Inverse Matrix (H1)

20160106

Solving a System of Linear Equations

3x3 A A^{-1} Inverse Matrix.

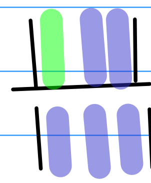
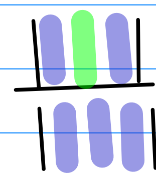
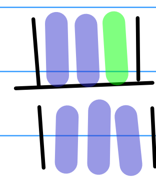
$$p_1 x + p_2 y + p_3 z = b_1$$
$$q_1 x + q_2 y + q_3 z = b_2$$
$$r_1 x + r_2 y + r_3 z = b_3$$

$$\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

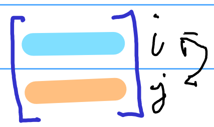
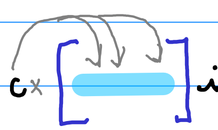
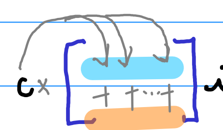
$$A \quad x = b$$

① A^{-1} $\qquad x = A^{-1} \cdot m$

② Cramer's Rule

$x =$  $y =$  $z =$ 

③ Gauss-Jordan Elimination RREF

R_{ij} =  cR_i = c×  $cR_i + R_j$ = c× 

Minor

The **minor** of entry a_{ij} M_{ij}

The determinant of the submatrix
that remains after deleting i-th row and j-th column

Given matrix | | Sub-matrix | Minor

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} \; & \; & \; \\ \; & a_{22} & a_{23} \\ \; & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \quad M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$n \times n$ $(n-1)\times(n-1)$

Cofactor

The **cofactor** of entry a_{ij} $C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} M_{ij}

The determinant of the submatrix
that remains after deleting i-th row and j-th column

Given matrix | | Sub-matrix | Minor

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} \; & \; & \; \\ \; & a_{22} & a_{23} \\ \; & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \quad M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$n \times n$ $(n-1)\times(n-1)$

Cofactor

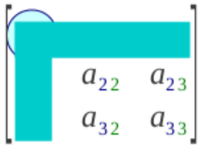
$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix} \quad \begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} = \begin{bmatrix} (-1)^{2} & (-1)^{3} & (-1)^{4} \\ (-1)^{3} & (-1)^{4} & (-1)^{5} \\ (-1)^{4} & (-1)^{5} & (-1)^{6} \end{bmatrix}$$

Minor Example (1)

Given matrix | Sub-matrix | Minor

$$\begin{bmatrix} \textcircled{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

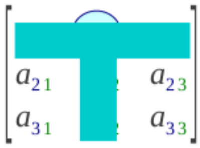


$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = +M_{11}$$

$$\begin{bmatrix} a_{11} & \textcircled{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

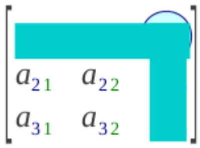


$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = -M_{12}$$

$$\begin{bmatrix} a_{11} & a_{12} & \textcircled{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

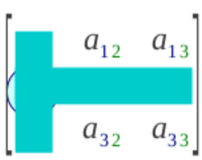
$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = +M_{13}$$

Minor Example (2)

Given matrix | Sub-matrix | Minor

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \textcircled{a_{21}} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

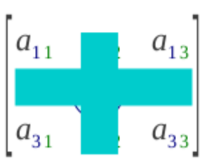


$$\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = -M_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \textcircled{a_{22}} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

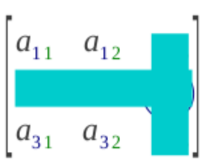


$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = +M_{22}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \textcircled{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$$\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

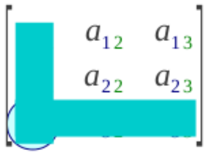
$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{23} = -M_{23}$$

Minor Example (2)

Given matrix | Sub-matrix | Minor / Cofactor

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

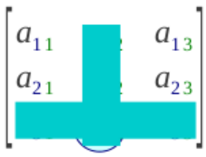


$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = +M_{31}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

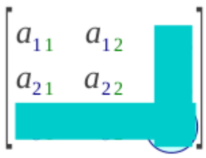


$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = -M_{32}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = +M_{33}$$

Determinant

The **determinant** of an $n \times n$ matrix $\boldsymbol{A}$ $\quad det(\boldsymbol{A})$

