

Vector Calculus (H.1)

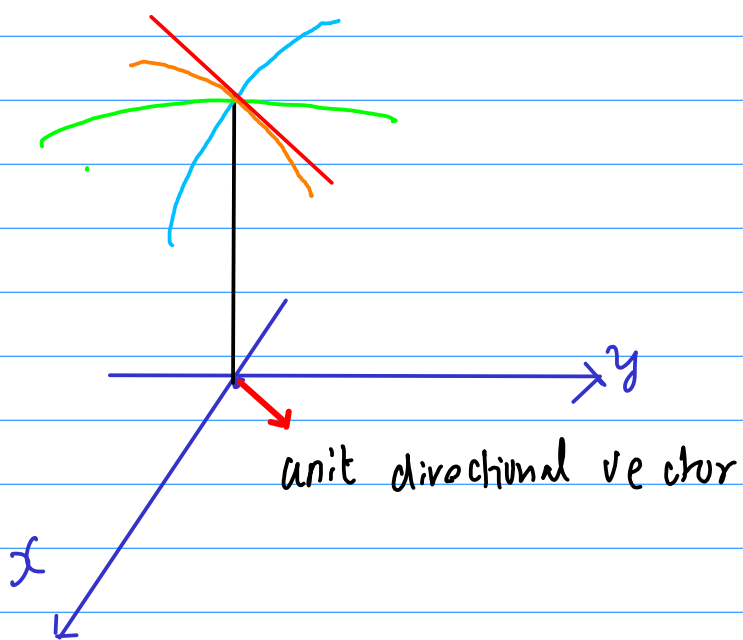
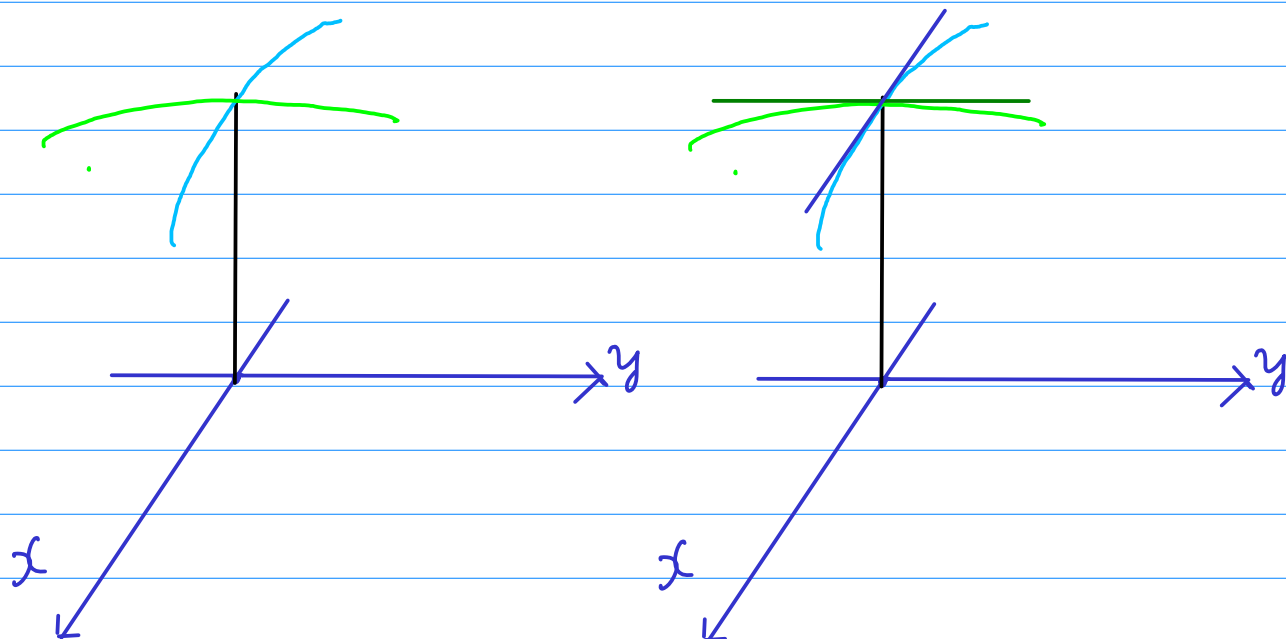
Partial Derivatives

20160109

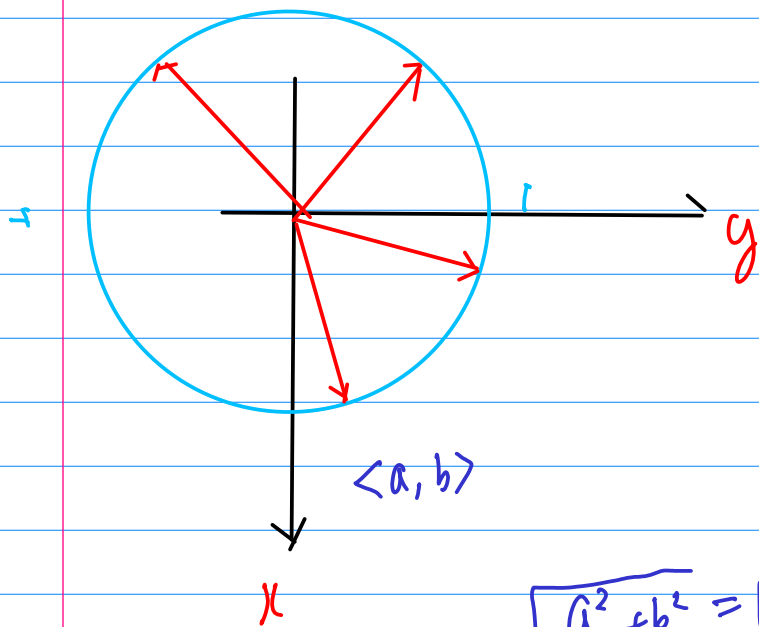
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Directional Derivatives



