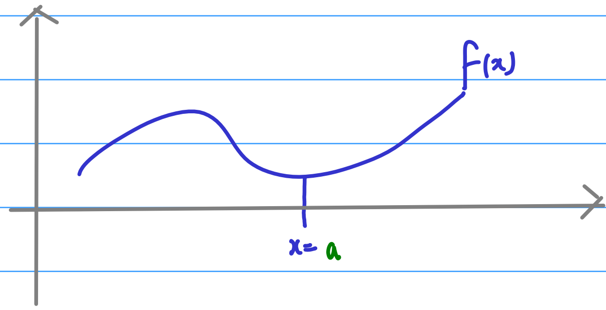
Complex Series (H.1)

20160921

Real Number Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n$$

$$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n\,(x-a)^n \\ a_n &= \frac{1}{n!} f^{(n)}(a) \end{aligned}}$$

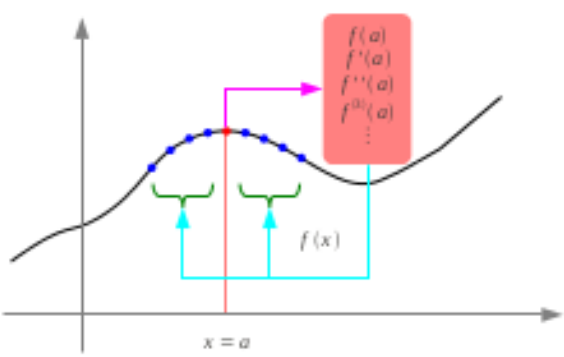


$n \geq 0$ $\qquad n = 0, 1, 2, \cdots$

$$\cancel{\frac{1}{(x-a)}}, \quad \cancel{\frac{1}{(x-a)^2}}, \quad \cancel{\frac{1}{(x-a)^3}}$$

Taylor Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$



Power Series → Taylor Series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$ Power Series

$$= a_0 + a_1(x-a)^1 + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_2(x-a)^2 + 4a_3(x-a)^3 + \cdots$$

$$f''(x) = 2!\,a_2 + 3\cdot 2\,a_2(x-a) + 4\cdot 3\,a_3(x-a)^2 + 5\cdot 4\,a_4(x-a)^3 + \cdots$$

$$f^{(3)}(x) = 3!\,a_2 + 4\cdot 3\cdot 2\,a_3(x-a) + 5\cdot 4\cdot 3\,a_4(x-a)^2 + 6\cdot 5\cdot 4\,a_4(x-a)^3 + \cdots$$

$$\vdots \qquad \vdots$$

$$f'(a) = a_1$$

$$f''(a) = 2!\,a_2$$

$$f^{(3)}(a) = 3!\,a_2$$

$$\vdots \qquad \vdots$$

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

$$a_n = \boxed{\frac{1}{n!} f^{(n)}(a)}$$

$\sin(x)$
$\cos(x)$
e^x

analytic always

$$\boxed{\frac{1}{1-x}}$$

not analytic at $x=1$

Taylor series + <u>ROC</u>

Complex Number Taylor Series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(z-a)^n$$

$$\boxed{\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-z_0)^n \\ a_n &= \frac{1}{n!} f^{(n)}(z_0) \end{aligned}}$$

z : a complex variable

a : a complex constant (eg, ± 1, $\pm i$, ...)

$n \geq 0$

only positive powers $(z-a)^0$, $(z-a)^1$, $(z-a)^2$, $(z-a)^3$, ...

No negative powers $(z-a)^{-1}$, $(z-a)^{-2}$, $(z-a)^{-3}$, $(z-a)^{-4}$, ...

$$= \frac{1}{(z-a)^1},\ \frac{1}{(z-a)^2},\ \frac{1}{(z-a)^3},\ \frac{1}{(z-a)^4},\ \ldots$$

→ no pole (denominator = 0) →

analytic over the entire complex domain

(always converge)

$\sum_{n=0}^{\infty} a_n (z-a)^n$ ⇒ entire function

z must not be any poles of $f(z)$

Geometric Series

real $a + ar + ar^2 + ar^3 + ar^4 \cdots\cdots = \dfrac{a}{1-r}$

$|r|<1$

complex $\boxed{a + az + az^2 + az^3 + \cdots} = \boxed{\dfrac{a}{1-z}}$

under the condition $|z|<1$ Converge

$$
\begin{array}{rrcl}
 & a + az + az^2 + \cdots + az^{n-1} & = & S_n \\
-) & az + az^2 + \cdots + az^{n-1} + az^n & = & zS_n \\
\hline
 & a \qquad - az^n & = & (1-z)S_n
\end{array}
$$

$$S_n = \frac{a(1-z^n)}{1-z}$$

under the condition $|z|<1$:

$$\lim_{n\to\infty} z^n = 0$$

$$\lim_{n\to\infty} S_n = \frac{a}{1-z}$$

Constraint

$f(z) = \frac{1}{1-z}$ not entire function

pole: $z = 1 \rightarrow$ not analytic

$$\underbrace{\frac{1}{1-z}}_{\text{not analytic at } z=1} = \underbrace{1 + z + z^2 + z^3 + \cdots\cdots}_{\text{always analytic}},$$

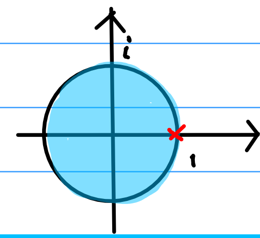
equal only with the condition $|z| < 1$

constraint

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} 1 \cdot (z-0)^n,$$

$$a_n = \frac{1}{n!} f^{(n)}(0) = 1$$

ROC



Taylor Series expanded at the origin $z = 0$

only within this ROC

$\boxed{\underbrace{\dfrac{1}{1-z}}_{\text{not analytic at } z=1}} = \boxed{\underbrace{1+z+z^2+z^3+\cdots\cdots}_{\text{always analytic}}}$, $\boxed{\text{equal only with the condition } \underline{|z|<1} \quad \text{constraint}}$

$$|z|<1 \begin{cases} \dfrac{1}{1-0} = 1+0+0^2+0^3+\cdots\cdots \\ \dfrac{1}{1-\left(\frac{1}{2}\right)} = 1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots\cdots \end{cases}$$

$$f(z)\text{'s pole} \quad z=1 \begin{cases} \underbrace{\dfrac{1}{1-1}}_{\infty} \overset{?}{=} 1+\underbrace{1+1^2+1^3+\cdots\cdots}_{\infty} \end{cases}$$

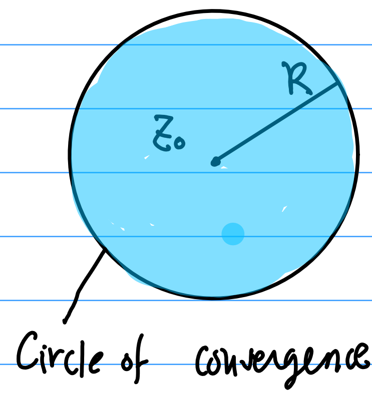
$$|z|>1 \begin{cases} \dfrac{1}{1-2} \cancel{=} 1+2+2^2+2^3+\cdots\cdots \end{cases}$$

$$\boxed{\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-z_0)^n \\ a_n &= \frac{1}{n!} f^{(n)}(z_0) \end{aligned}}$$

Convergence Circle

radius of Convergence R

$|z - z_0| < R$



$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1 (z - z_0)^1 + a_2 (z - z_0)^2 + \cdots\cdots$$

power series

Taylor Series

$$a_k = \frac{f^{(k)}(z_0)}{k!} + |z - z_0| < R$$

Complex Valued Function Example

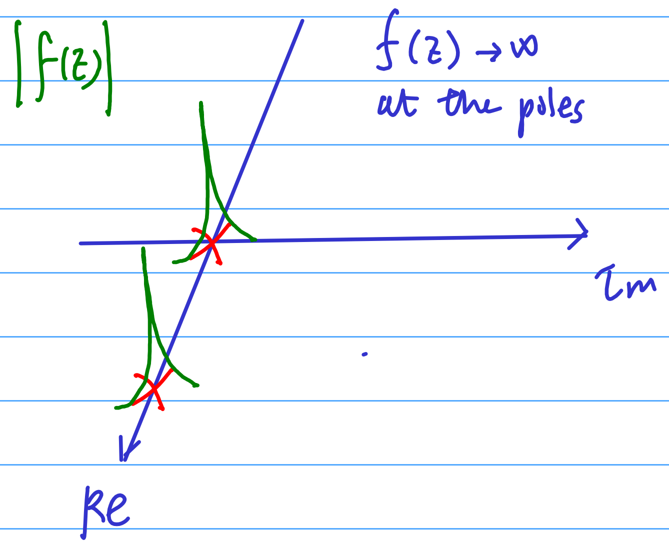
Laplace transform $F(s) = \frac{1}{s(s+1)}$

$F(s) = \frac{1}{s} - \frac{1}{s+1}$

$s = \sigma + i\omega$ in Laplace trans.
complex variable

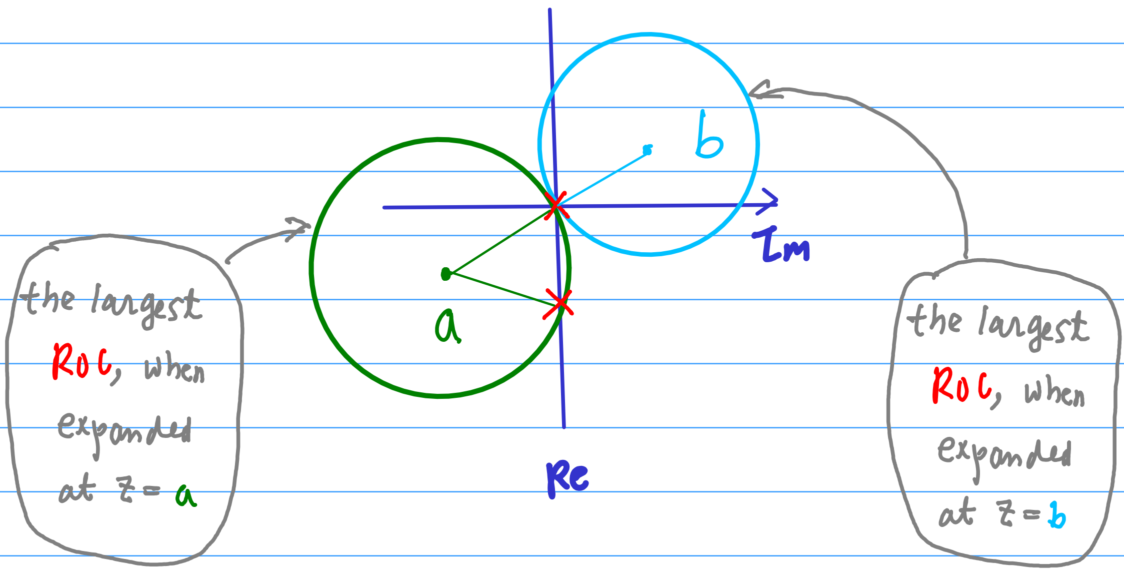
$f(z) = \frac{1}{z} - \frac{1}{z+1}$

z a usual variable
in the complex analysis



No pole is allowed in ROC

Taylor Series' ROC must exclude any pole.

