

Inverse Matrix (H1)

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Solving a System of Linear Equations

3x3 A A^{-1} Inverse Matrix.

$$p_1 x + p_2 y + p_3 z = b_1$$

$$q_1 x + q_2 y + q_3 z = b_2$$

$$r_1 x + r_2 y + r_3 z = b_3$$

$$\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A x = b$$

① $A^{-1} \quad x = A^{-1} \cdot b$

② Cramer's Rule $x = \frac{\begin{vmatrix} \text{green} & \text{purple} & \text{purple} \\ \text{purple} & \text{purple} & \text{purple} \end{vmatrix}}{\begin{vmatrix} \text{purple} & \text{purple} & \text{purple} \\ \text{purple} & \text{purple} & \text{purple} \\ \text{purple} & \text{purple} & \text{purple} \end{vmatrix}} \quad y = \frac{\begin{vmatrix} \text{purple} & \text{green} & \text{purple} \\ \text{purple} & \text{purple} & \text{purple} \\ \text{purple} & \text{purple} & \text{purple} \end{vmatrix}}{\begin{vmatrix} \text{purple} & \text{purple} & \text{purple} \\ \text{purple} & \text{purple} & \text{purple} \\ \text{purple} & \text{purple} & \text{purple} \end{vmatrix}} \quad z = \frac{\begin{vmatrix} \text{purple} & \text{purple} & \text{green} \\ \text{purple} & \text{purple} & \text{purple} \\ \text{purple} & \text{purple} & \text{purple} \end{vmatrix}}{\begin{vmatrix} \text{purple} & \text{purple} & \text{purple} \\ \text{purple} & \text{purple} & \text{purple} \\ \text{purple} & \text{purple} & \text{purple} \end{vmatrix}}$

③ Gauss-Jordan Elimination RREF

$R_{ij} = \begin{bmatrix} \text{blue} \\ \text{orange} \end{bmatrix} \begin{matrix} i \\ j \end{matrix}$ $cR_i = c \times \begin{bmatrix} \text{blue} \end{bmatrix} \begin{matrix} i \\ \end{matrix}$ $cR_i + kR_j = c \times \begin{bmatrix} \text{blue} \\ \text{orange} \end{bmatrix} \begin{matrix} i \\ \end{matrix}$

Minor

The **minor** of entry a_{ij} M_{ij}

The determinant of the submatrix that remains after **deleting** i -th row and j -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Determinant (3A)

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Cofactor

The **cofactor** of entry a_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The **minor** of entry a_{ij}

$$M_{ij}$$

The determinant of the submatrix that remains after **deleting** i -th row and j -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

$$\begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} = \begin{bmatrix} (-1)^2 & (-1)^3 & (-1)^4 \\ (-1)^3 & (-1)^4 & (-1)^5 \\ (-1)^4 & (-1)^5 & (-1)^6 \end{bmatrix}$$

Minor Example (1)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = +M_{11}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = -M_{12}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = +M_{13}$$

Determinant (3A)

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Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = -M_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = +M_{22}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{23} = -M_{23}$$

Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Minor / Cofactor

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = +M_{31}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = -M_{32}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = +M_{33}$$

Determinant (3A)

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Determinant

The **determinant** of an $n \times n$ matrix \mathbf{A} $\det(\mathbf{A})$

✧ Cofactor expansion along the i -th row
(elements of the i -th row) · (cofactors at the i -th row)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ \det(\mathbf{A}) &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ \det(\mathbf{A}) &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \end{aligned}$$

✧ Cofactor expansion along the j -th column
(elements of the j -th column) · (cofactors at the j -th column)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ \det(\mathbf{A}) &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ \det(\mathbf{A}) &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$

Determinant (3A)

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Adjoint

The **cofactor** of entry a_{ij} $C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} M_{ij}

The determinant of the submatrix
that remains after deleting i -th row and j -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{array}{lll} a_{11} \Leftrightarrow C_{11} & a_{12} \Leftrightarrow C_{12} & a_{13} \Leftrightarrow C_{13} \\ a_{21} \Leftrightarrow C_{21} & a_{22} \Leftrightarrow C_{22} & a_{23} \Leftrightarrow C_{23} \\ a_{31} \Leftrightarrow C_{31} & a_{32} \Leftrightarrow C_{32} & a_{33} \Leftrightarrow C_{33} \end{array}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose

✧ Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

Inverse Matrix

The **cofactor** of entry a_{ij} $C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} M_{ij}

The determinant of the submatrix that remains after deleting i -th row and j -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose \rightarrow

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A}) = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Determinant (3A)

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Inverse Matrix

Given matrix

$$\begin{bmatrix} +2 & +2 & 0 \\ -2 & +1 & +1 \\ +3 & 0 & +1 \end{bmatrix}$$

$$\left(\begin{array}{l} + \begin{vmatrix} +1 & +1 \\ 0 & +1 \end{vmatrix} - \begin{vmatrix} -2 & +1 \\ +3 & +1 \end{vmatrix} + \begin{vmatrix} -2 & +1 \\ +3 & 0 \end{vmatrix} \\ - \begin{vmatrix} +2 & 0 \\ 0 & +1 \end{vmatrix} + \begin{vmatrix} +2 & 0 \\ +3 & +1 \end{vmatrix} - \begin{vmatrix} +2 & +2 \\ +3 & 0 \end{vmatrix} \\ + \begin{vmatrix} +2 & 0 \\ +1 & +1 \end{vmatrix} - \begin{vmatrix} +2 & 0 \\ -2 & +1 \end{vmatrix} + \begin{vmatrix} +2 & +2 \\ -2 & +1 \end{vmatrix} \end{array} \right)$$

Matrix of Cofactors

$$\begin{bmatrix} +1 & +5 & -3 \\ -2 & +2 & +6 \\ +2 & -2 & +6 \end{bmatrix}$$

Adjoint $\text{adj}(\mathbf{A})$

$$\begin{bmatrix} +1 & -2 & +2 \\ +5 & +2 & -2 \\ -3 & +6 & +6 \end{bmatrix}$$

$$+2 \cdot \begin{vmatrix} +1 & +1 \\ 0 & +1 \end{vmatrix} + 2 \cdot \begin{vmatrix} -2 & +1 \\ +3 & +1 \end{vmatrix} + 0 \cdot \begin{vmatrix} -2 & +1 \\ +3 & 0 \end{vmatrix} = 12$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A})$$

a number a matrix

$$\mathbf{A}^{-1} = \frac{1}{12} \begin{bmatrix} +1 & -2 & +2 \\ +5 & +2 & -2 \\ -3 & +6 & +6 \end{bmatrix}$$

	$\det(A) \neq 0$	$\det(A) = 0$
A^{-1}	A^{-1} exists	No inverse
$Ax = 0$	the only solution $x = A^{-1}0 = 0$ trivial	many solutions parameterized
RREF	I	# zero row ≥ 1

$$A \cdot A^{-1} = A^{-1} A = I$$

$$(A^{-1})^{-1} = A$$

$$(A B)^{-1} = B^{-1} A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$