Unit directional vector



$$\vec{u} = \langle a, b \rangle$$



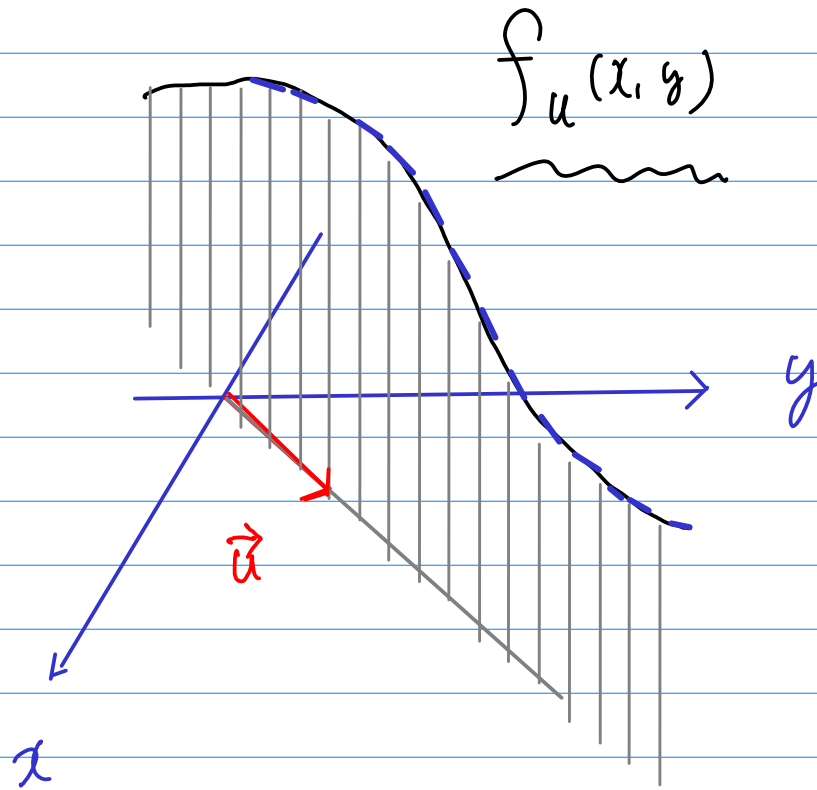
unit directional vector

magnitude = length = 1

$$\sqrt{a^2 + b^2} = 1$$

$$a^2 + b^2 = 1$$

directional derivatives



$$\vec{u} = \langle a, b \rangle$$

$$D_{\vec{u}} f(x, y) = \lim_{h \rightarrow 0} \frac{f(x+ah, y+bh) - f(x, y)}{h}$$

$$D_{\vec{u}} f(x, y) = a D_x f(x, y) + b D_y f(x, y)$$

$$= a \boxed{f_x(x, y)} + b \boxed{f_y(x, y)}$$

$$\langle \boxed{f_x(x, y)}, \boxed{f_y(x, y)} \rangle \cdot \underbrace{\langle a, b \rangle}_{\text{vector}}$$

$f(x, y)$
scalar
field

↑
partial
derivative

$$\frac{\partial}{\partial x} f(x, y)$$

↑
partial
derivative

$$\frac{\partial}{\partial y} f(x, y)$$

$$\vec{u} = \langle a, b \rangle$$

$$\begin{aligned} D_{\vec{u}} f(x, y) &= a \boxed{f_x(x, y)} + b \boxed{f_y(x, y)} \\ &= \langle f_x, f_y \rangle \cdot \langle a, b \rangle \end{aligned}$$

$$\vec{u} = \langle a, b, c \rangle$$

$$\begin{aligned} D_{\vec{u}} f(x, y, z) &= a \cdot f_x(x, y, z) + b \cdot f_y(x, y, z) + c \cdot f_z(x, y, z) \\ &= \langle f_x, f_y, f_z \rangle \cdot \langle a, b, c \rangle \end{aligned}$$

$f \rightarrow 3$ variables x, y, z

$$f(x, y, z)$$

consider this
as a vector.

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

gradient (vector) of f

∇

$$\nabla f \triangleq \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

x 방향의 성분
 y 방향의 성분
 z 방향의 성분

Operator

$$\nabla \triangleq \left\{ \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial b} \vec{j} \right\}$$

Operand.

$$\nabla f(x, b) \triangleq \left\{ \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial b} \vec{j} \right\} \equiv \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial b} \right\rangle$$