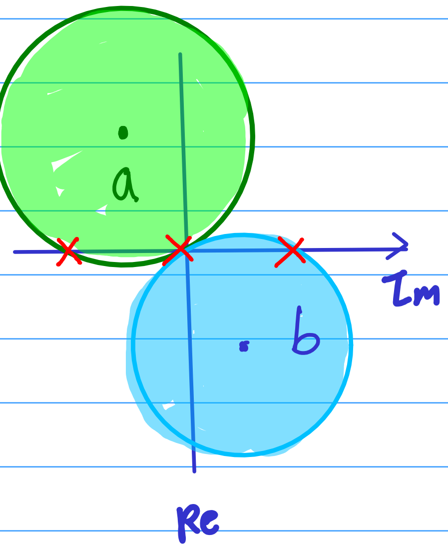


Not expandable around any pole : Taylor

* ROC must not include any pole.

Taylor Series expansion at a

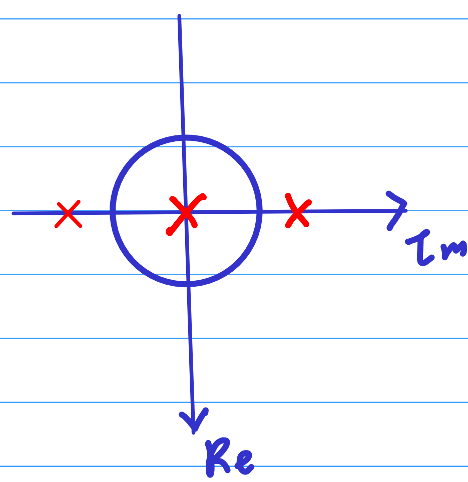
$f(z) = \sum_{k=0}^{\infty} \boxed{a_k} (z-a)^k$



Taylor Series expasion at b

$f(z) = \sum_{k=0}^{\infty} \boxed{b_k} (z-b)^k$

At $\boxed{z=0}$ taylor series expansion → McLaurin Series



No Taylor series around 0.

ROC should not contain any singularities must be analytic

In this case, McLaurin series expansion is not allowed

Expand at an arbitrary point

$$f(z) = \frac{1}{1-z}$$ expand at $z = 3$ need $(z-3)$ term

$$\frac{1}{1-3+3-z} = \frac{1}{-2-(z-3)} = \frac{1}{-2\left(1+\frac{z-3}{2}\right)}$$

$$= \frac{-\frac{1}{2}}{1-\left(\frac{3-z}{2}\right)}$$

$$\left|\frac{3-z}{2}\right| < 1$$

$$|z-3| < 2$$

$$f(z) = -\frac{1}{2}\left(1 - \frac{1}{2}(z-3) + \frac{1}{2^2}(z-3)^2 - \frac{1}{2^3}(z-3)^3 + \frac{1}{2^4}(z-3)^4 + \cdots\right)$$

$$= -\frac{1}{2} + \frac{1}{2^2}(z-3) - \frac{1}{2^3}(z-3)^2 + \frac{1}{2^4}(z-3)^3 - \frac{1}{2^5}(z-3)^5 + \cdots$$

expansion points

$z = 3$

```
(%i6) taylor(1/(1-z), z, 3, 7);
```

$$\text{(\%o6)/T/}\quad -\frac{1}{2}+\frac{z-3}{4}-\frac{(z-3)^2}{8}+\frac{(z-3)^3}{16}-\frac{(z-3)^4}{32}+\frac{(z-3)^5}{64}-\frac{(z-3)^6}{128}+\frac{(z-3)^7}{256}+...$$

$z = i$

```
(%i7) taylor(1/(1-z), z, %i, 7);
```

$$\text{(\%o7)/T/}\quad \frac{\%i+1}{2}+\frac{\%i\,(z-\%i)}{2}+\frac{(\%i-1)(z-\%i)^2}{4}-\frac{(z-\%i)^3}{4}-\frac{(\%i+1)(z-\%i)^4}{8}-\frac{\%i\,(z-\%i)^5}{8}-\frac{(\%i-1)(z-\%i)^6}{16}+\frac{(z-\%i)^7}{16}+...$$

$z = 1+i$

```
(%i8) taylor(1/(1-z), z, 1+%i, 7);
```

$$\text{(\%o8)/T/}\quad \%i-(z-\%i-1)-\%i\,(z-\%i-1)^2+(z-\%i-1)^3+\%i\,(z-\%i-1)^4-(z-\%i-1)^5-\%i\,(z-\%i-1)^6+(z-\%i-1)^7+...$$

Simple Pole → Infinite Geometric Series

$f(z) = \frac{1}{1-z}$ expand at $z = 3$ ⇒ need $(z-3)$ term

$$\frac{1}{1-3+3-z} = \frac{1}{-2-(z-3)} = \frac{1}{-2\left(1+\frac{z-3}{2}\right)}$$

$$= \frac{1}{-2\left\{1-\left(\frac{z-3}{-2}\right)\right\}} = \frac{1}{(1-3)\left\{1-\frac{(z-3)}{(1-3)}\right\}}$$

$$\boxed{z'-z = (z'-z_0)+(z_0-z) = (z'-z_0)\left\{1-\frac{(z-z_0)}{(z'-z_0)}\right\}}$$

$$\frac{1}{1-z} = \frac{1}{(1-3)+(3-z)} = \frac{1}{(1-3)\left\{1-\frac{(z-3)}{(1-3)}\right\}} = \frac{-\frac{1}{2}}{1-\left(\frac{z-3}{2}\right)}$$

$$\boxed{\left|\frac{z-z_0}{z'-z_0}\right| < 1}$$

To expand at z_0 : $(z'-z) \rightarrow (z-z_0)$

$$\boxed{\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-z_0)^n \\ a_n &= \frac{1}{n!} f^{(n)}(z_0) \end{aligned}}$$

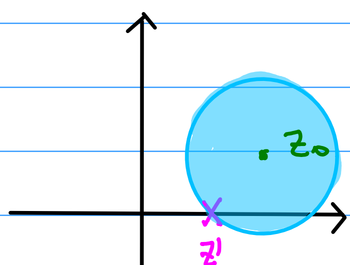
if we want a Taylor series <u>at $z = z_0$</u>

Convert real poles at $z = z'$ as follows

$$(z' - z) \to (z - z_0)$$

$$\boxed{z' - z = (z' - z_0) + (z_0 - z) = (z' - z_0) \cdot \left\{ 1 - \frac{(z - z_0)}{(z' - z_0)} \right\}}$$

$$\boxed{\left| \frac{z - z_0}{z' - z_0} \right| < 1 \; : \; \text{ROC}}$$



$$|z - z_0| < |z' - z_0|$$

inside a circle

$$\boxed{\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n \\ a_n &= \frac{1}{n!} f^{(n)}(z_0) \end{aligned}}$$

Two Infinite Geometric Series

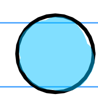
① $$z' - z = (z'-z_0) - (z-z_0) = (z'-z_0)\left\{1 - \frac{(z-z_0)}{(z'-z_0)}\right\}$$

$z_0=3$ $$\frac{1}{1-z} = \frac{1}{(1-3)-(z-3)} = \frac{1}{(1-3)\left\{1-\frac{(z-3)}{(1-3)}\right\}} = \frac{-\frac{1}{2}}{1-\left(\frac{z-3}{2}\right)}$$

z in the numerator

$$\frac{1}{\left(1-\frac{z}{\square}\right)}$$

inside a circle



$|z| < \square$

the only allowed ROC
in the Taylor Series

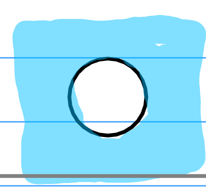
② $$z' - z = (z'-z_0) - (z-z_0) = \left\{\frac{(z'-z_0)}{(z-z_0)} - 1\right\}\cdot(z-z_0)$$

$z_0=3$ $$\frac{1}{1-z} = \frac{1}{(1-3)-(z-3)} = \frac{1}{\left\{\frac{(1-3)}{(z-3)}-1\right\}(z-3)} = \frac{-1}{1+\left(\frac{2}{z-3}\right)}\cdot\frac{1}{(z-3)}$$

z in the denominator

$$\frac{1}{\left(1-\frac{\square}{z}\right)}$$

outside a circle



$|z| > \square$

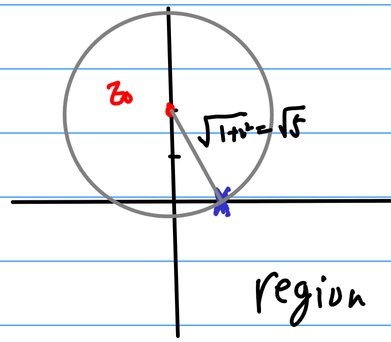
negative powers → not allowed ROC in the Taylor Series
but allowed in the Laurent Series

2014 f

Ex 2

$$f(z) = \frac{1}{1-z}$$

$$z_0 = 2i$$



region of circle

$$f(z) = \frac{1}{(1-z)} \qquad f(z_0) = f(2i) = \frac{1}{(1-2i)}$$

$$f'(z) = \frac{1}{(1-z)^2} \qquad f^{(1)}(z_0) = f^{(1)}(2i) = \frac{1!}{(1-2i)^2}$$

$$f''(z) = \frac{2!}{(1-z)^3} \qquad f^{(2)}(z_0) = f^{(2)}(2i) = \frac{2!}{(1-2i)^3}$$

$$f'''(z) = \frac{3!}{(1-z)^4} \qquad f^{(3)}(z_0) = f^{(3)}(2i) = \frac{3!}{(1-2i)^4}$$

$$f^{(k)}(z) = \frac{k!}{(1-z)^{k+1}}$$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{(1-2i)^{k+1}} (z-2i)^k$$

Simple pole conversion method

$$\frac{1}{1-z} = \frac{1}{(1-2i)-(z-2i)} = \frac{1}{(1-2i)\left\{1-\frac{(z-2i)}{(1-2i)}\right\}}$$

$$= \frac{1}{(1-2i)} \cdot \boxed{\frac{1}{\left\{1-\frac{(z-2i)}{(1-2i)}\right\}}}$$

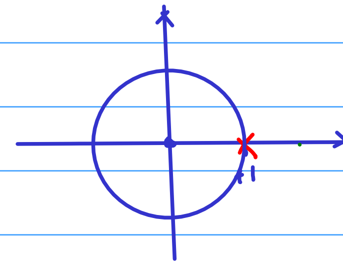
$$= \frac{1}{(1-2i)} \cdot \left\{1+\left(\frac{z-2i}{1-2i}\right)^1+\left(\frac{z-2i}{1-2i}\right)^2+\left(\frac{z-2i}{1-2i}\right)^3+\cdots\right\}$$

$$= \frac{1}{(1-2i)} + \frac{(z-2i)^1}{(1-2i)^2} + \frac{(z-2i)^2}{(1-2i)^3} + \frac{(z-2i)^3}{(1-2i)^4} + \cdots$$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{(1-2i)^{k+1}}(z-2i)^k$$

M^cLaurin Expansion $f(z) = \dfrac{1}{(1-z)^2}$



Taylor Series Expansion at $z=0$

Method Ⅰ

$$f(z) = (1-z)^{-2}$$

$$f^{(1)}(z) = -2(1-z)^{-3}\cdot(-1) \qquad f^{(1)}(0) = 2!$$

$$f^{(2)}(z) = 2\cdot 3(1-z)^{-4}(-1) \qquad f^{(2)}(0) = 3!$$

$$f^{(3)}(z) = -2\cdot 3\cdot 4(1-z)^{-5}\cdot(-1) \qquad f^{(3)}(0) = 4!$$

$$f^{(k)}(z) = (k+1)!\,(1-z)^{k+2} \qquad f^{(k)}(0) = (k+1)!$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(z-a)^n$$

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)}$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{n!}(n+1)! = (n+1)$$

$$f(z) = \sum_{n=0}^{\infty} (n+1) z^n = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

