Substitution (4A)

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Substitution Method

Substitution Method (1)

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

$$y = h(x, u) \qquad u = \Phi(x)$$

$$z = f(x(t), y(t))$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x}\frac{dx}{dx} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx} = h_x(x,u) + h_u(x,u)\frac{du}{dx}$$

$$\frac{dy}{dx} = g(x, y) = g(x, h(x,u))$$

$$\frac{dy}{dx} = g(x, y) = g(x, h(x,u))$$

$$g(x, h(x,u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$g(x, h(x,u)) = h_x(x,y) + h_u(x,u)\frac{du}{dx}$$

Substitution Method (2)

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

$$y = h(x, u)$$

$$u = \Phi(x)$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{d}{dx} y = \frac{d}{dx} h(x, u)$$

$$g(x, h(x, u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$g(x, h(x, u)) = h_x(x, y) + h_u(x, u)\frac{du}{dx}$$

Substitution Method Example

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = s\left(\frac{y}{x}\right)$$

$$y' = s\left(\frac{y}{x}\right)$$

$$y = ux \qquad u = \frac{y}{x}$$

$$y = h(x, u) \Leftarrow u = \Phi(x)$$

$$y' = u + xu'$$

$$y' = g(x, ux) = s(u)$$

$$\frac{dy}{dx} \Leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$s(u) = u + xu'$$

$$y \Leftarrow h(x, u)$$

$$s(u) - u = xu'$$

$$\Rightarrow \frac{du}{s(u) - u} = \frac{dx}{x}$$

Substitute Equation

a new literal　　a function of x

$$u = \Phi(x)$$

contains x and y literals
(y is also a function of x)

a new literal u is introduced
using old literals x and y :
a new function of x

$$u = \frac{y}{x} = \frac{y(x)}{x}$$

a old literal　　a function of x and u

$$y = h(x, u)$$

the old literal y can be replaced by
the new literal u and the old literal x:
a new function of u and x

$$y = u\,x$$

New Differential Equation

(1) replace y'

$$\frac{dy}{dx} \Leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$y' = u + xu' \Leftarrow y = ux \Leftarrow u = \frac{y}{x}$$

(2) replace y

$$g(x, y) \Leftarrow g(x, h(x, u))$$

$$y' = g(x, ux) = s(u) \Leftarrow \frac{dy}{dx} = s\left(\frac{y}{x}\right)$$

Find $y = f(x)$ **in**

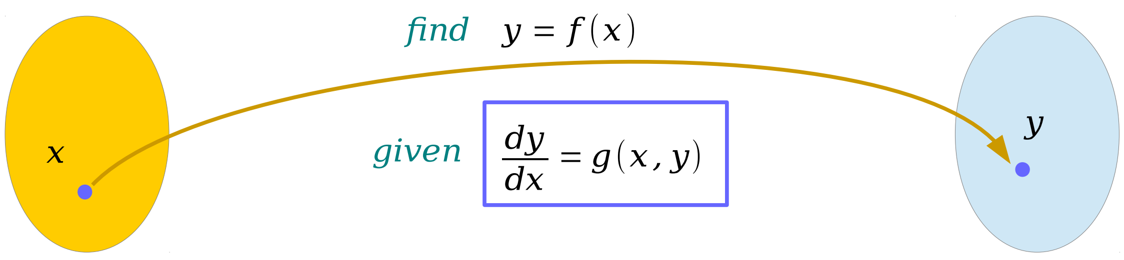
$$\frac{dy}{dx} = g(x, y)$$

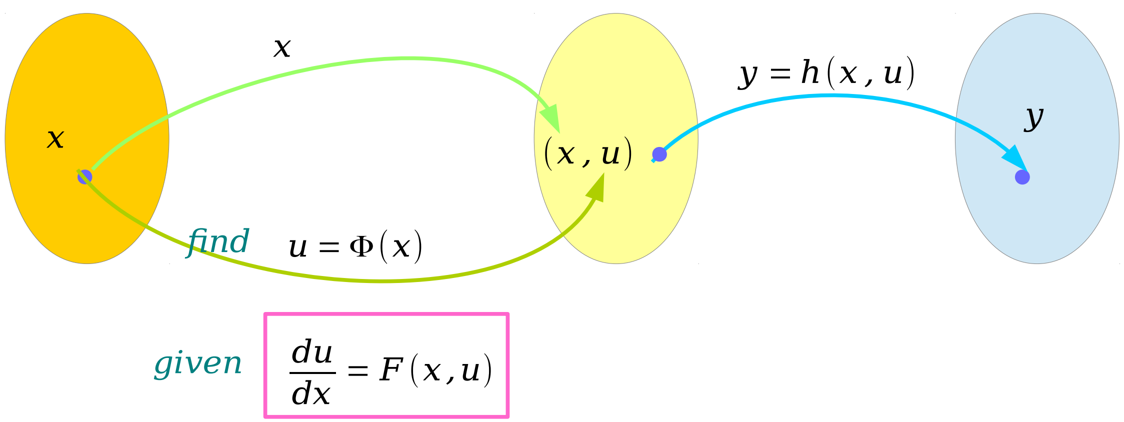
Find $u = \Phi(x)$ **in**

$$\frac{du}{dx} = F(x, u)$$

new differential equation

Function point of view





$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

(1) replace y'