①

$$\frac{\partial f}{\partial x} = +1$$

$$\frac{\partial f}{\partial b} = +1$$

②

$$\frac{\partial f}{\partial x} = -1$$

$$\frac{\partial f}{\partial b} = +1$$

③

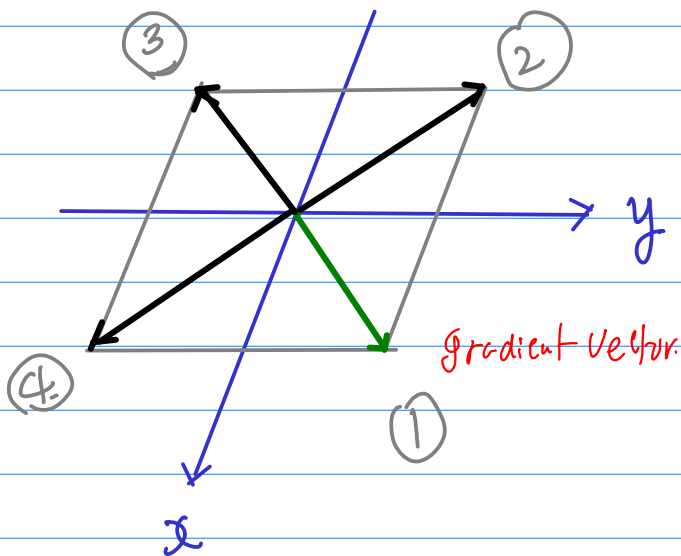
$$\frac{\partial f}{\partial x} = -1$$

$$\frac{\partial f}{\partial b} = -1$$

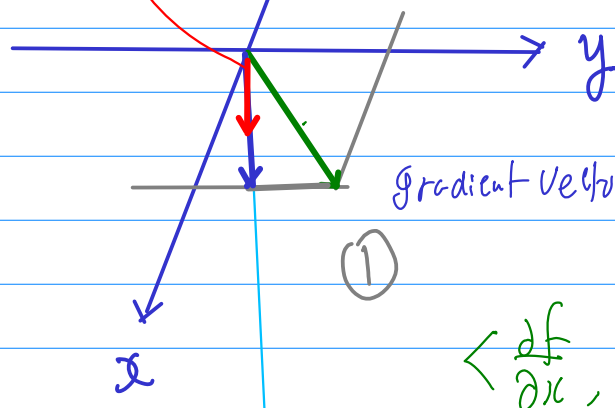
④

$$\frac{\partial f}{\partial x} = +1$$

$$\frac{\partial f}{\partial b} = -1$$

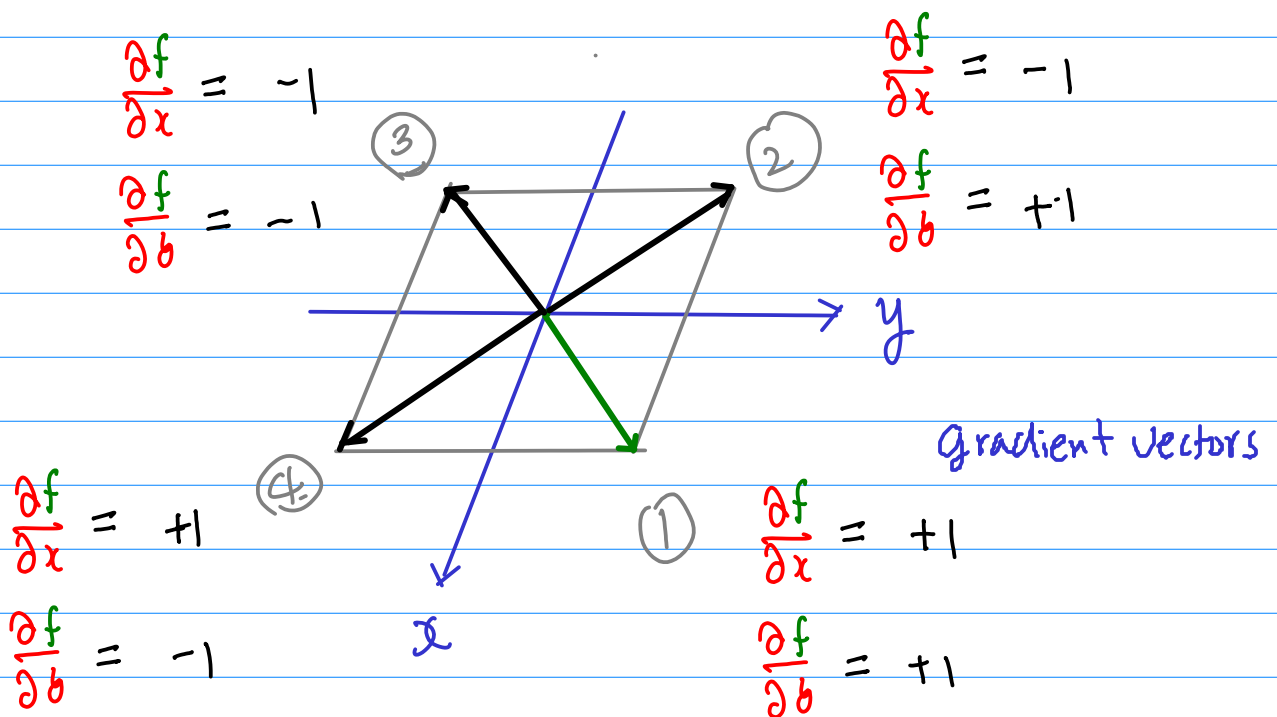
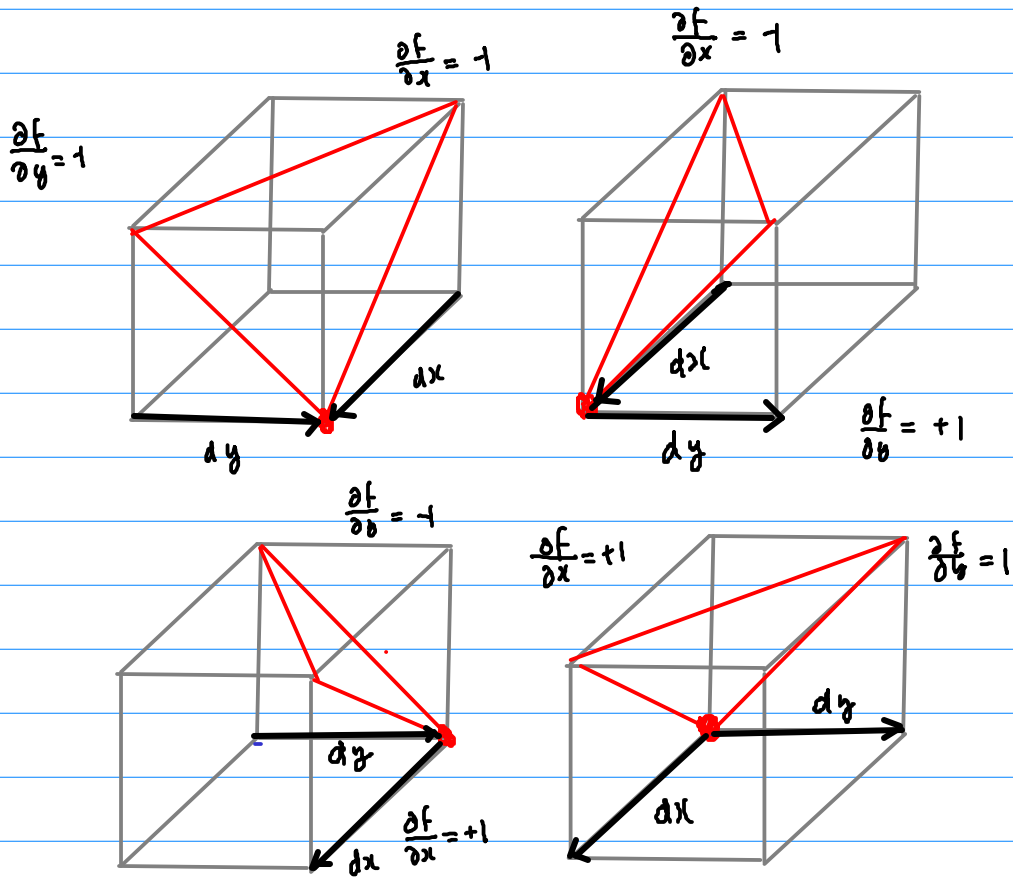