Method Ⅱ

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

$$\frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}\left(1 + z + z^2 + z^3 + \cdots\right)$$

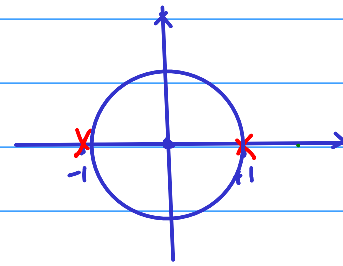
$$\left.\begin{aligned} \frac{1}{(1-z)^2} &= 1 + 2z + 3z^2 + 4z^3 + \cdots \\ &= \sum_{k=1}^{\infty} k z^{k-1} \end{aligned}\right\} \qquad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad z_0 = 0$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) \Rightarrow n$$

$$= \sum_{k=0}^{\infty} (k+1) z^k$$

M^cLaurin Expansion $f(z) = \dfrac{1}{1-z^2}$

Taylor Series Expansion at $z=0$



Method (I)

$$f(z) = \boxed{(1-z^2)^{-1}}$$

$$f^{(1)}(z) = \boxed{-(1-z^2)^{-2}(-2z)}$$

$$f^{(2)}(z) = 2(1-z^2)^{-3}(-2z)^2 + 2(1-z^2)^{-2}$$

$$= \boxed{8(1-z^2)^{-3}z^2 + 2(1-z^2)^{-2}}$$

$$f^{(3)}(z) = -24(1-z^2)^{-4}z^2(-2z) + 16(1-z^2)^{-3}z - 4(1-z^2)^{-3}(-2z)$$

$$= \boxed{48(1-z^2)^{-4}z^3 + 24(1-z^2)^{-3}z}$$

$$f^{(4)}(z) = 48(-4)(1-z^2)^{-5}(-2z)z^3 + 48(1-z^2)^{-4}\cdot 3z^2 + 24(-3)(1-z^2)^{-4}(-2z^2) + 24(1-z^2)^{-3}$$

$$= 48\cdot 8(1-z^2)^{-5}z^4 + (1-z^2)^{-4}z^2(48\cdot 3 + 24\cdot 6) + 24(1-z^2)^{-3}$$

$$= \boxed{48\cdot 8(1-z^2)^{-5}z^4 + 48\cdot 6(1-z^2)^{-4}z^2 + 24(1-z^2)^{-3}}$$

$$f(0) = 1 = 1!$$
$$f^{(1)}(0) = 0$$
$$f^{(2)}(0) = 2 = 2!$$
$$f^{(3)}(0) = 0$$
$$f^{(4)}(0) = 24 = 4!$$

$$f^{(2n)}(0) = (2n)!$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n!)} z^{2n} = \sum_{n=0}^{\infty} z^{2n}$$

Method (II)

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

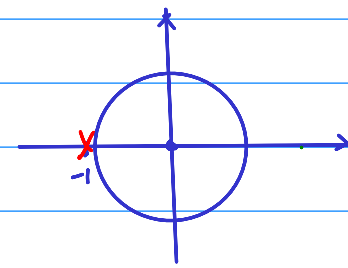
$$\frac{1}{1-z^2} = 1 + (z^2) + (z^2)^2 + (z^2)^3 + \cdots = 1 + z^2 + z^4 + z^6 + \cdots$$

$$\left.\begin{aligned}\frac{1}{1-z^2} &= 1 + (z^2) + (z^2)^2 + (z^2)^3 + \cdots \\ &= \sum_{n=0}^{\infty}(z^2)^n = \sum_{n=0}^{\infty} z^{2n}\end{aligned}\right\}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad z_0 = 0$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) \Rightarrow 1$$

McLaurin Expansion $f(z) = \dfrac{1}{1+z}$



Taylor Series Expansion at $z = 0$

Method ①

$$f(z) = (1+z)^{-1} \qquad f^{(0)}(0) = 1$$

$$f^{(1)}(z) = -(1+z)^{-2} \qquad f^{(1)}(0) = -1!$$

$$f^{(2)}(z) = 2(1+z)^{-3} \qquad f^{(2)}(0) = 2!$$

$$f^{(3)}(z) = -6(1+z)^{-4} \qquad f^{(3)}(0) = -3!$$

$$f^{(k)}(z) = (-1)^k k!\,(1-z)^{k+1} \qquad f^{(k)}(0) = (-1)^k \cdot k!$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(z-a)^n$$

$$\boxed{\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-z_0)^n \\ a_n &= \frac{1}{n!} f^{(n)}(z_0) \end{aligned}}$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{n!}(-1)^n n! = (-1)^n$$

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + \cdots$$

Method ②

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

$$\frac{1}{1-(-z)} = 1 + (-z) + (-z)^2 + (-z)^3 + (-z)^4 + \cdots$$

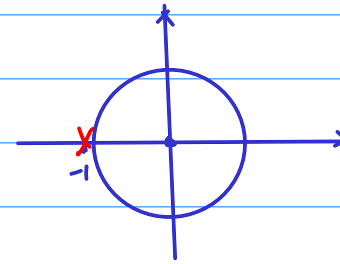
$$\left.\begin{aligned} \frac{1}{1+z} &= 1 - z + z^2 - z^3 + z^4 + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k z^k \end{aligned}\right\}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad z_0 = 0$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) \Rightarrow (-1)^n$$

M^cLaurin Expansion $f(z) = \ln(1+z)$

Taylor Series Expansion at $z=0$



Method (I)

$$f(z) = \ln(1+z) \qquad f^{(0)}(0) = 0$$
$$f^{(1)}(z) = (1+z)^{-1} \qquad f^{(1)}(0) = 1 = (-1)^0 \cdot 0!$$
$$f^{(2)}(z) = -(1+z)^{-2} \qquad f^{(2)}(0) = -1 = (-1)^1 \cdot 1!$$
$$f^{(3)}(z) = 2(1+z)^{-3} \qquad f^{(3)}(0) = 2 = (-1)^2 \cdot 2!$$
$$f^{(3)}(z) = -6(1+z)^{-4} \qquad f^{(4)}(0) = -6 = (-1)^3\, 3!$$

$$f^{(k)}(z) = (-1)^{k-1}(k-1)!\,(1+z)^{k+1} \qquad f^{(k)}(0) = (-1)^{k-1}(k-1)!$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(z-a)^n$$

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)}$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{n!}(-1)^{n-1}(n-1)!$$

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)} z^{k+1} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

Method (II)

$$\frac{1}{1+z} = 1 + z + z^2 + z^3 + \cdots$$

$$\int \frac{1}{1+z}\,dz = \int 1 - z + z^2 - z^3 + \cdots\, dz$$

$$\left.\begin{aligned} \ln(1+z) &= z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)} z^{k+1} \end{aligned}\right\}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad z_0 = 0$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) \Rightarrow \frac{(-1)^n}{(n+1)}$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n$$

$$\frac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \cdots = \sum_{n=0}^{\infty} z^{2n}$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots = \sum_{n=0}^{\infty} (-z)^n$$

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots = \sum_{n=0}^{\infty} (n+1) z^n$$

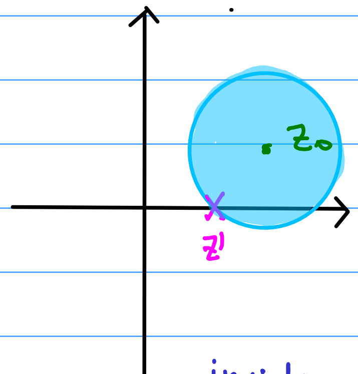
$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1}$$

$f(z) = \sum_{k=0}^{\infty} a_k z^k$ expand at $z = 0$

Two ways of converting simple poles

$$\left|\frac{z-z_0}{z'-z_0}\right|<1 \; : \text{ROC}$$

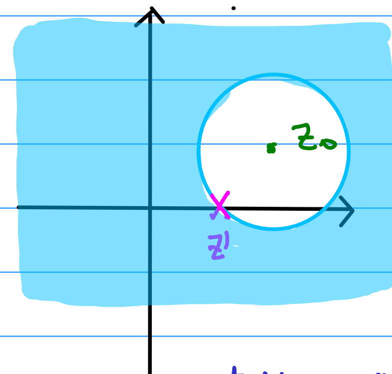
$$|z-z_0| < |z'-z_0|$$



inside a circle

$$\left|\frac{z'-z_0}{z-z_0}\right|<1 \; : \text{ROC}$$

$$|z-z_0| > |z'-z_0|$$



outside a circle

z in the numerator

(1) $$\boxed{z'-z = (z'-z_0)-(z-z_0) = (z'-z_0)\cdot\left\{1-\frac{(z-z_0)}{(z'-z_0)}\right\}}$$

$$\left|\frac{z-z_0}{z'-z_0}\right|<1 \; : \text{ROC}$$

inside a circle

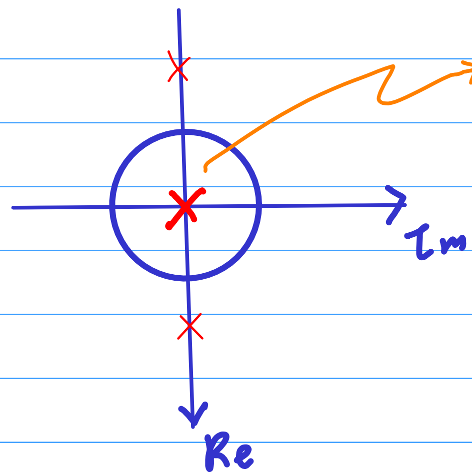
z in the denominator

(2) $$\boxed{z'-z = (z'-z_0)-(z-z_0) = \left\{\frac{(z'-z_0)}{(z-z_0)}-1\right\}\cdot(z-z_0)}$$

$$\left|\frac{z'-z_0}{z-z_0}\right|<1 \; : \text{ROC} \;\Rightarrow\; \left|\frac{z-z_0}{z'-z_0}\right|>1$$

outside a circle

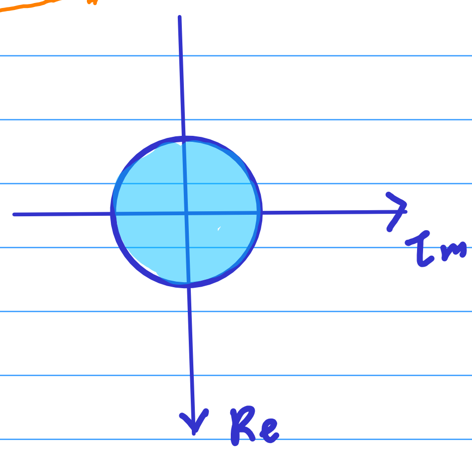
Intersection of two ROC's



at $z=0$, Taylor Series ✗

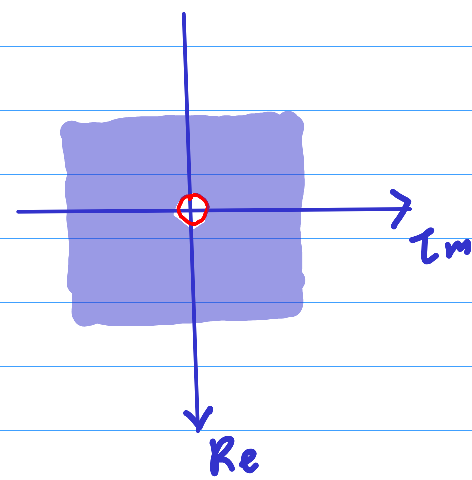
ROC should not include any singularities

