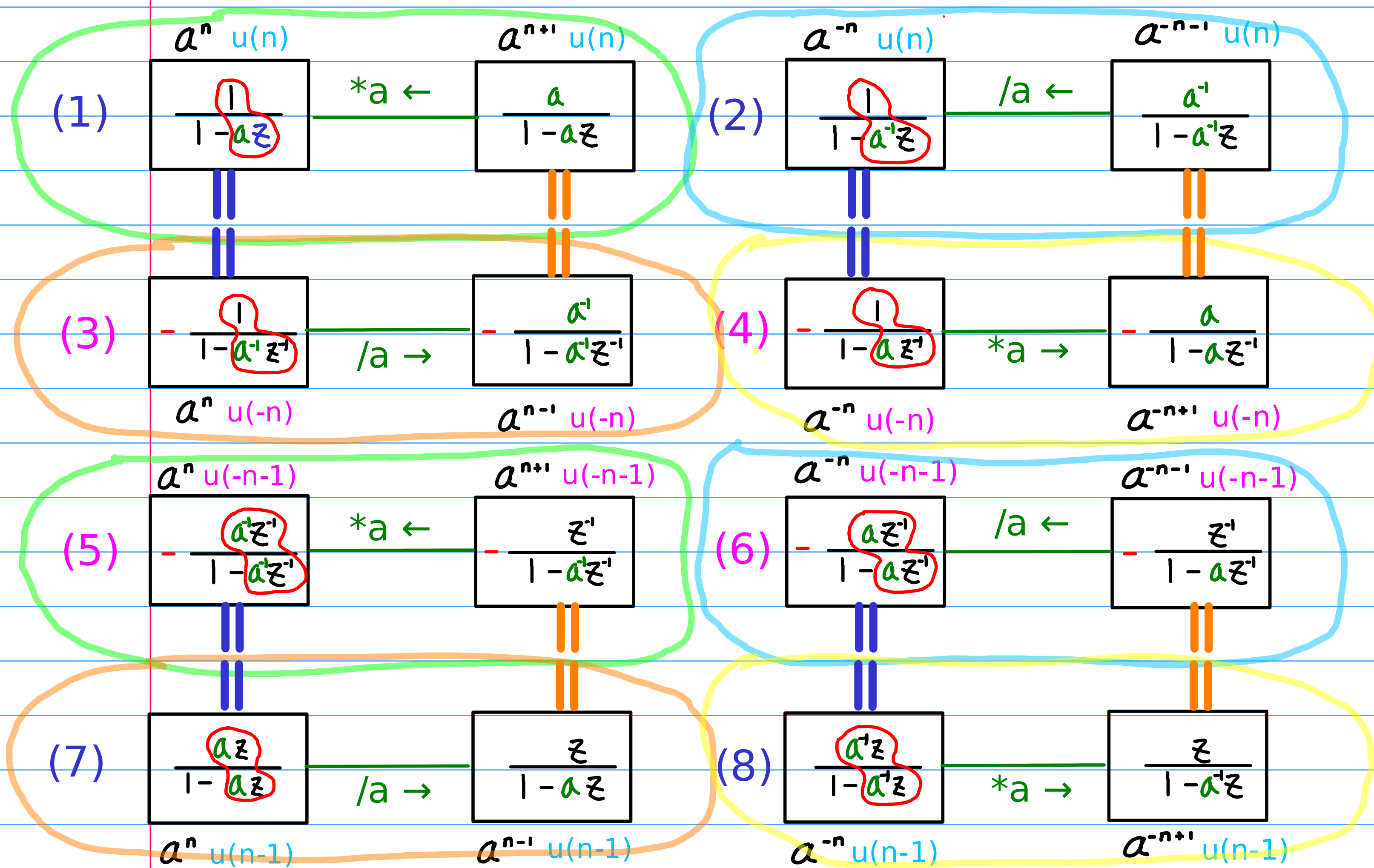
Shift(Rng)