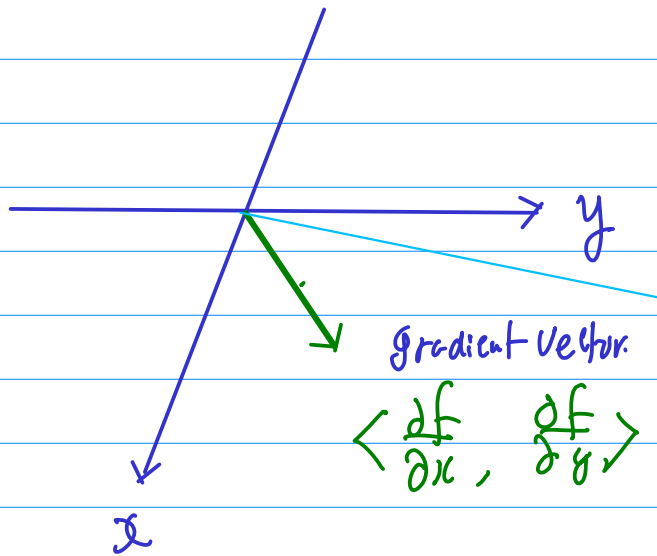
Unit directional vector $\langle a, b \rangle$



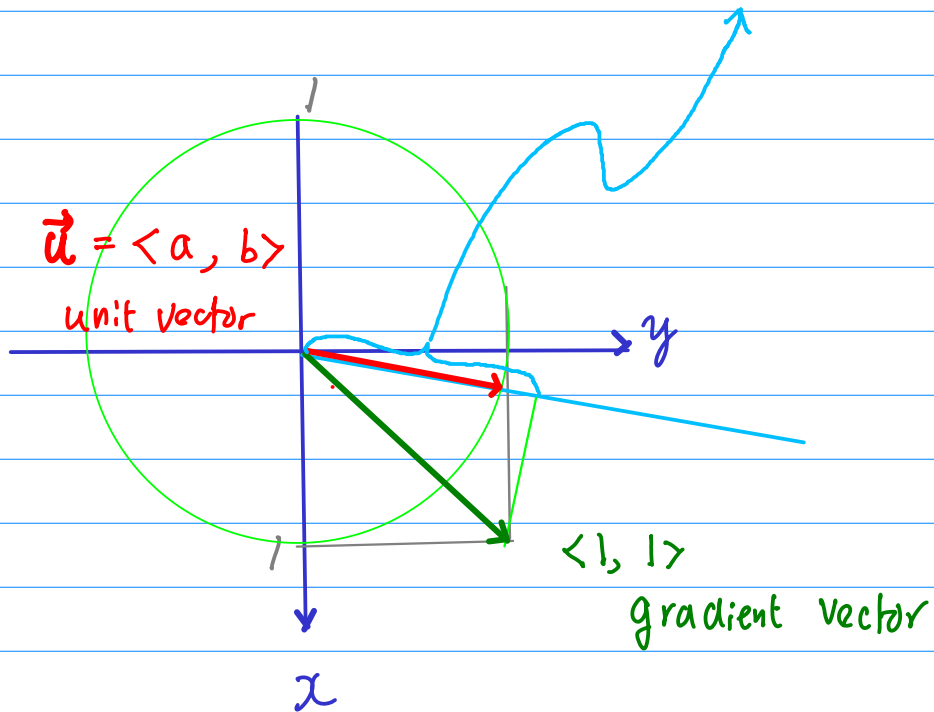
$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 1, 1 \rangle$$

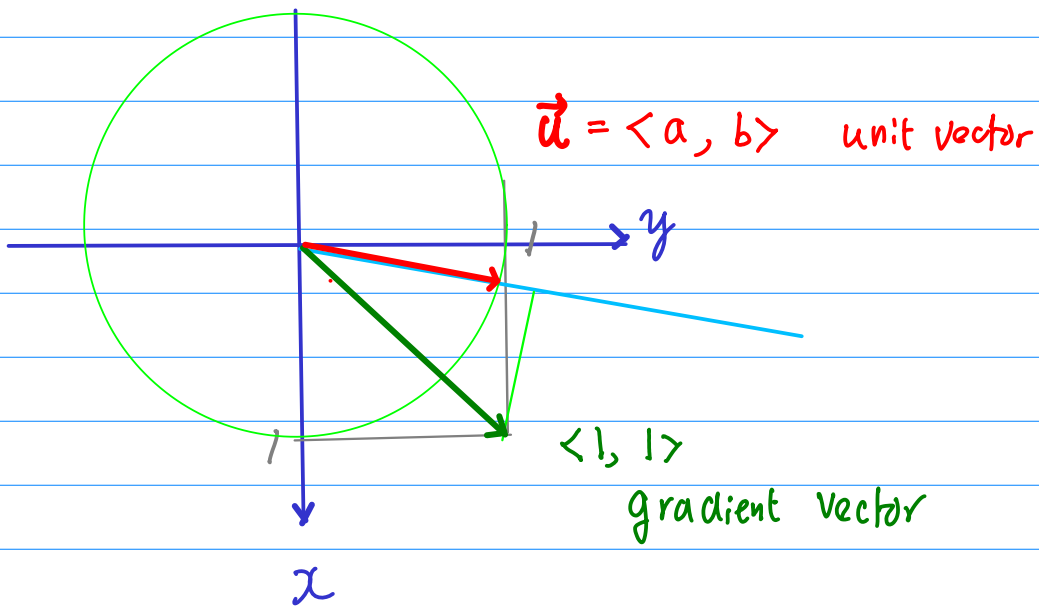
$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle$$



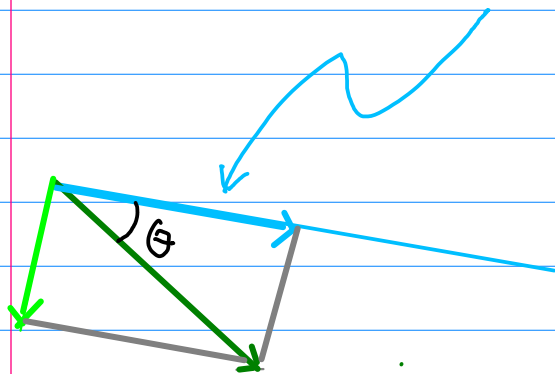


$$\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \cdot \langle a, b \rangle$$





projection of the gradient vector
onto the unit directional vector



the length of the projection

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle$$

$$\begin{aligned} &\langle +1, +1 \rangle \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \end{aligned}$$