$$\frac{dy}{dx} \Leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

(2) replace y

$$g(x, y) \Leftarrow g(x, h(x, u))$$

ODE point of view

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

$$y = h(x, \Phi(x))$$

$$y = h(x, u) \qquad u = \Phi(x)$$

$$g(x, h(x,u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$g(x, h(x,u)) = h_x(x,u) + h_u(x,u)\frac{du}{dx}$$

$$\frac{du}{dx} = \left\{ g(x, h(x,u)) - \frac{\partial h}{\partial x} \right\} / \frac{\partial h}{\partial u}$$

$$\frac{du}{dx} = \{ g(x, h(x,u)) - h_x(x,u) \} / h_u(x,u)$$

$$\frac{du}{dx} = F(x, u)$$

$$\frac{du}{dx} = F(x, u)$$

$$u = \Phi(x)$$

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4/7/15

Homogeneous First Order ODEs

Homogeneous Functions

A homogeneous function of degree α

$$f(tx, ty) = t^{\alpha} f(x, y)$$

$$f(x, y) = x^2 + y^2$$

$$f(tx, ty) = (tx)^2 + (ty)^2$$

$$= t^2 (x^2 + y^2)$$

$$= t^2 f(x, y)$$

A homogeneous Equations of degree α

$$M(x, y)dx + N(x, y)dy = 0$$

$$M(tx, ty) = t^{\alpha} M(x, y)$$
$$N(tx, ty) = t^{\alpha} N(x, y)$$

$$M(x, y) = M(x, x \cdot y/x) = x^{\alpha} M(1, y/x)$$
$$M(x, y) = M(y \cdot x/y, y) = y^{\alpha} M(x/y, 1)$$

$$N(x, y) = N(x, x \cdot y/x) = x^{\alpha} N(1, y/x)$$
$$N(x, y) = N(y \cdot x/y, y) = y^{\alpha} N(x/y, 1)$$

Homogeneous Equations (1)

A *homogeneous* Equations of *degree α*

$$M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) = x^{\alpha} M(1, y/x)$$

$$N(x,y) = x^{\alpha} N(1, y/x)$$

$$u = y/x \qquad y = u\,x$$

$$x^{\alpha} M(1, y/x)dx + x^{\alpha} N(1, y/x)dy = 0$$

$$M(1,u)dx + N(1,u)dy = 0$$

$$dy = \frac{\partial}{\partial x}(u\,x)dx + \frac{\partial}{\partial u}(u\,x)du$$

$$dy = u\,dx + x\,du$$

$$M(1,u)dx + N(1,u)(u\,dx + x\,du) = 0$$

A *homogeneous* Equations of *degree α*

$$M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) = y^{\alpha} M(x/y, 1)$$

$$N(x,y) = y^{\alpha} N(x/y, 1)$$

$$v = x/y \qquad x = v\,y$$

$$y^{\alpha} M(x/y, 1)dx + y^{\alpha} N(x/y, 1)dy = 0$$

$$M(v,1)dx + N(v,1)dy = 0$$

$$dx = \frac{\partial}{\partial y}(v\,y)dy + \frac{\partial}{\partial v}(vy)dv$$

$$dx = v\,dy + y\,dv$$

$$M(v,1)(v\,dy + y\,dv) + N(v,1)dy = 0$$

Homogeneous Equations (2)