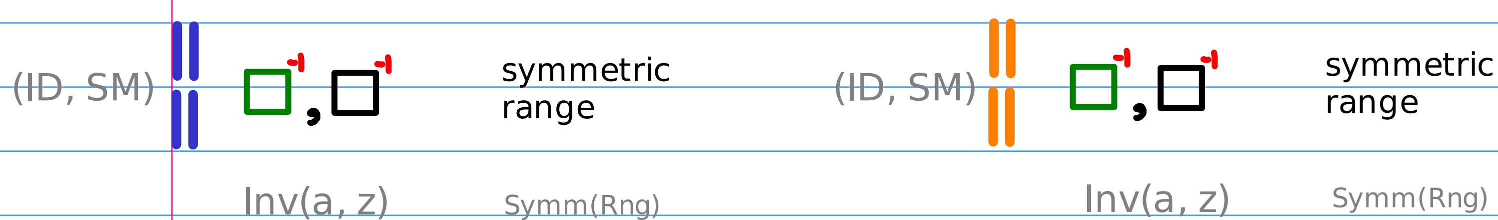
a ↔ z;

u(n) & u(-n)

u(n-1) & u(-n-1)

Symmetric Region

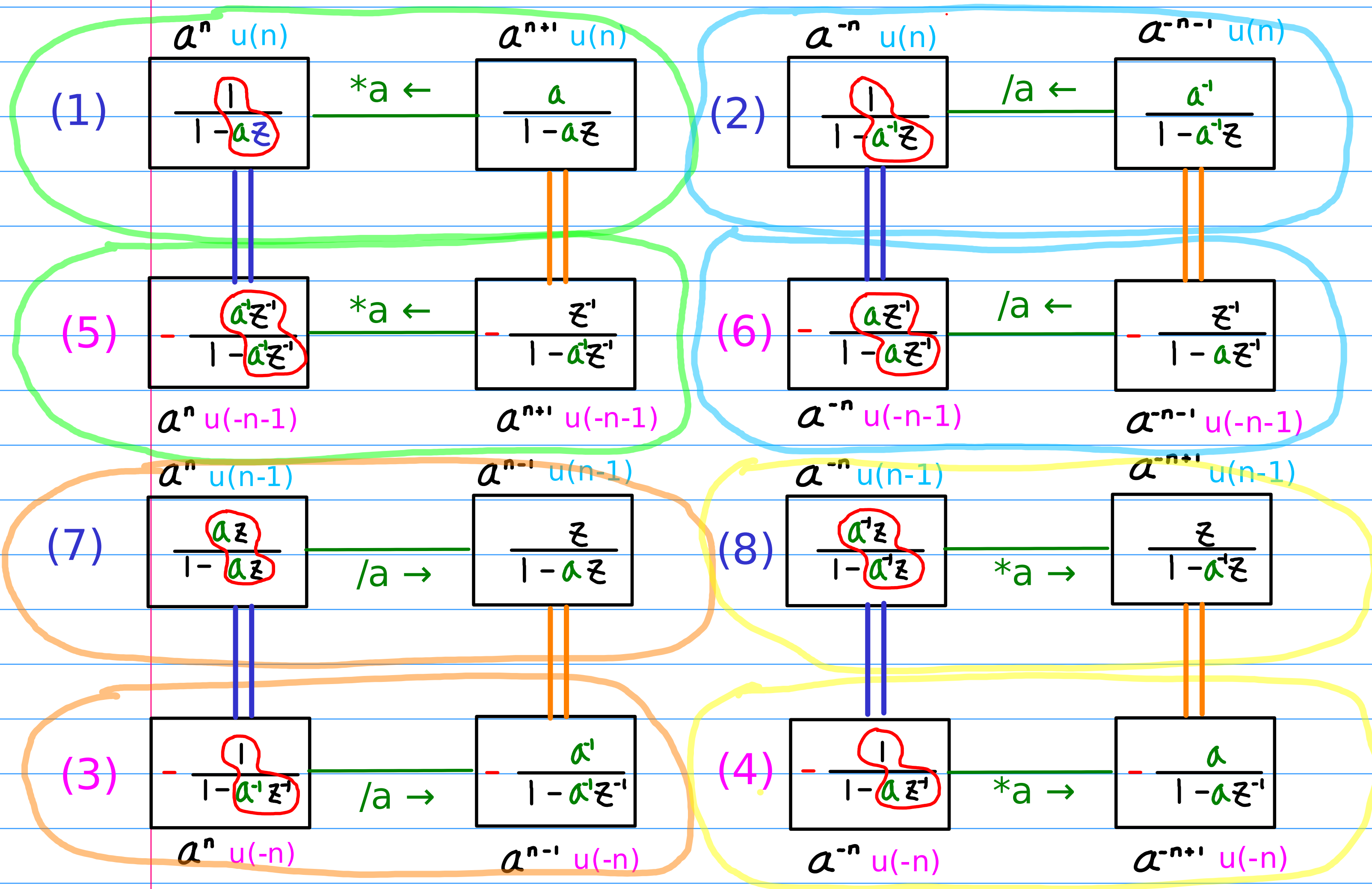




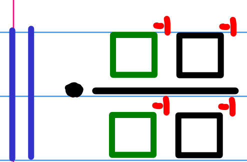
u(n) & u(-n-1)

u(n-1) & u(-n)

Complementary Region



(ID, CP)

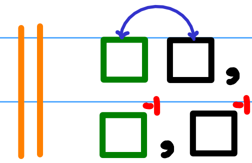


complementary range

dual(CR)

Comp(Rng)

(ID, CP)



complementary range

nSwap(a, z), Inv(a, z)

Comp(Rng)

nSwap(a, z) numerator swap: a ↔ z for a common ration az

inv(a, z) inverse(a) and inverse(b)

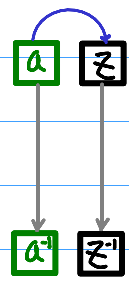
$$\frac{a}{1-az}$$

nSwap(a, z)

$$\frac{z}{1-az}$$

inv(a, z)

$$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$$



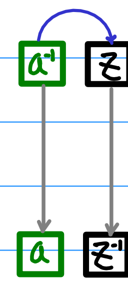
$$\frac{a^{-1}}{1-a^{-1}z}$$

nSwap(a, z)

$$\frac{z}{1-a^{-1}z}$$

inv(a, z)

$$\frac{z^{-1}}{1-az^{-1}}$$



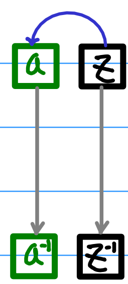
$$\frac{z}{1-az}$$

nSwap(a, z)

$$\frac{a}{1-az}$$

inv(a, z)

$$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$$



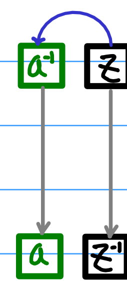
$$\frac{z}{1-a^{-1}z}$$

nSwap(a, z)

$$\frac{a^{-1}}{1-a^{-1}z}$$

inv(a, z)