✳ intersection of ROC's



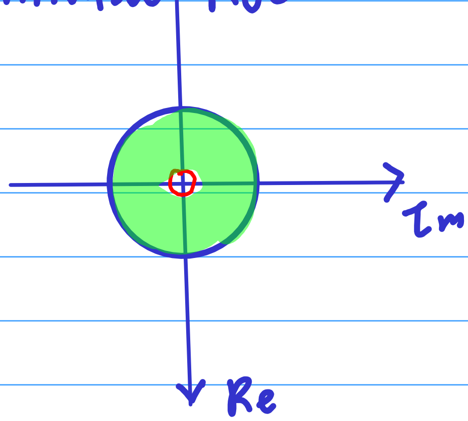
inside the circle

$\cap$



outside the circle

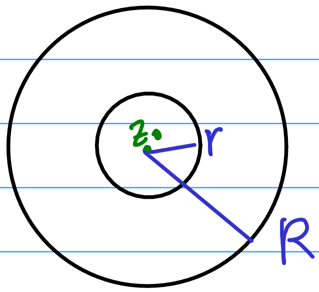
Annular ROC



→ Laurent Series

Laurent Series has two parts

$$f(z) = a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$
$$+ \frac{b_1}{(z-z_0)^1} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots$$



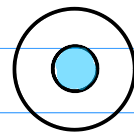
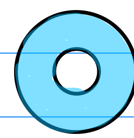
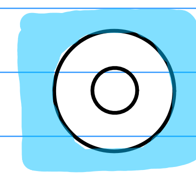
converge

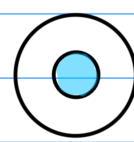
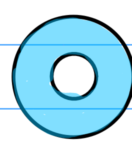
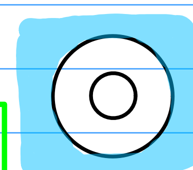
$$r < |z - z_0| < R$$

in view of a_k & b_k

	$\sum_{k=1}^{\infty} \frac{b_k}{(z-z_0)^k}$	$\sum_{k=0}^{\infty} a_k (z-z_0)^k$
$\lvert z-z_0 \rvert < x$	?	converge
$\lvert z-z_0 \rvert > y$	converge	?
	negative powers ($k < 0$)	positive powers ($k \geq 0$)

In view of ROC's

	$\lvert z-z_0\rvert < R$	$\lvert z-z_0\rvert > R$
$\lvert z-z_0\rvert < r$	$\lvert z-z_0\rvert < r$ 	no intersection $\times$
$\lvert z-z_0\rvert > r$	$r < \lvert z-z_0\rvert < R$ 	$\lvert z-z_0\rvert > R$ 

	$\lvert z-z_0\rvert < R$	$\lvert z-z_0\rvert > R$
$\lvert z-z_0\rvert < r$	$\boxed{\sum_{k=0}^{\infty} a_k (z-z_0)^k}$ 	no intersection $\times$
$\lvert z-z_0\rvert > r$	$\boxed{\sum_{k=0}^{\infty} a_k (z-z_0)^k}$ $\boxed{\sum_{k=1}^{\infty} \frac{b_k}{(z-z_0)^k}}$ 	$\boxed{\sum_{k=1}^{\infty} \frac{b_k}{(z-z_0)^k}}$ 

Two Types of Series Example

$$f(z) = \frac{z}{(z-1)(z-3)} = \frac{1}{2}\left[\boxed{-\frac{1}{z-1}} + \boxed{\frac{3}{z-3}}\right]$$

partial fraction expansion

at $z=0$

① $\boxed{\frac{-1}{z-1}}$	ⓐ	$\frac{1}{1-z} = [1+z+z^2+z^3+\cdots]$	$\lvert z\rvert<1$
	ⓑ	$\frac{-1}{z(1-(\frac{1}{z}))} = -\frac{1}{z}\left[1+(\frac{1}{z})+(\frac{1}{z})^2+(\frac{1}{z})^3+\cdots\right]$	$\lvert z\rvert>1$
② $\boxed{\frac{3}{z-3}}$	ⓐ	$\frac{-1}{1-(\frac{z}{3})} = -\left[1+(\frac{z}{3})+(\frac{z}{3})^2+(\frac{z}{3})^3+\cdots\right]$	$\lvert z\rvert<3$
	ⓑ	$\frac{3}{z(1-(\frac{3}{z}))} = \frac{3}{z}\left[1+(\frac{3}{z})+(\frac{3}{z})^2+(\frac{3}{z})^3+\cdots\right]$	$\lvert z\rvert>3$

Ⅰ $\lvert z\rvert<1$ ①-ⓐ, ②-ⓐ converge $\lvert z\rvert<1$, $\lvert z\rvert<3$

$$\frac{1}{1-z} - \frac{1}{1-(\frac{z}{3})}$$

Ⅱ $1<\lvert z\rvert<3$ ①-ⓑ, ②-ⓐ converge $\lvert z\rvert>1$, $\lvert z\rvert<3$

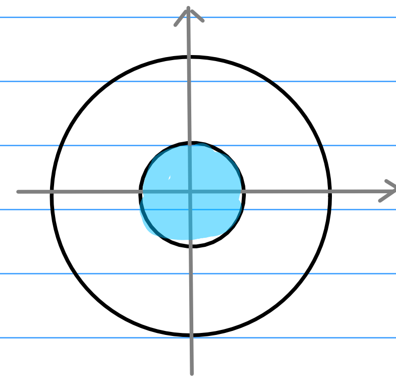
$$\frac{-1}{z(1-(\frac{1}{z}))} - \frac{1}{1-(\frac{z}{3})}$$

Ⅲ $\lvert z\rvert>3$ ①-ⓑ, ②-ⓑ converge $\lvert z\rvert>1$, $\lvert z\rvert>3$

$$\frac{-1}{z(1-(\frac{1}{z}))} + \frac{3}{z(1-(\frac{3}{z}))}$$

$$f(z)=\frac{z}{(z-1)(z-3)}=\frac{1}{2}\left[-\frac{1}{z-1}+\frac{3}{z-3}\right]$$

(a) $|z|<1$

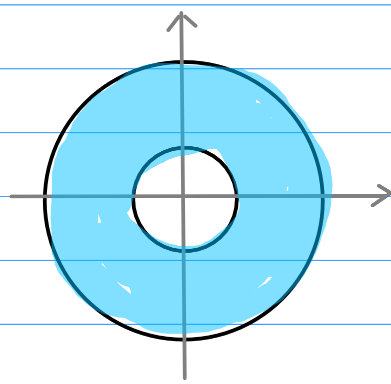


$$f(z)=\frac{1}{2}\left[\frac{1}{1-z}-\frac{1}{1-\left(\frac{z}{3}\right)}\right]$$

$$=\frac{1}{2}\left[\begin{array}{l}1+z+z^2+z^3+\cdots\\ -1-\frac{z}{3}-\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\end{array}\right]$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\left(1-\frac{1}{3^n}\right)z^n$$

(b) $1<|z|<3$

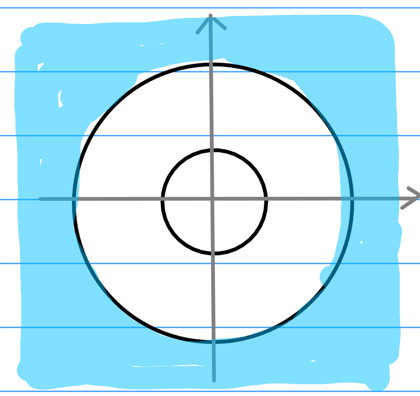


$$f(z)=\frac{1}{2}\left[-\frac{1}{z\left(1-\frac{1}{z}\right)}-\frac{1}{1-\left(\frac{z}{3}\right)}\right]$$

$$=\frac{1}{2}\left[\begin{array}{l}-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)\\ -\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right)\end{array}\right]$$

$$=\frac{1}{2}\left[-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n-\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right]$$

(c) $|z|>3$



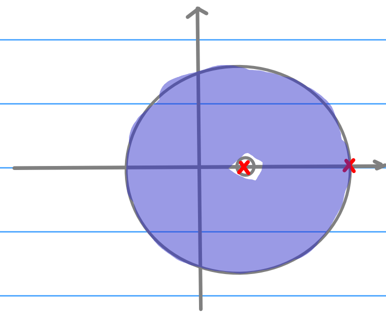
$$f(z)=\frac{1}{2}\left[-\frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{3}{z\left(1-\frac{3}{z}\right)}\right]$$

$$=\frac{1}{2}\left[\begin{array}{l}-\frac{1}{z}\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right)\\ +\frac{3}{z}\left(1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right)\end{array}\right]$$

$$=\frac{1}{2}\left[-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n+\frac{3}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n\right]$$

2nd Order pole

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$



2nd order pole

(a) $0 < |z-1| < 2$ expand at $z=1$

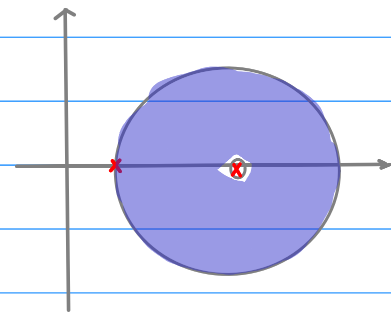
$$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-1)^2}\cdot\frac{1}{-2+(z-1)}$$

$$= \frac{1}{2(z-1)^2}\cdot\frac{1}{-1+\frac{(z-1)}{2}} = \frac{-1}{2(z-1)^2}\cdot\frac{1}{1-\frac{(z-1)}{2}}$$

$$= \frac{-1}{2(z-1)^2}\cdot\left[1+\frac{(z-1)^1}{2^1}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\cdots\right]$$

$$= -\frac{1}{2}\frac{1}{(z-1)^2} - \frac{1}{4}\frac{1}{(z-1)^1} - \frac{1}{8} - \frac{1}{16}(z-1)^1 + \cdots$$

2nd order pole



Simple pole

(b) $0 < |z-3| < 2$ expand at $z=3$

$$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-3)(2+(z-3))^2}$$

$$= \frac{1}{2(z-3)}\frac{1}{\left(1+\frac{(z-3)}{2}\right)^2} = \frac{1}{2(z-3)}\frac{d}{dz}\left[\frac{-1}{\left(1+\frac{(z-3)}{2}\right)}\right]$$

$$\frac{1}{\left(1+\frac{(z-3)}{2}\right)} = 1 - \frac{(z-3)^1}{2^1} + \frac{(z-3)^2}{2^2} - \frac{(z-3)^3}{2^3} + \cdots$$

$$\frac{d}{dz}\left[\frac{-1}{\left(1+\frac{(z-3)}{2}\right)}\right] = \left[+\frac{1}{2^1} - 2\frac{(z-3)^1}{2^2} + 3\frac{(z-3)^2}{2^3} - \cdots\right]$$

$$f(z) = \frac{1}{2(z-3)}\left[+\frac{1}{2^1} - 2\frac{(z-3)^1}{2^2} + 3\frac{(z-3)^2}{2^3} - \cdots\right]$$

$$= \frac{1}{4}\frac{1}{(z-3)} - \frac{1}{4} + \frac{9}{16}(z-3) - \cdots$$

Simple pole

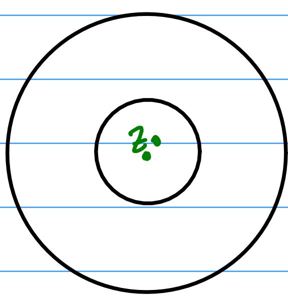
Laurent Series Coefficients a_n, b_n

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0) + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$
$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