A homogeneous Equations of degree α

$$M(x,y)dx + N(x,y)dy = 0$$

$$x^{\alpha} M(1, y/x)dx + x^{\alpha} N(1, y/x)dy = 0$$

$$M(1,u)dx + N(1,u)dy = 0$$

$$u = y/x \qquad y = ux$$

$$dy = u\,dx + x\,du$$

$$M(1,u)dx + N(1,u)(u\,dx + x\,du) = 0$$

$$[M(1,u) + u\,N(1,u)]dx + x\,N(1,u)du = 0$$

$$\frac{dx}{x} + \frac{N(1,u)\,du}{[M(1,u) + u\,N(1,u)]} = 0$$

A homogeneous Equations of degree α

$$M(x,y)dx + N(x,y)dy = 0$$

$$y^{\alpha} M(x/y, 1)dx + y^{\alpha} N(x/y, 1)dy = 0$$

$$M(v,1)dx + N(v,1)dy = 0$$

$$v = x/y \qquad x = vy$$

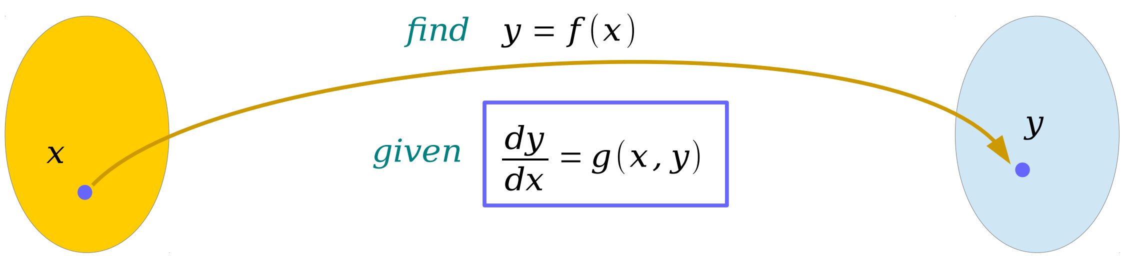
$$dx = v\,dy + y\,dv$$

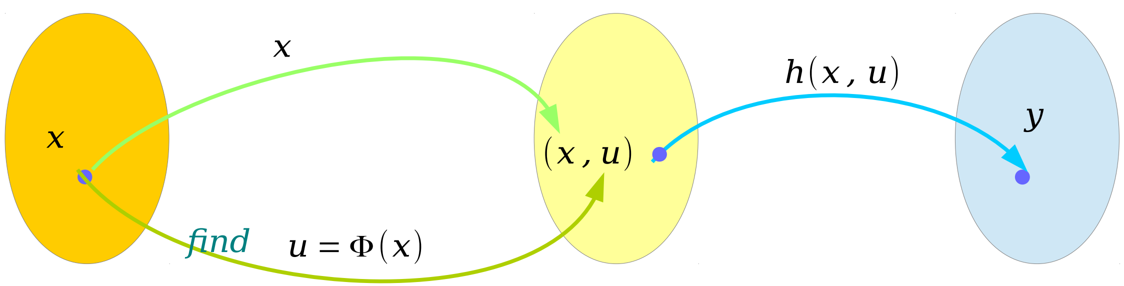
$$M(v,1)(v\,dy + y\,dv) + N(v,1)dy = 0$$

$$[v\,M(v,1) + N(v,1)]dy + y\,M(v,1)dv = 0$$

$$\frac{dy}{y} + \frac{M(v,1)\,dv}{[v\,M(v,1) + N(v,1)]} = 0$$

Homogeneous Equations (3)





given $\frac{du}{dx} = F(x, u)$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx}dx = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial u}\frac{du}{dx}dx$$

$$dy = u\,dx + x\,du$$

$$M(x, y)dx + N(x, y)dy = 0$$

$$M(x, y) = x^{\alpha} M(1, y/x)$$

$$N(x, y) = x^{\alpha} N(1, y/x)$$

$$u = \Phi(x) = y/x$$

★ $y = h(x, u) = u\,x$

★ $dy = u\,dx + x\,du$

$$\frac{dx}{x} + \frac{N(1, u)\,du}{[M(1, u) + u\,N(1, u)]} = 0$$

Homogeneous Equations (4)

all are functions of x

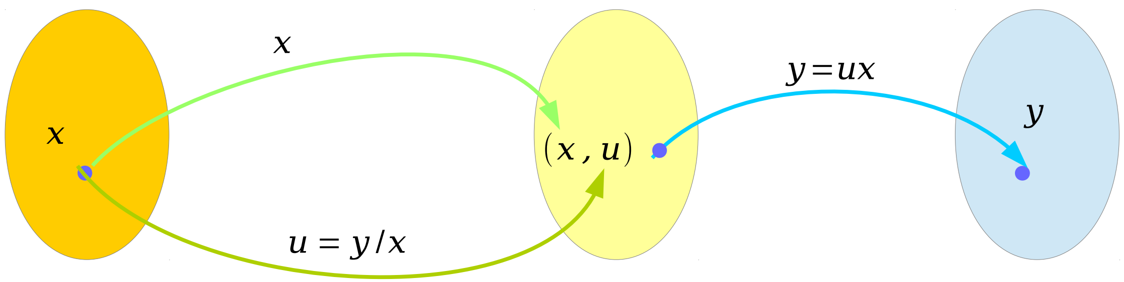
$y = f(x)$	⇨	$y(x)$
$u = \Phi(x)$	⇨	$u(x)$
$u = y/x$	⇨	$u(x) = y(x)/x$
$y = ux$	⇨	$y(x) = u(x)x$