↑
the slope of
x-directional
tangent

↑
the slope of
y-directional
tangent

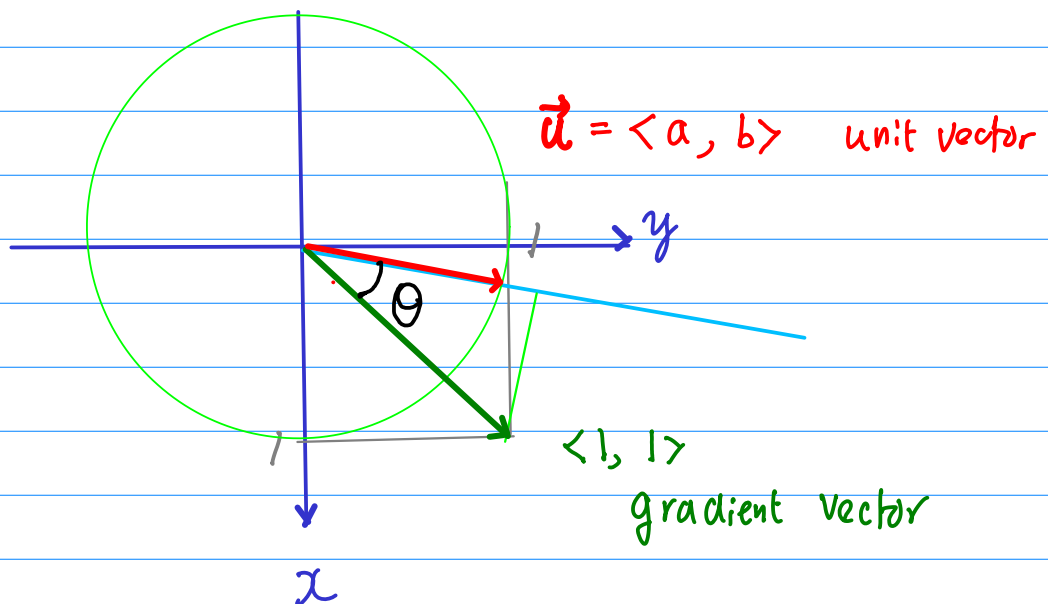
the length of the projection

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle = f_{\vec{u}}(x, y)$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

방향 미분
값
기울기

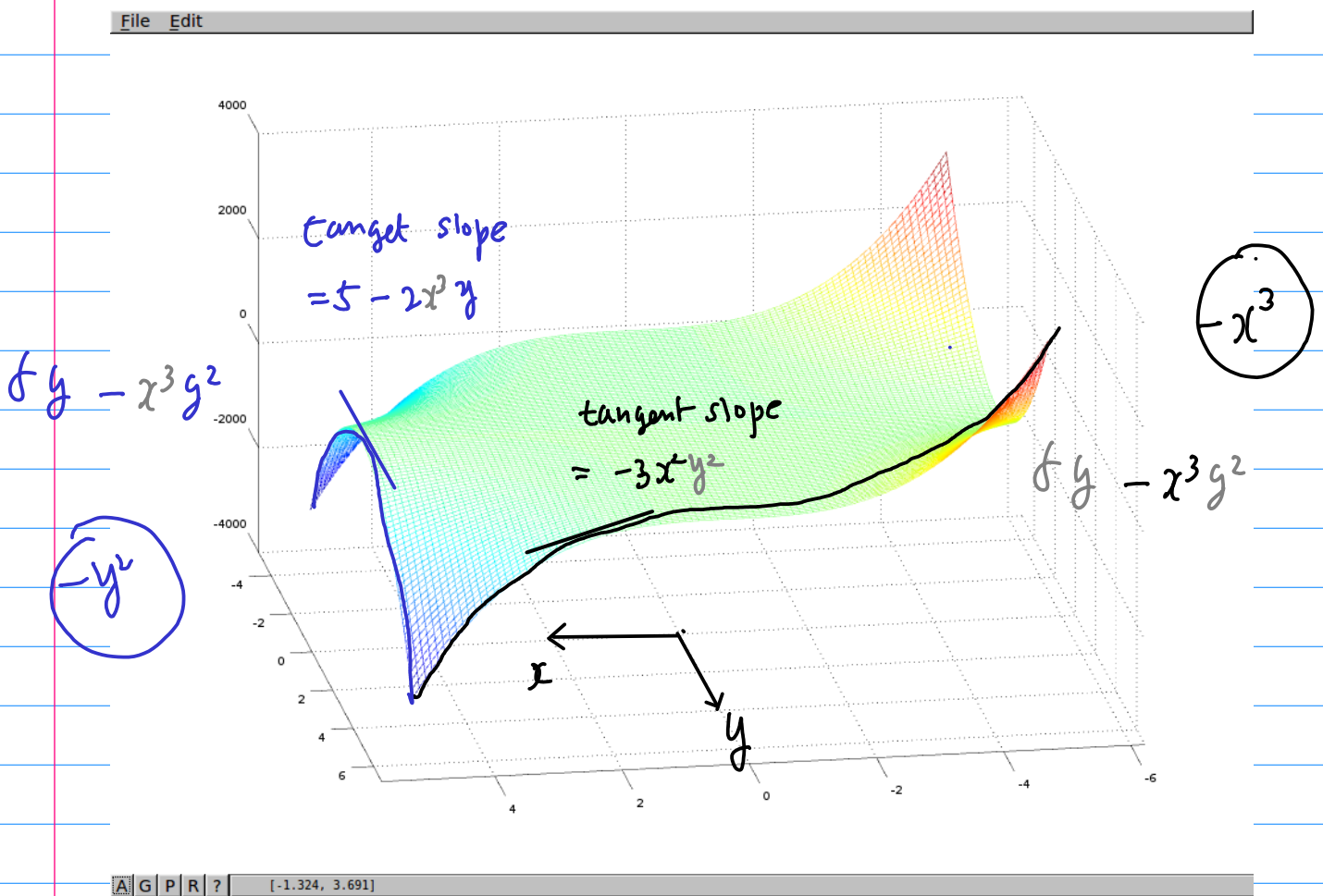
$$\begin{aligned} \nabla f \cdot \vec{u} &= \|\nabla f\| \cdot \|\vec{u}\| \cdot \cos \theta \\ &= \|\nabla f\| \cdot \cos \theta \end{aligned}$$