$$f(z) = a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$
$$+ \boxed{\frac{b_1}{(z-z_0)^1}} + \boxed{\frac{b_2}{(z-z_0)^2}} + \boxed{\frac{b_3}{(z-z_0)^3}}$$
$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$
$$\boxed{a_{-1}(z-z_0)^{-1}} \quad \boxed{a_{-2}(z-z_0)^{-2}} \quad \boxed{a_{-3}(z-z_0)^{-3}}$$



$$\boxed{b_i = a_{-i}}$$

Line Integration & Laurent Series

$$\oint_c f(z)\,dz = ?$$

③ ② ①

$$f(z) = \boxed{\cdots \frac{b_2}{(z-z_0)^2}} + \boxed{\frac{b_1}{(z-z_0)^1}} + \boxed{a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \cdots}$$

$$= \boxed{\cdots a_{-2}(z-z_0)^{-2}} + \boxed{a_{-1}(z-z_0)^{-1}} + \boxed{a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \cdots}$$

$$= \boxed{f_3(z)} + \boxed{f_2(z)} + \boxed{f_1(z)}$$

① no singularities inside c

$$\oint_c f_1(z)\,dz = 0$$

② pole $z = z_0$

$$\oint_c f_2(z)\,dz = 2\pi i \cdot a_{-1}$$

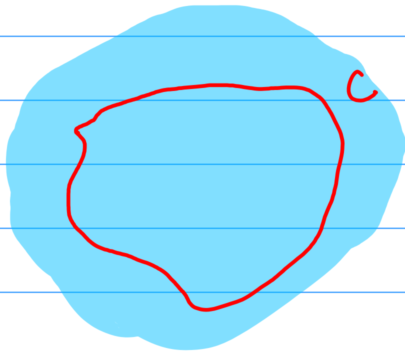
the only remaining (non-zero) Residue at $z = z_0$

③ pole $z = z_0$

$$\oint_c f_3(z)\,dz = 0$$

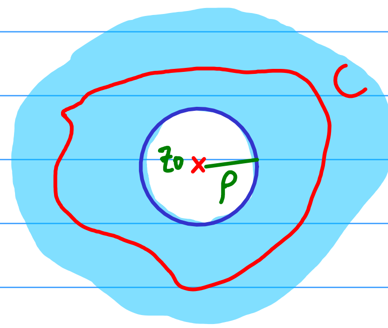
$(n > 0)$

① no pole inside C



$(n < 0)$

②, ③ a pole inside C



$$\boxed{\begin{aligned} z &= z_0 + \rho e^{i\theta} \\ dz &= i\rho e^{i\theta}\, d\theta \end{aligned}}$$

① $\oint (z-z_0)^1\, dz = \int_0^{2\pi} i(\rho e^{i\theta})^2\, d\theta = 0$ $\qquad n = 1 \quad (n>0)$

$(1, 2, 3, \cdots)$

② $\oint \frac{1}{(z-z_0)^1}\, dz = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}}\, d\theta = 2\pi i$ $\qquad n = -1$

$(n<0)$

③ $\oint \frac{1}{(z-z_0)^2}\, dz = \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho^2 e^{i2\theta}}\, d\theta = 0$ $\qquad n \neq -1$

$(-2, -3, \cdots)$

Laurent Series Coefficients

$$f(z) = \boxed{\cdots \frac{b_2}{(z-z_0)^2}} + \boxed{\frac{b_1}{(z-z_0)^1}} + \boxed{a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \cdots}$$

$$= \boxed{\cdots a_{-2}(z-z_0)^{-2}} + \boxed{a_{-1}(z-z_0)^{-1}} + \boxed{a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \cdots}$$

$$a_{-1} = \frac{1}{2\pi i}\oint f(z)\,dz$$

$$\frac{f(z)}{(z-z_0)} = \boxed{\cdots \frac{b_1}{(z-z_0)^2}} + \boxed{\frac{a_0}{(z-z_0)^1}} + \boxed{a_1 + a_2(z-z_0)^1 + a_3(z-z_0)^2 + \cdots}$$

$$= \boxed{\cdots a_{-1}(z-z_0)^{-2}} + \boxed{a_0(z-z_0)^{-1}} + \boxed{a_1 + a_2(z-z_0)^1 + a_3(z-z_0)^2 + \cdots}$$

$$a_0 = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)}\,dz$$

$$\boxed{a_k = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{k+1}}\,dz}$$

z_0 : the center of Laurent series expansion

Finding a_k

$$f(z) = \cdots \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)^1} + a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \cdots$$

$$= \cdots a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \cdots$$

$$\frac{f(z)}{(z-z_0)^3} = \frac{a_0}{(z-z_0)^3} + a_1\frac{(z-z_0)^1}{(z-z_0)^3} + a_2\frac{(z-z_0)^2}{(z-z_0)^3} + a_3\frac{(z-z_0)^3}{(z-z_0)^3} +$$

$$+ \frac{b_1}{(z-z_0)^{1+3}} + \frac{b_2}{(z-z_0)^{2+3}} + \frac{b_3}{(z-z_0)^{3+3}}$$

$$\frac{f(z)}{(z-z_0)^3} = \frac{a_0}{(z-z_0)^3} + \frac{a_1}{(z-z_0)^2} + \frac{a_2}{(z-z_0)} + a_3 + a_4(z-z_0) + \cdots$$

$$+ \frac{b_1}{(z-z_0)^4} + \frac{b_2}{(z-z_0)^5} + \frac{b_3}{(z-z_0)^6} + \cdots$$

$$\oint \frac{f(z)}{(z-z_0)^3}\,dz = \oint \frac{a_2}{(z-z_0)}\,dz = 2\pi i\, a_2$$

$$\underset{k}{a_2} = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{\underset{k+1}{3}}}\,dz$$

$a_k, \quad k = \cdots -2, -1, 0, 1, 2, \cdots$

$\boxed{a_k} \qquad \boxed{\dfrac{f(z)}{(z-z_0)^{k+1}}}$

$\cdots$

$$a_3 = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^4} dz$$

$$a_2 = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^3} dz$$

$$a_1 = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^2} dz$$

$$a_0 = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^1} dz$$

$$a_{-1} = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^0} dz$$

$$a_{-2} = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{-1}} dz = \frac{1}{2\pi i} \oint f(z)(z-z_0)^1 dz$$

$$a_{-3} = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{-2}} dz = \frac{1}{2\pi i} \oint f(z)(z-z_0)^2 dz$$

$\cdots$

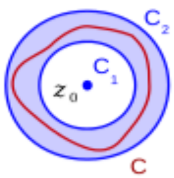
Coefficients a_n & b_n

$$f(z) = \cdots + \boxed{a_n (z-z_0)^n} + \cdots$$

$$\frac{f(z)}{(z-z_0)^{n+1}} = \cdots + \boxed{\frac{a_n}{(z-z_0)}} + \cdots$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \oint_C \frac{a_n}{(z-z_0)}\, dz$$

$$= 2\pi i \cdot a_n$$



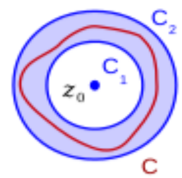
$$\boxed{a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz}$$

$$f(z) = \cdots + \boxed{\frac{b_n}{(z-z_0)^n}} + \cdots$$

$$\frac{f(z)}{(z-z_0)^{-n+1}} = \cdots + \boxed{\frac{b_n}{(z-z_0)}} + \cdots$$

$$\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\, dz = \oint_C \frac{b_n}{(z-z_0)}\, dz$$

$$= 2\pi i \cdot b_n$$



$$\boxed{b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\, dz}$$

$\parallel$

a_{-n}

$$-\infty \longleftarrow \qquad \longrightarrow +\infty$$

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0) + \cdots$$

$$\oint \frac{f(z)}{(z-z_0)^{2+1}}\, dz = \quad 0 \quad 0 \quad \cdots\cdots \quad 0 \quad \cdots + \oint \frac{a_2}{(z-z_0)}\, dz + \cdots$$

$$a_2 = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{2+1}}\, dz$$

$$\oint f(z)\, dz = \oint \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0) \cdots dz$$

$$= \oint \frac{a_{-1}}{(z-z_0)}\, dz = 2\pi i \cdot a_{-1}$$

Taylor Series Coefficients

A **power series** in powers of $(z-z_0)$ non-negative powers

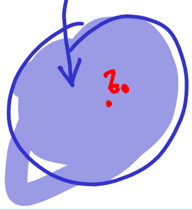
$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

The **Taylor series** of a function $f(z)$ non-negative powers

$n \geq 0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad a_n = \frac{1}{n!}\boxed{f^{(n)}(z_0)}$$

f(z) analytic



$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{n+1}}\,dw$$

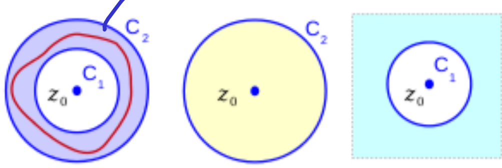
$$\boxed{a_n = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw}$$

the same form

Laurent's Theorem and Coefficients

f(z)

$f(z)$: **analytic** in the annular domain D
between concentric circles C_1 and C_2
centered at z_0 $\qquad r < |z-z_0| < R$



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$$

convergent in the domain D

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$$

any simple closed path C in D

$n \in (-\infty, +\infty)$

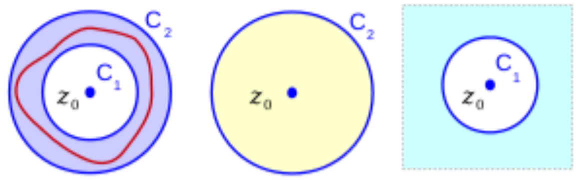
$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z-z_0)^k \qquad \boxed{a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz}$$

Laurent's Theorem and Coefficients

$f(z)$: **analytic** in the annular domain D

between concentric circles C_1 and C_2

centered at z_0 $\qquad r < |z - z_0| < R$



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$$

convergent in the domain D

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$$

any simple closed path C in D

$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z-z_0)^k$$

$$\boxed{a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz}$$

Laurent Series $\quad a_k = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{(z-z_0)^{k+1}}\,dz \qquad -\infty < k < +\infty$

Taylor Series $\quad a_n = \dfrac{1}{2\pi i}\oint_C \dfrac{f(w)}{(w-z_0)^{k+1}}\,dw \qquad n \geq 0$

$$f(z) = \frac{f(0)}{0!}z^0 + \frac{f^{(1)}(0)}{1!}z^1 + \frac{f^{(2)}(0)}{2!}z^2 + \frac{f^{(3)}(0)}{3!}z^3 + \frac{f^{(4)}(0)}{4!}z^4 + \cdots$$