Cofactor expansion along the i-th row
(elements of the i-th row) · (cofactors at the i-th row)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$det(\boldsymbol{A}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$det(\boldsymbol{A}) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$det(\boldsymbol{A}) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

Cofactor expansion along the j-th column
(elements of the j-th column) · (cofactors at the j-th column)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$det(\boldsymbol{A}) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$
$$det(\boldsymbol{A}) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$
$$det(\boldsymbol{A}) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$$

Adjoint

The **cofactor** of entry a_{ij} $\quad C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} $\quad M_{ij}$

The determinant of the submatrix
that remains after deleting i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad n \times n$$

$$a_{11} \Leftrightarrow C_{11} \quad a_{12} \Leftrightarrow C_{12} \quad a_{13} \Leftrightarrow C_{13}$$
$$a_{21} \Leftrightarrow C_{21} \quad a_{22} \Leftrightarrow C_{22} \quad a_{23} \Leftrightarrow C_{23}$$
$$a_{31} \Leftrightarrow C_{31} \quad a_{32} \Leftrightarrow C_{32} \quad a_{33} \Leftrightarrow C_{33}$$

Matrix of Cofactors $\quad$ Adjoint

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \xrightarrow{\text{transpose}} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$ $\quad$ $n \times n$

Inverse Matrix

The **cofactor** of entry a_{ij} $\quad C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} $\quad M_{ij}$

The determinant of the submatrix that remains after deleting i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad n \times n$$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \quad n \times n$$

transpose →

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \quad n \times n$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} \qquad \mathbf{A}^{-1} = \frac{1}{det(\mathbf{A})} adj(\mathbf{A}) = \frac{1}{det(\mathbf{A})} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Inverse Matrix

Given matrix

$$\begin{bmatrix} +2 & +2 & 0 \\ -2 & +1 & +1 \\ +3 & 0 & +1 \end{bmatrix}$$

$$\begin{pmatrix} +\begin{vmatrix} +1 & +1 \\ 0 & +1 \end{vmatrix} & -\begin{vmatrix} -2 & +1 \\ +3 & +1 \end{vmatrix} & +\begin{vmatrix} -2 & +1 \\ +3 & 0 \end{vmatrix} \\ -\begin{vmatrix} +2 & 0 \\ 0 & +1 \end{vmatrix} & +\begin{vmatrix} +2 & 0 \\ +3 & +1 \end{vmatrix} & -\begin{vmatrix} +2 & +2 \\ +3 & 0 \end{vmatrix} \\ +\begin{vmatrix} +2 & 0 \\ +1 & +1 \end{vmatrix} & -\begin{vmatrix} +2 & 0 \\ -2 & +1 \end{vmatrix} & +\begin{vmatrix} +2 & +2 \\ -2 & +1 \end{vmatrix} \end{pmatrix}$$

Matrix of Cofactors

$$\begin{bmatrix} +1 & +5 & -3 \\ -2 & +2 & +6 \\ +2 & -2 & +6 \end{bmatrix}$$

Adjoint $adj(\mathbf{A})$

$$\begin{bmatrix} +1 & -2 & +2 \\ +5 & +2 & -2 \\ -3 & +6 & +6 \end{bmatrix}$$

$$+2 \cdot \begin{vmatrix} +1 & +1 \\ 0 & +1 \end{vmatrix} + 2 \cdot \begin{vmatrix} -2 & +1 \\ +3 & +1 \end{vmatrix} + 0 \cdot \begin{vmatrix} -2 & +1 \\ +3 & 0 \end{vmatrix} = 12$$

$$\mathbf{A}^{-1} = \frac{1}{det(\mathbf{A})} adj(\mathbf{A})$$

a number $\quad$ a matrix

$$\mathbf{A}^{-1} = \frac{1}{12} \begin{bmatrix} +1 & -2 & +2 \\ +5 & +2 & -2 \\ -3 & +6 & +6 \end{bmatrix}$$

	$\det(A) \neq 0$	$\det(A) = 0$
A^{-1}	A^{-1} exists	No inverse
$Ax = 0$	the only solution $x = A^{-1}0 = 0$ trivial	many solutions parameterized
RREF	I	# zero row ≥ 1

$$A \cdot A^{-1} = A^{-1} A = I$$

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$(A^{T})^{-1} = (A^{-1})^{T}$$