$$f(x, y) = 5y - x^3y^2$$

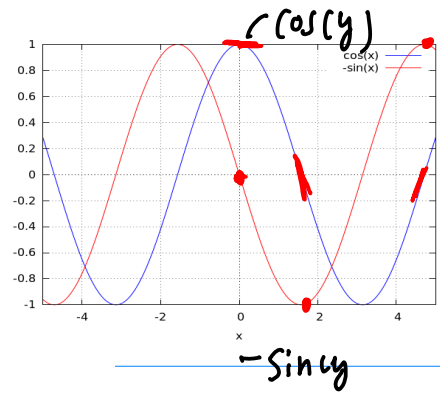
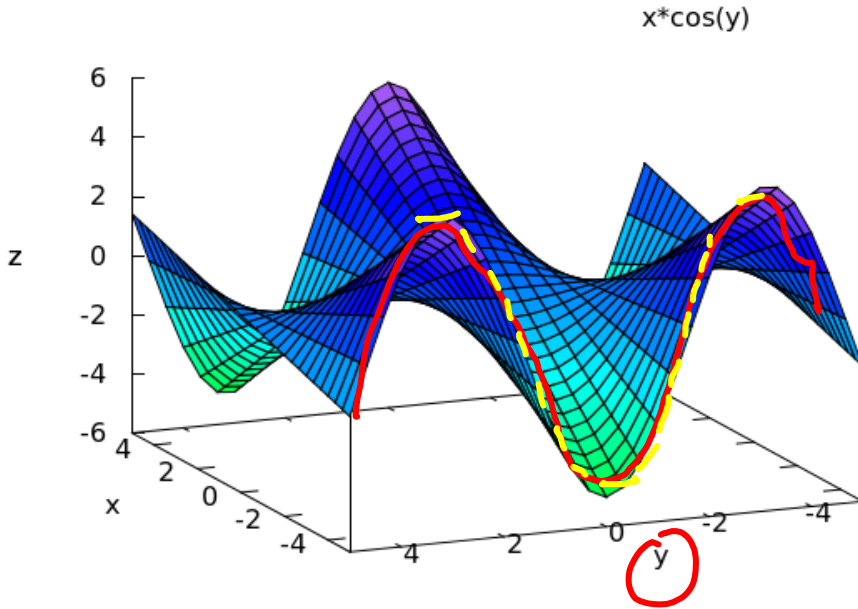
$$\frac{\partial f}{\partial x} = -3x^2y^2 \quad \frac{\partial f}{\partial y} = 5 - 2x^3y$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle -3x^2y^2, 5 - 2x^3y \rangle$$

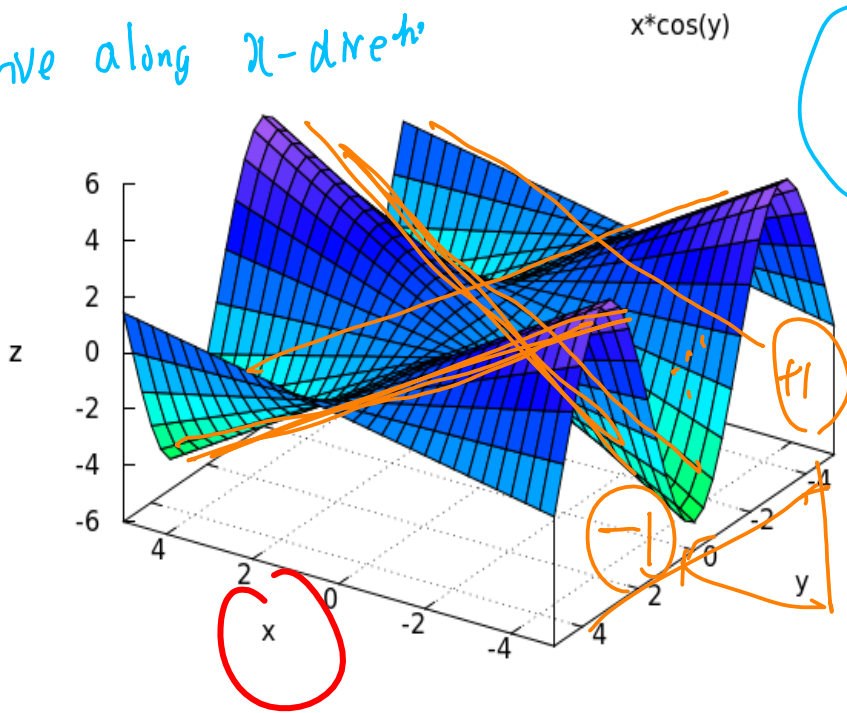


$$f(x, y) = x \cdot \cos(y)$$

$$\frac{\partial}{\partial y} f(x, y) = -x \sin(y)$$



derivative along x -direction



$$\frac{\partial}{\partial x} f(x, y) = \cos(y)$$

기울기가 상수
↑
y-방향

$$f(x, y) = x \cdot \cos(y)$$

$$\begin{aligned} \nabla f(x, y) &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= \left\langle \cos(y), -x \sin(y) \right\rangle \end{aligned}$$

gradient vector of $f(x, y)$

$$\begin{aligned} \nabla f(x, y) &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} \end{aligned}$$

<Vector field>

..... vector valued function

for each point



2-d

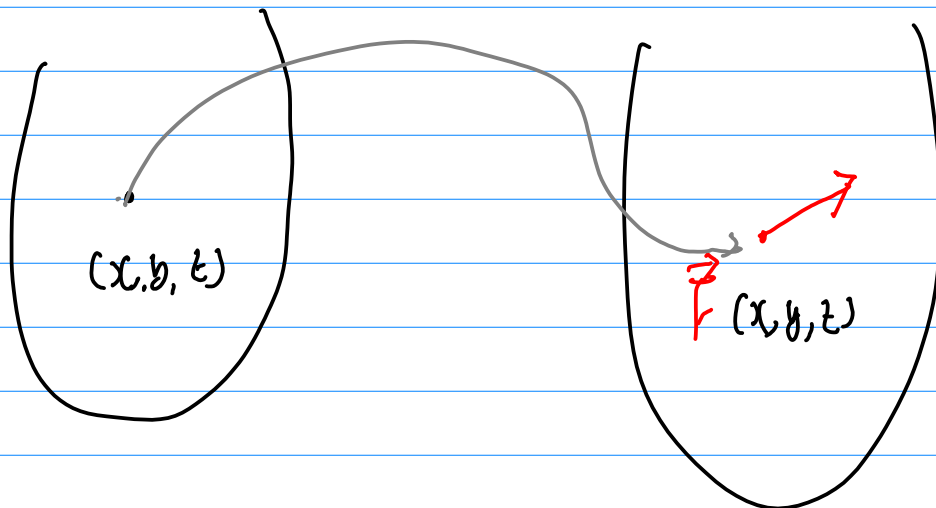
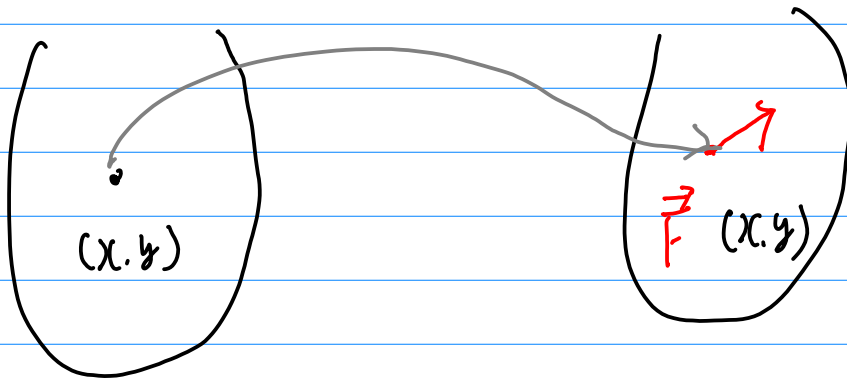
(x, y)