Taylor Series : z^k $\{k \geq 0\}$ → $\boxed{\text{ROC: inside a circle}}$

→ analytic everywhere in the ROC

$$\sin(z) = z^1 - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots$$

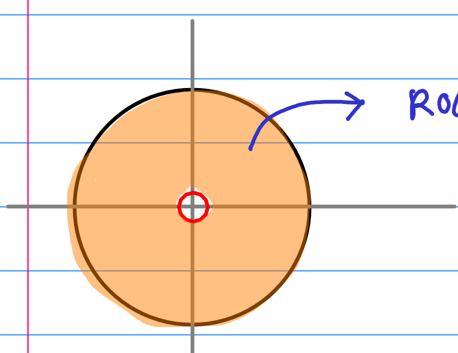
→ entire function : any R, analytic → Taylor Expansion possible

$$\frac{\sin(z)}{z^3} = \underbrace{\frac{1}{z^2}}_{\text{b-series}} \underbrace{- \frac{1}{3!} + \frac{1}{5!}z^2 - \frac{1}{7!}z^4 + \cdots}_{\text{a-series}}$$

b-series: principal part, negative powers

a-series: analytic part

isolated singularity at $z=0$



ROC : excluding $z=0$

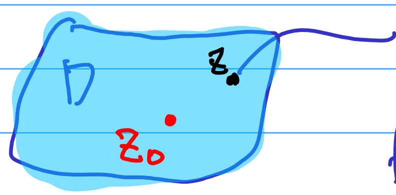
inside a circle ROC

negative power → Taylor Series ✗

→ must use Laurent Series

* Taylor Series Expansion

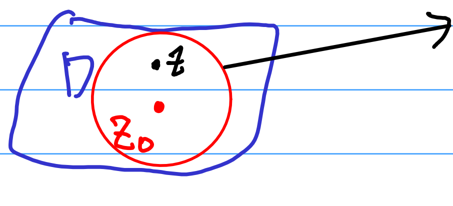
~~Singular~~



z any point in D

$f(z)$: analytic in D

no singularity in D



z_0 : expansion point

the largest circle centered at z_0

ROC: inside the largest circle

in this ROC, the Taylor series converges

$\Rightarrow$ $f(z) \neq \infty$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

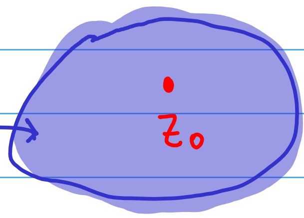
any point z in D : analytic

Cauchy's Integral Formula

$$f(z) = \frac{1}{2\pi i} \oint_C \boxed{\frac{f(s)}{s - z}} ds$$

Cauchy's Integration Formula

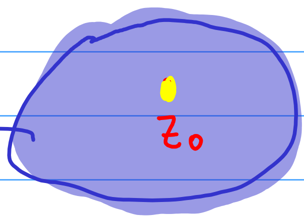
$f(z)$ is analytic here



$$f(z_0) = \frac{1}{2\pi i} \oint_C \boxed{\frac{f(z)}{z - z_0}} \, dz \qquad (\mathrm{I})$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \boxed{\frac{f(z)}{(z - z_0)^{n+1}}} \, dz \qquad (\mathrm{II})$$

$\frac{f(z)}{z - z_0}$ is analytic, excluding z_0



$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)} \, dz$$

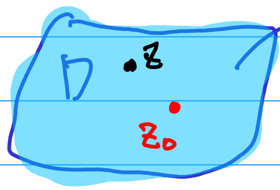
$$f^{(1)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \, dz$$

$$f^{(2)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3} \, dz$$

$$f^{(3)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^4} \, dz$$

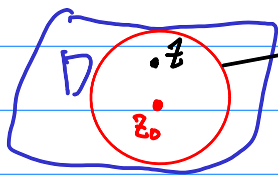
at z_0, $\boxed{\frac{f(z)}{z - z_0}}$ not analytic, nor differentiable, nor continuous

C may any simple closed path enclosing the point of evaluation z



$f(z)$ is analytic, here

z any point in D



the largest circle C centered at z_0

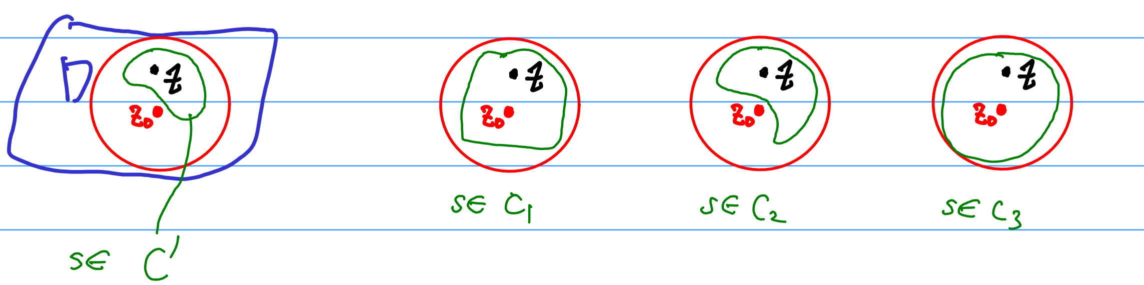
ROC: inside this circle

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

Taylor series expansion

* Taylor Series expansion at z_0 inside the circle C

* a point of evaluation : z enclosed by the contour C'

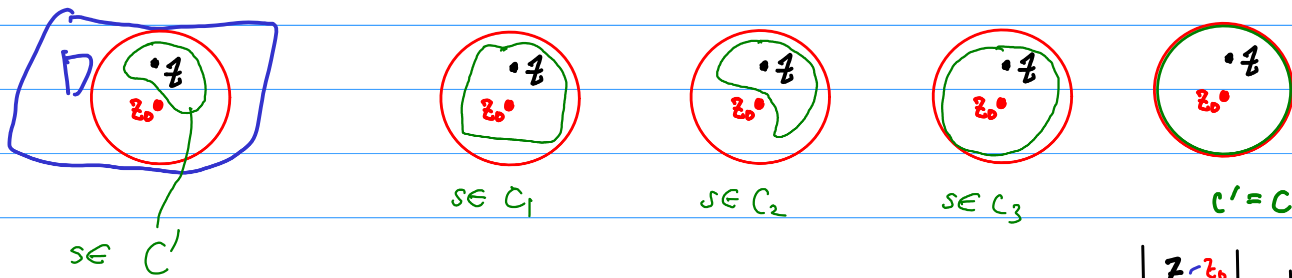


$$f(z) = \frac{1}{2\pi i} \oint_{C'} \boxed{\frac{f(s)}{s - z}} ds$$

$$= \frac{1}{2\pi i} \oint_{C_i} \boxed{\frac{f(s)}{s - z}} ds$$

for any Countour C' in D enclosing z

any point z in D ← Cauchy Int.



$$\left|\frac{z-z_0}{s-z_0}\right|<1$$

* $(s-z)$

$$= (s-z_0) - (z-z_0)$$

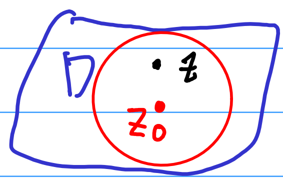
$$= (s-z_0)\left\{1-\left(\frac{z-z_0}{s-z_0}\right)\right\}$$

* Taylor Series expansion at z_0 inside the circle C
* a point of evaluation : z enclosed by the contour C'
* s is any points on the contour C'

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$$

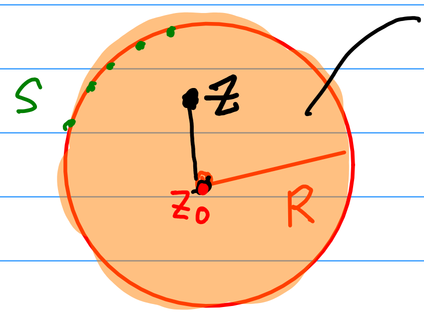
$$\left[\frac{1}{1-\left(\frac{z-z_0}{\zeta-z_0}\right)}\right] = 1 + \left(\frac{z-z_0}{\zeta-z_0}\right) + \left(\frac{z-z_0}{\zeta-z_0}\right)^2 + \left(\frac{z-z_0}{\zeta-z_0}\right)^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \left(\frac{z-z_0}{\zeta-z_0}\right)^k \qquad \left|\frac{z-z_0}{\zeta-z_0}\right| < 1$$



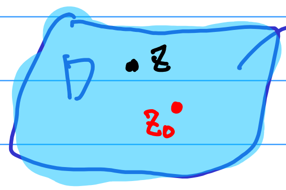
Taylor Series $z \in D$

$D: |z - z_0| < R$ ROC



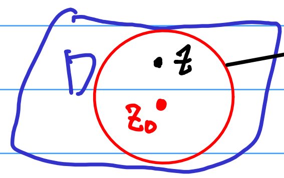
ROC – Taylor Series converges

let's select $C' = C$



$f(z)$: analytic in D

z any point in D



the largest circle C

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

any point z in ROC — a point of evaluation

$$f(z) = \frac{1}{2\pi i} \oint_C \boxed{\frac{f(s)}{s - z}} ds \quad \Leftarrow \text{ Cauchy Integration}$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(s)}{s - z_0} \left[\frac{1}{1 - \left(\frac{z - z_0}{s - z_0} \right)} \right] ds$$

$$\boxed{\left| \frac{z - z_0}{s - z_0} \right| < 1}$$

$$\Downarrow$$

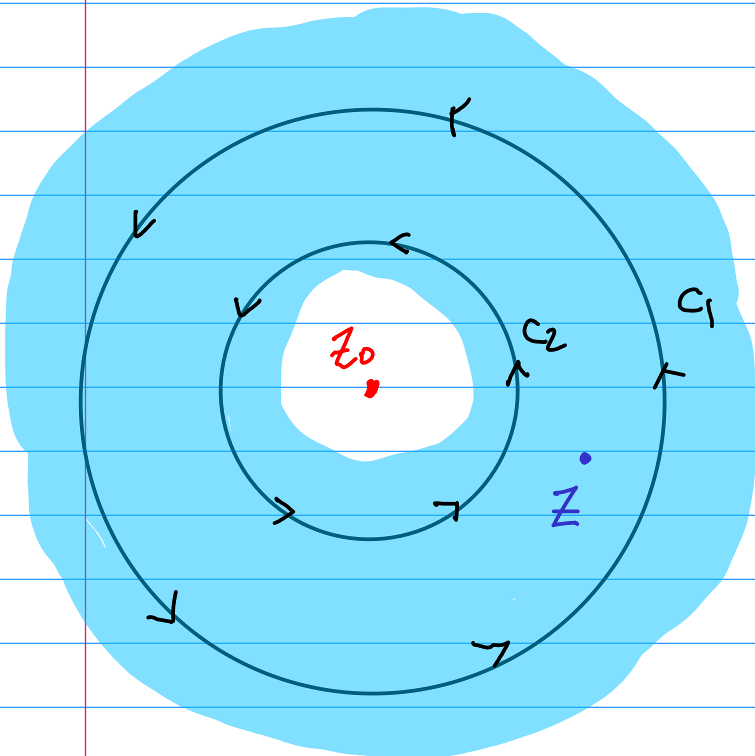
$$= \frac{1}{2\pi i} \oint_C \frac{f(s)}{s - z_0} \left[\sum_{k=0}^{\infty} \left(\frac{z - z_0}{s - z_0} \right)^k \right] ds$$

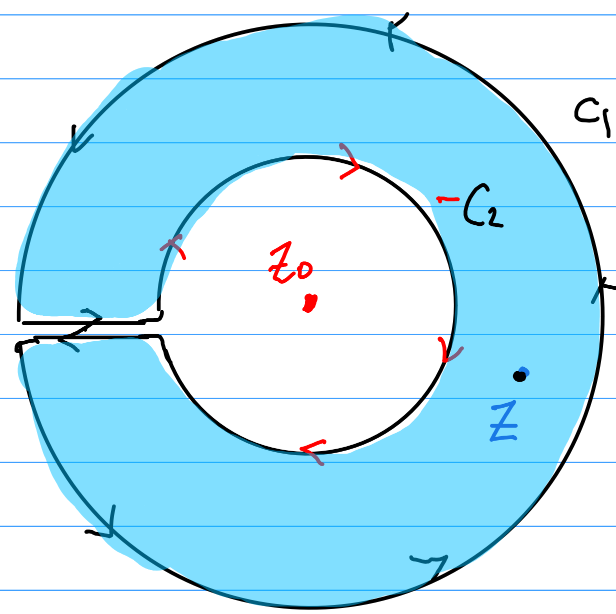
$$= \sum_{k=0}^{\infty} \boxed{\frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds} \; (z - z_0)^k$$

$$\parallel$$

$$\sum_{k=0}^{\infty} \boxed{\frac{f^{(n)}(z_0)}{n!}} (z - z_0)^k$$

Laurent's Theorem





$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(s)}{s-z}\,ds - \frac{1}{2\pi i}\oint_{C_2} \frac{f(s)}{s-z}\,ds$$