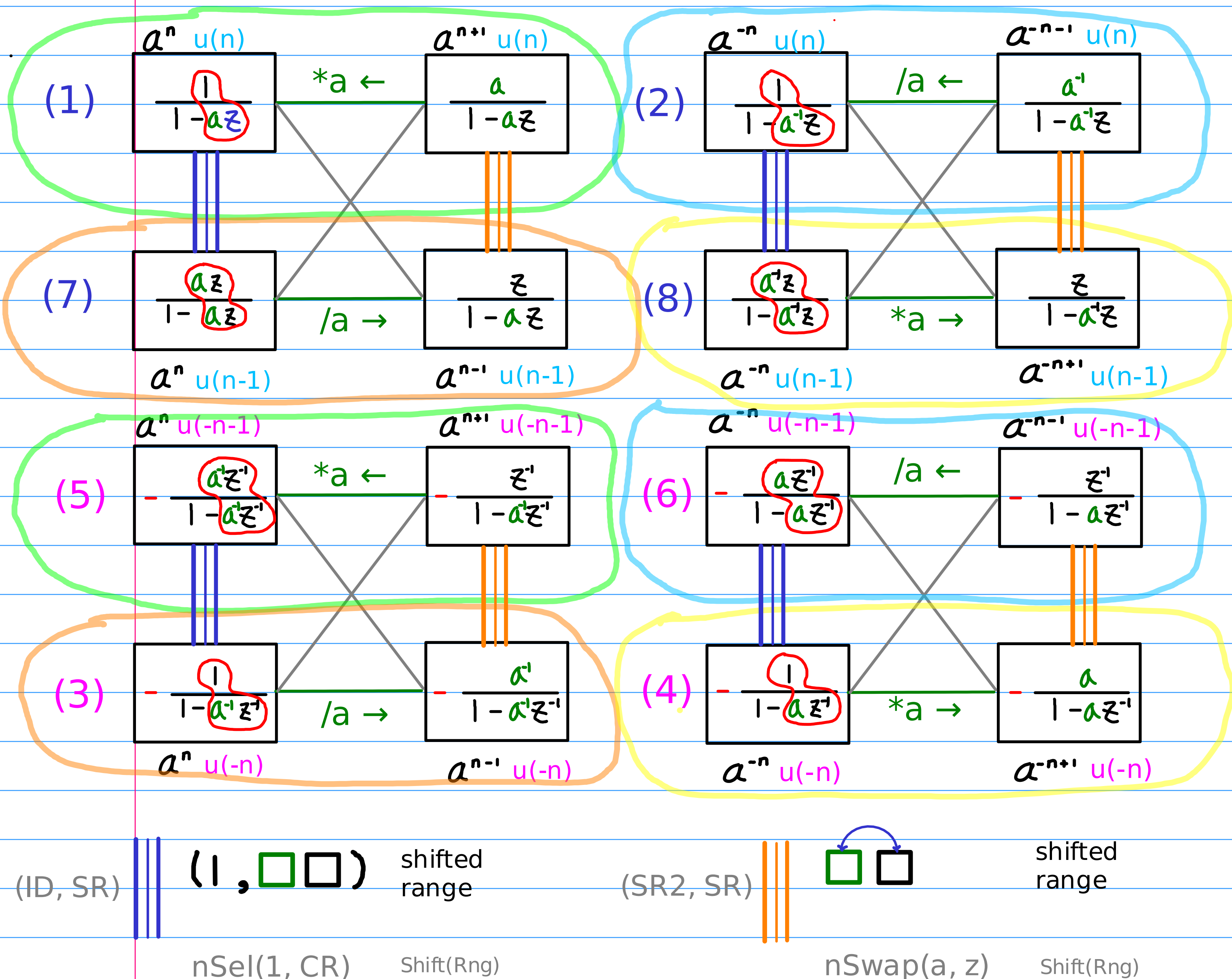
$$-\frac{a}{1-az^{-1}}$$



u(n) & u(n-1)

u(-n-1) & u(-n)

Right Shifted Region



nSwap(a, z) numerator swap: a ↔ z for a common ration az

$$\frac{a}{1-az}$$

nSwap(a, z) az $\boxed{a} \leftrightarrow \boxed{z}$

$$\frac{z}{1-az}$$

$$\frac{a^{-1}}{1-a^{-1}z}$$

nSwap(a, z) $a^{-1}z$ $\boxed{a^{-1}} \leftrightarrow \boxed{z}$

$$\frac{z}{1-a^{-1}z}$$

$$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$$

nSwap(a, z) $a^{-1}z^{-1}$ $\boxed{a^{-1}} \leftrightarrow \boxed{z^{-1}}$

$$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$$

$$\frac{z^{-1}}{1-az^{-1}}$$

nSwap(a, z) az^{-1} $\boxed{a} \leftrightarrow \boxed{z^{-1}}$

$$\frac{a}{1-az^{-1}}$$