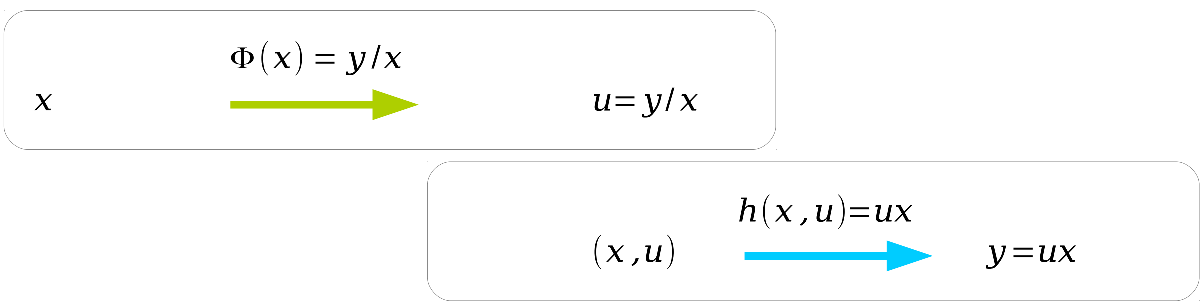
$$y = h(x, u) = h(x, \Phi(x))$$
$$= ux \qquad = \Phi(x)x$$
$$= \frac{y}{x}x$$

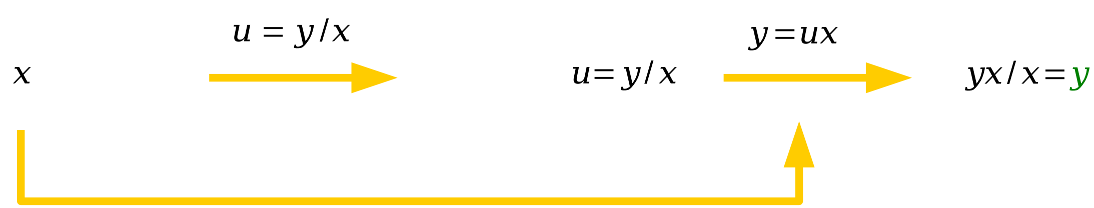
★ $y = ux$

★ $dy = u\,dx + x\,du$

Homogeneous Equations (5)







$$M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) = x^{\alpha} M(1, y/x)$$

$$N(x,y) = x^{\alpha} N(1, y/x)$$

$$u = \Phi(x) = y/x$$

$$y = h(x,u) = u\,x$$

$$dy = u\,dx + x\,du$$

$$\frac{dx}{x} + \frac{N(1,u)\,du}{[M(1,u) + u\,N(1,u)]} = 0$$

Bernoulli's First Order ODEs

Bernoulli's Equations (1)

***Bernoulli's* Equation**

$$\frac{dy}{dx} + P(x)\,y \;=\; Q(x)\,y^n$$

$$\frac{dy}{dx} + P(x)\,y \;=\; Q(x)\,y^0$$

$$\frac{dy}{dx} + P(x)\,y \;=\; Q(x) \qquad n = 0$$

***Linear* Equation**

$$\frac{dy}{dx} + P(x)\,y \;=\; Q(x)\,y^1$$

$$\frac{dy}{dx} + [P(x) - Q(x)]\,y \;=\; 0 \qquad n = 1$$

***Linear* Equation**

***Bernoulli's* Equation**

$$y' + P(x)\,y \;=\; Q(x)\,y^n$$

$$y' + P(x)\,y \;=\; Q(x)\,y^0$$

$$y' + P(x)\,y \;=\; Q(x) \qquad n = 0$$

***Linear* Equation**

$$y' + P(x)\,y \;=\; Q(x)\,y^1$$

$$y' + [P(x) - Q(x)]\,y \;=\; 0 \qquad n = 1$$

***Linear* Equation**

Bernoulli's Equations (2)

***Bernoulli's* Equation**

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{y}{y^n} = Q(x)$$

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

$$u = y^{1-n} \qquad \frac{du}{dx} = (1-n)\boxed{y^{-n}\frac{dy}{dx}}$$

$$\frac{1}{(1-n)}\frac{du}{dx} + P(x)u = Q(x)$$

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

***Linear* Equation**

***Bernoulli's* Equation**

$$y' + P(x)y = Q(x)y^n$$

$$\frac{1}{y^n}y' + P(x)\frac{y}{y^n} = Q(x)$$

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$

$$u = y^{1-n} \qquad u' = (1-n)\boxed{y^{-n}y'}$$

$$\frac{1}{(1-n)}u' + P(x)u = Q(x)$$

$$u' + (1-n)P(x)u = (1-n)Q(x)$$

***Linear* Equation**

References

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[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
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[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[5] www.chem.arizona.edu/~salzmanr/480a