Geometric Series

$$\frac{1}{z'-z} = \frac{1}{z'-z_0+z_0-z} = \frac{1}{(z'-z_0)-(z-z_0)}$$

I $$\frac{1}{(z'-z_0)-(z-z_0)} = \frac{1}{(z'-z_0)\left[1-\frac{(z-z_0)}{(z'-z_0)}\right]}$$

☆ $\boxed{\text{under the condition } \frac{(z-z_0)}{(z'-z_0)}<1 \quad z'\neq z_0}$

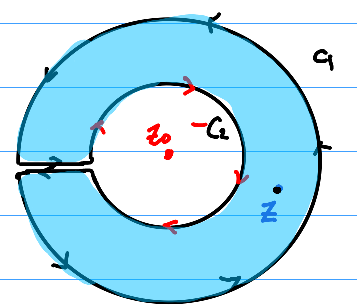
inside the large circle

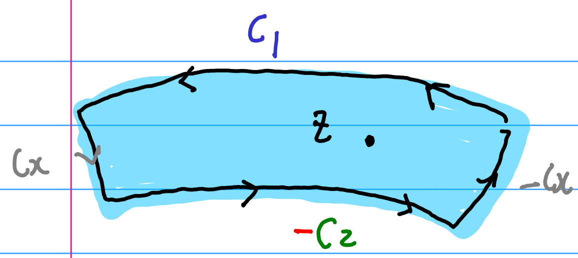
II $$\frac{1}{(z'-z_0)-(z-z_0)} = \frac{1}{\left[\frac{(z'-z_0)}{(z-z_0)}-1\right](z-z_0)}$$

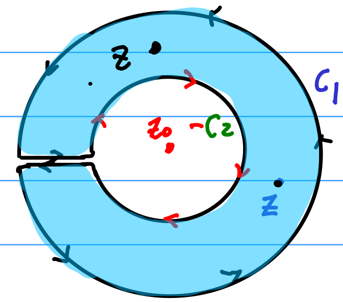
☆ $\boxed{\text{under the condition } \frac{(z'-z_0)}{(z-z_0)}<1 \quad z\neq z_0}$

outside the small circle

$$\frac{1}{1-r} \Rightarrow 1+r+r^2+r^3+\cdots\cdots$$





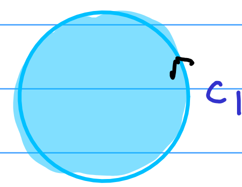


$$f(z) = \boxed{\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{s-z}\,ds} - \boxed{\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s-z}\,ds}$$

$s \in C_1$ $\qquad$ $s \in C_2$

$$\boxed{\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{s-z}\,ds} = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

positive powers

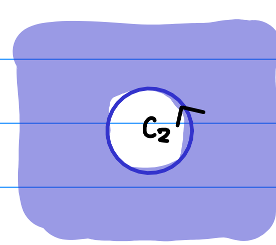


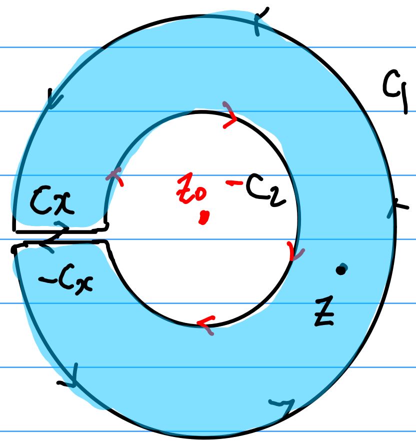
inside the large circle

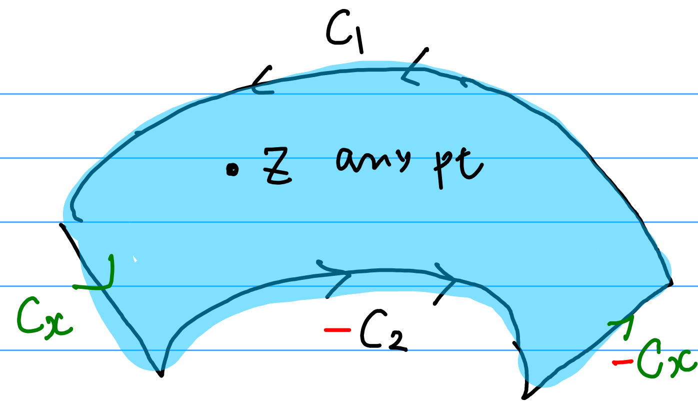
$$-\boxed{\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s-z}\,ds} = \sum_{k=1}^{\infty} b_k \frac{1}{(z-z_0)^k}$$

negative power

outside the small circle







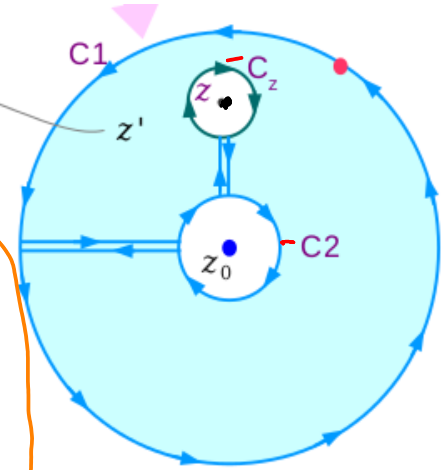
$$f(z) = \frac{1}{2\pi i} \oint_C \boxed{\frac{f(z)}{z - z}} \, dz$$

$$f(z) = \frac{1}{2\pi i} \oint_{C_1 + C_x - C_2 - C_x} \boxed{\frac{f(s)}{s - z}} \, ds$$

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(s)}{s - z} \, ds - \frac{1}{2\pi i} \oint_{C_2} \frac{f(s)}{s - z} \, ds$$

z_0 the center
z the evaluation point
z' the contour point

$f(z')$	: **analytic**	for all z'	in this region
$\dfrac{f(z')}{(z'-z)}$	: **analytic**	for all z'	in this region z is excluded

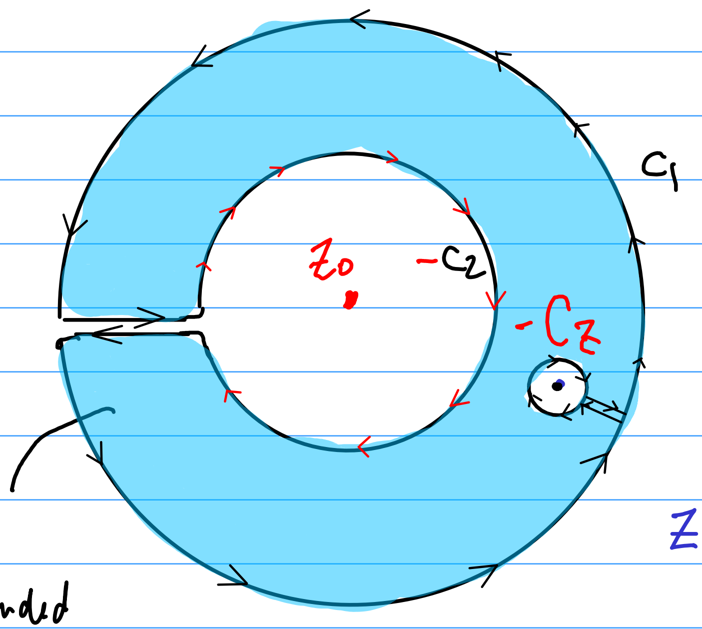


$$\underbrace{\boxed{\frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}dz'}}_{C_1} = \underbrace{\boxed{\frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}dz'}}_{C_2} + \underbrace{\boxed{\frac{1}{2\pi i}\oint_{Cz}\frac{f(z')}{z'-z}dz'}}_{C_z} \Rightarrow \boxed{f(z)}$$

apply geometric series

Cauchy Integration Formula

a point of evaluation z

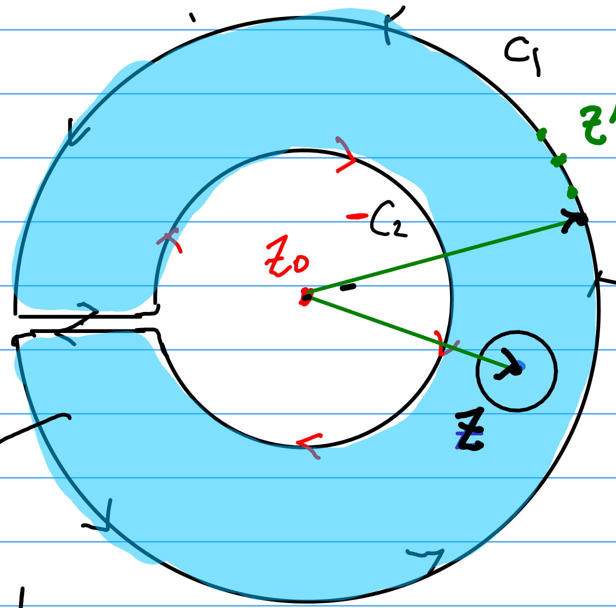


here, z & z_0 are excluded

$\boxed{z' \in C_2}$ Big Circle

a point of evaluation z

$$\left| \frac{z - z_0}{z' - z_0} \right| < 1$$

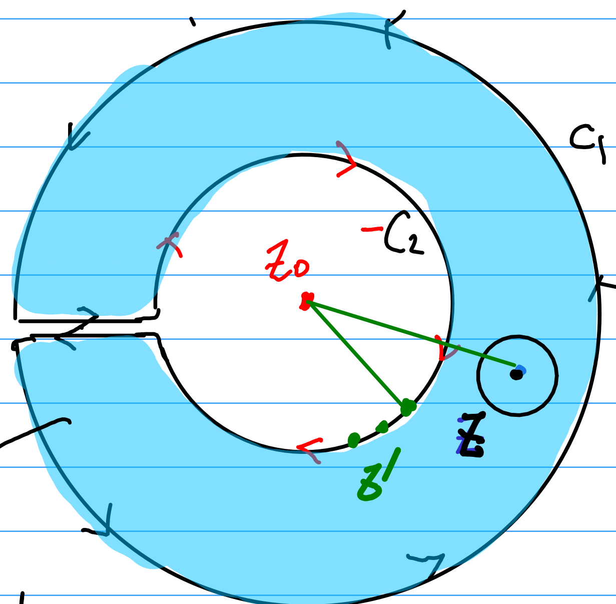


here, z & z_0 are excluded

$\boxed{z' \in C_1}$ small circle

a point of evaluation z

$$\left| \frac{z' - z_0}{z - z_0} \right| < 1$$



here, z & z_0 are excluded

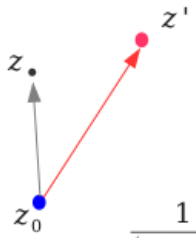
$$z'-z = (z'-z_0)+(z_0-z) = (z'-z_0)\left\{1-\frac{(z-z_0)}{(z'-z_0)}\right\}$$

$$-(z'-z) = (z_0-z')+(z-z_0) = \left\{-\frac{(z'-z_0)}{(z-z_0)}+1\right\}\cdot(z-z_0)$$