a vector (2-d) is assigned

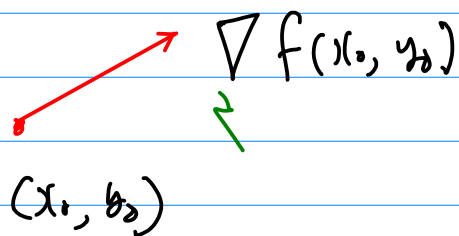
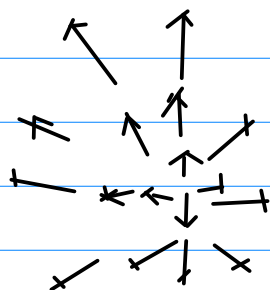
3-d

(x, y, z)

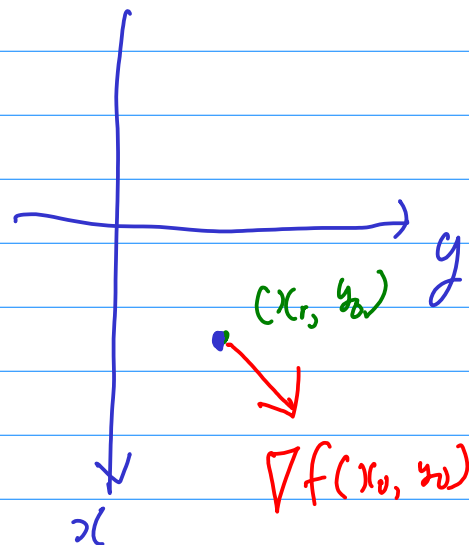
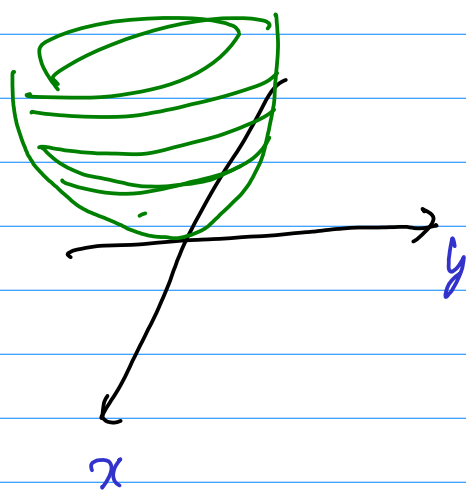
a vector (3-d) is assigned



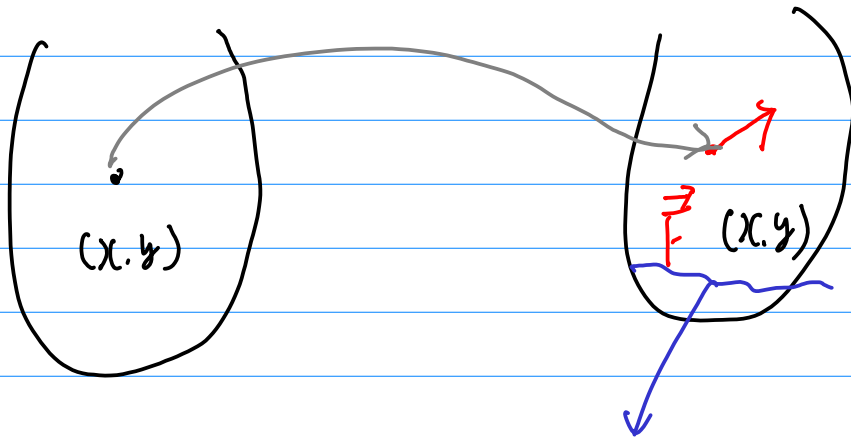
Gradient vector field



$$z = f(x, y) : \text{3D의 2D}$$



2-d vector field



at a 2-d point (x, y)

the value of a function \vec{F}
: a 2-d vector .

2 component

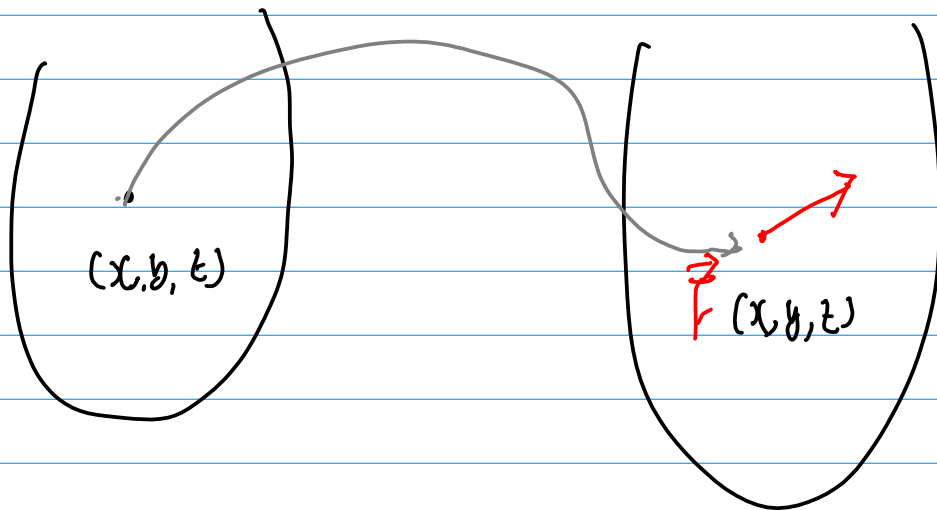
$$\langle P(x, y), Q(x, y) \rangle$$

$$= P(x, y) \vec{i} + Q(x, y) \vec{j}$$

P, Q :

scalar function

Vector valued function



at a 3-d point (x, y, z) the value of a function \vec{F}
 : a 3-d vector.

3 component

$$\langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

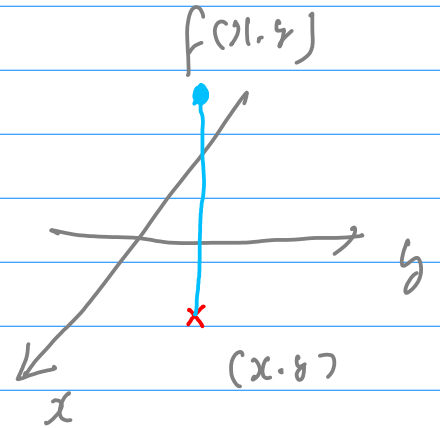
$$= P(x, y, z) \vec{i} + Q(x, y, z) \vec{j} + R(x, y, z) \vec{k}$$

P, Q, R
 scalar function

Vector valued function

2 variable functions

$$f(x, y) = x^2 + y^2$$

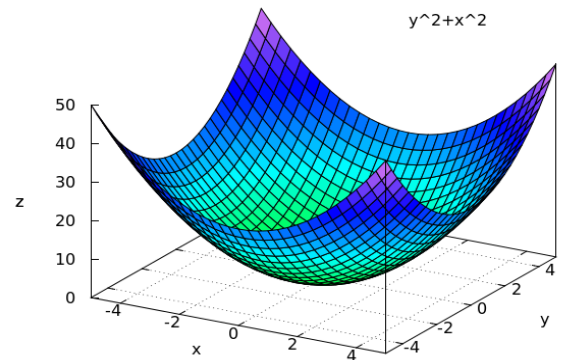


at a 2-d point (x, y) ,

a value is assigned

$$x^2 + y^2$$

Scalar function $f(x, y)$
~~Vector~~ function



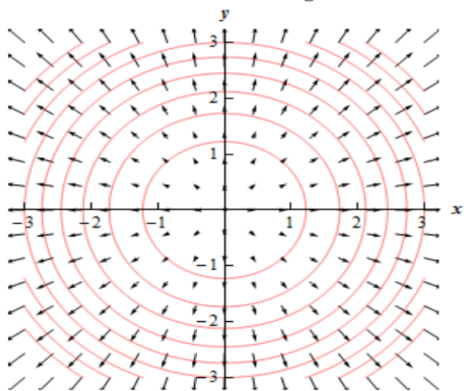
Gradient Vector

$$\vec{F}(x, y) = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle$$

at a 2-d point (x, y) ,

a vector is assigned

$$\left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle$$



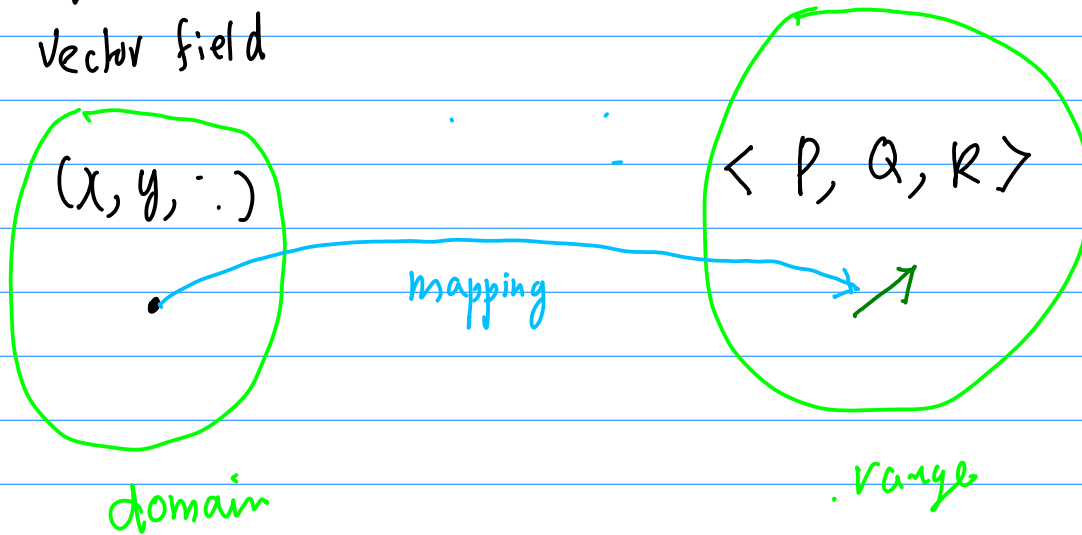




$$\vec{F} = P \vec{i} + Q \vec{j} + R \vec{k} \quad \mathbb{R}^3$$

$$\downarrow = \langle P, Q, R \rangle$$

vector field



$$P = P(x, y, z)$$

$$Q = Q(x, y, z)$$

$$R = R(x, y, z)$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\begin{aligned} & \frac{\partial R}{\partial y} \vec{i} + \frac{\partial P}{\partial z} \vec{j} + \frac{\partial Q}{\partial x} \vec{k} \\ - & \frac{\partial Q}{\partial z} \vec{i} - \frac{\partial R}{\partial x} \vec{j} - \frac{\partial P}{\partial y} \vec{k} \end{aligned}$$

$$\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

$$\nabla = \left(\frac{\partial}{\partial x} \right) \vec{i} + \left(\frac{\partial}{\partial y} \right) \vec{j} + \left(\frac{\partial}{\partial z} \right) \vec{k}$$

$$\vec{F} = P \vec{i} + Q \vec{j} + R \vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\vec{a} = \langle x_1, y_1, z_1 \rangle = x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}$$

$$\vec{b} = \langle x_2, y_2, z_2 \rangle = x_2 \vec{i} + y_2 \vec{j} + z_2 \vec{k}$$

$$\vec{a} \times \vec{b} \triangleq \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

Vec Vec

Vec

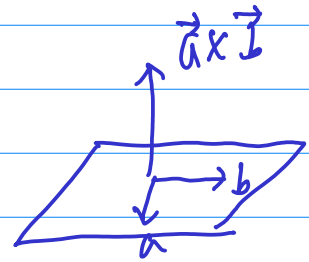
\vec{a} 에 대한

속성

\vec{b} 에 대한

속성

오른손 법칙





$$f(x, y)$$

$$\int_C f ds$$