Cz C1 C2

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_{Cz}\frac{f(z')}{z'-z}\,dz'} = \frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}\,dz'$$

Big Circle (C_1)



$$\left|\frac{(z-z_0)}{(z'-z_0)}\right| < 1$$

$$\frac{1}{(z'-z)} = \frac{1}{(z'-z_0)}\cdot\frac{1}{1-\left(\frac{z-z_0}{z'-z_0}\right)}$$

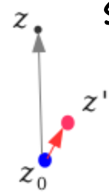
$$= \frac{1}{(z'-z_0)}\cdot\sum_{n=0}^{+\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$$

$$= \boxed{\sum_{n=0}^{+\infty}\frac{(z-z_0)^n}{(z'-z_0)^{n+1}}}$$

$$= \sum_{k=0}^{+\infty}\frac{(z-z_0)^k}{(z'-z_0)^{k+1}}$$

$$\boxed{\frac{1}{(z'-z)} = \sum_{k=0}^{+\infty}\frac{(z-z_0)^k}{(z'-z_0)^{k+1}}}$$

Small Circle (C_2)



$$\left|\frac{(z'-z_0)}{(z-z_0)}\right| < 1$$

$$-\frac{1}{(z'-z)} = \frac{1}{(z-z_0)}\cdot\frac{1}{1-\left(\frac{z'-z_0}{z-z_0}\right)}$$

$$= \frac{1}{(z-z_0)}\cdot\sum_{n=0}^{+\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n$$

$$m = n+1 \Rightarrow = \boxed{\sum_{n=0}^{+\infty}\frac{(z'-z_0)^n}{(z-z_0)^{n+1}}} = \sum_{m=1}^{+\infty}\frac{(z'-z_0)^{m-1}}{(z-z_0)^m}$$

$$k = -m \Rightarrow = \sum_{k=-\infty}^{-1}\frac{(z-z_0)^k}{(z'-z_0)^{k+1}} \Leftarrow \sum_{m=1}^{\infty}\frac{(z-z_0)^{-m}}{(z'-z_0)^{-m-1}}$$

$$-m-1 = -k-1 = -(k+1)$$

$$\boxed{\frac{-1}{(z'-z)} = \sum_{k=-\infty}^{-1}\frac{(z-z_0)^k}{(z'-z_0)^{k+1}}}$$

☆ $\boxed{\text{under the condition } \frac{(z-z_0)}{(z'-z_0)} < 1 \quad z' \neq z_0}$

☆ $\boxed{\text{under the condition } \frac{(z'-z_0)}{(z-z_0)} < 1 \quad z \neq z_0}$

analyticity at z_0 was not required !!

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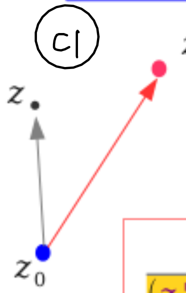
$$\overset{Cz}{\boxed{f(z) = \frac{1}{2\pi i}\oint_{Cz}\frac{f(z')}{z'-z}\,dz'}} = \overset{C_1}{\boxed{\frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}\,dz'}}\ \overset{C_2}{\boxed{-\frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}\,dz'}}$$

c1



$$\boxed{\frac{(z-z_0)}{(z'-z_0)} < 1}$$

$$\boxed{\frac{1}{(z'-z)} = \sum_{k=0}^{+\infty}\frac{(z-z_0)^k}{(z'-z_0)^{k+1}}}$$

c2

$$\boxed{\frac{(z'-z_0)}{(z-z_0)} < 1}$$

$$\boxed{\frac{-1}{(z'-z)} = \sum_{k=-\infty}^{-1}\frac{(z-z_0)^k}{(z'-z_0)^{k+1}}}$$

$$\frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}\,dz'$$

$$= \frac{1}{2\pi i}\oint_{C1} f(z')\boxed{\left[\sum_{k=0}^{+\infty}\frac{(z-z_0)^k}{(z'-z_0)^{k+1}}\right]_{0}^{\infty}}\,dz'$$

$$-\frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}\,dz'$$

$$= \frac{1}{2\pi i}\oint_{C2} f(z')\boxed{\left[\sum_{k=-\infty}^{-1}\frac{(z-z_0)^k}{(z'-z_0)^{k+1}}\right]_{-\infty}^{-1}}\,dz'$$

$$f(z) = \frac{1}{2\pi i}\oint_{Cz}\frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\oint_{C} f(z')\left[\sum_{k=-\infty}^{+\infty}\frac{(z-z_0)^k}{(z'-z_0)^{k+1}}\right]_{-\infty}^{+\infty}dz' = \boxed{\sum_{n=-\infty}^{+\infty}\left[\frac{1}{2\pi i}\oint_{C}\frac{f(z')}{(z'-z_0)^{k+1}}\,dz'\right]}(z-z_0)^k$$

From the geometric series properties,

$$\boxed{\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s-z}\,ds} = \sum_{k=0}^{\infty}\boxed{a_k}(z-z_0)^k \Rightarrow \boxed{a_k} = \boxed{\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z_0)^{k+1}}\,ds}$$

$$-\boxed{\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{s-z}\,ds} = \sum_{k=1}^{\infty}\frac{\boxed{a_{-k}}}{(z-z_0)^k} \Rightarrow \boxed{a_{-k}} = \boxed{\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{(s-z_0)^{-k+1}}\,ds}$$

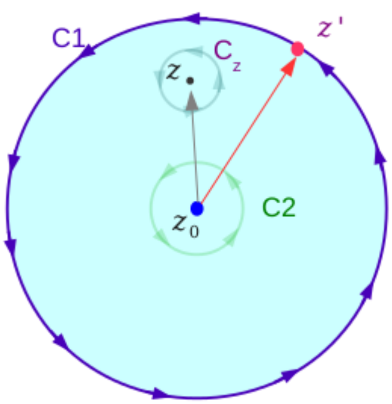
Laurent Series Coefficients

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}$$

analytic part — pricipal part

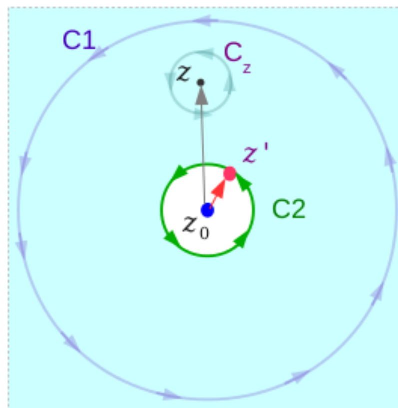
$$\boxed{\frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}dz'} = \boxed{\frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}dz'} + \boxed{\frac{1}{2\pi i}\oint_{Cz}\frac{f(z')}{z'-z}dz'}$$

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_{Cz}\frac{f(z')}{z'-z}dz'} = \boxed{\frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}dz'} - \boxed{\frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}dz'}$$



$$\frac{(z-z_0)}{(z'-z_0)} < 1$$

$$\boxed{\frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}dz'}$$



$$\frac{(z-z_0)}{(z'-z_0)} > 1$$

$$\frac{(z'-z_0)}{(z-z_0)} < 1$$

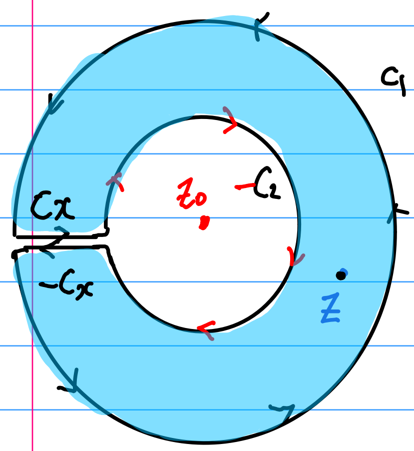
$$\boxed{-\frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}dz'}$$

a_k of the analytic part is analytic inside of C_1

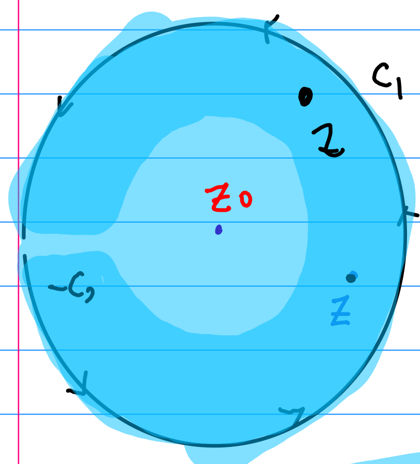
Taylor series expansion at z_0.

a_{-k} of the principal part is analytic outside of C_2

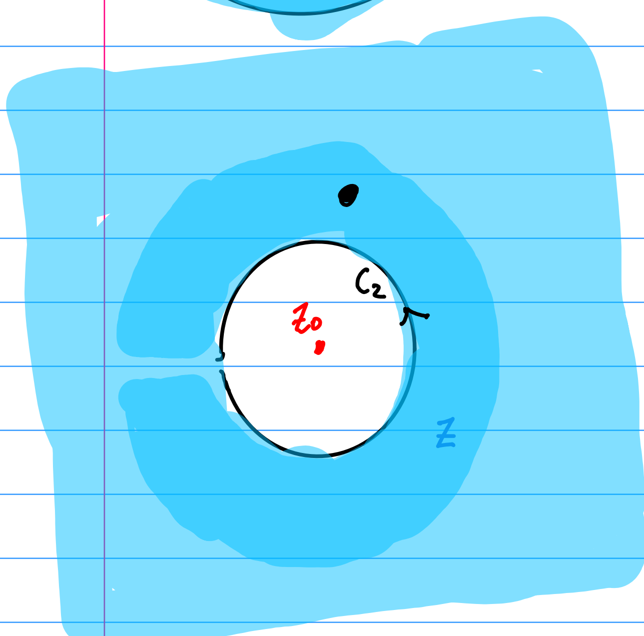
reciprocal – Taylor series



$$\boxed{f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{s-z}\,ds - \frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s-z}\,ds}$$

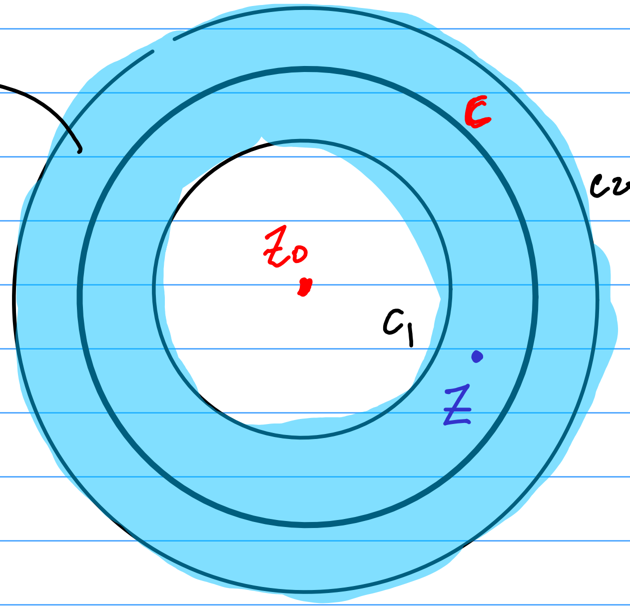


$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{s-z}\,ds \rightarrow \text{Taylor series}$$



$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s-z}\,ds \rightarrow \text{Reciprocal Taylor Series}$$

f(z) is analytic in D



$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \qquad \text{valid in } D$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}}\, ds$$

$$\boxed{\begin{array}{l}\text{Laurent expansion is never valid at } z_0 \\ \qquad\qquad\qquad\quad \text{never evaluated at } z_0\end{array}}$$

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