Laplace Transform Pairs (4A)

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1/8/15

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Selected Laplace Transform Pairs (1)

Function	Time domain $f(t)=\mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s)=\mathcal{L}\{f(t)\}$	Region of convergence	Reference
unit impulse	$\delta(t)$	1	all s	inspection
delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$		time shift of unit impulse
unit step	$u(t)$	$\frac{1}{s}$	Re(s) > 0	integrate unit impulse
delayed unit step	$u(t-\tau)$	$\frac{e^{-\tau s}}{s}$	Re(s) > 0	time shift of unit step
ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	Re(s) > 0	integrate unit impulse twice
nth power (for integer n)	$t^n \cdot u(t)$	$\frac{n!}{s^{n+1}}$	Re(s) > 0 ($n > -1$)	Integrate unit step n times
qth power (for complex q)	$t^q \cdot u(t)$	$\frac{\Gamma(q+1)}{s^{q+1}}$	Re(s) > 0 Re(q) > −1	[18][19]
nth root	$\sqrt[n]{t} \cdot u(t)$	$\frac{\Gamma(\frac{1}{n}+1)}{s^{\frac{1}{n}+1}}$	Re(s) > 0	Set $q = 1/n$ above.
nth power with frequency shift	$t^n e^{-\alpha t} \cdot u(t)$	$\frac{n!}{(s+\alpha)^{n+1}}$	Re(s) > −α	Integrate unit step, apply frequency shift
delayed nth power with frequency shift	$(t-\tau)^n e^{-\alpha(t-\tau)} \cdot u(t-\tau)$	$\frac{n! \cdot e^{-\tau s}}{(s+\alpha)^{n+1}}$	Re(s) > −α	Integrate unit step, apply frequency shift, apply time shift

http://en.wikipedia.org/wiki/Laplace_transform

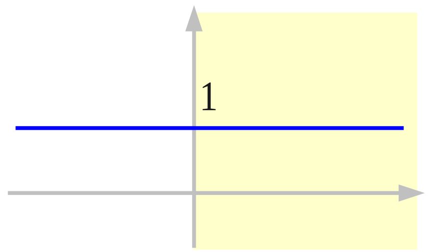
Selected Laplace Transform Pairs (2)

exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	Re(s) > $-\alpha$	Frequency shift of unit step
two-sided exponential decay	$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2 - s^2}$	$-\alpha$ < Re(s) < α	Frequency shift of unit step
exponential approach	$(1 - e^{-\alpha t}) \cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	Re(s) > 0	Unit step minus exponential decay
sine	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$	Re(s) > 0	Bracewell 1978, p. 227
cosine	$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2+\omega^2}$	Re(s) > 0	Bracewell 1978, p. 227
hyperbolic sine	$\sinh(\alpha t) \cdot u(t)$	$\frac{\alpha}{s^2-\alpha^2}$	Re(s) > $\|\alpha\|$	Williams 1973, p. 88
hyperbolic cosine	$\cosh(\alpha t) \cdot u(t)$	$\frac{s}{s^2-\alpha^2}$	Re(s) > $\|\alpha\|$	Williams 1973, p. 88
exponentially decaying sine wave	$e^{-\alpha t}\sin(\omega t) \cdot u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$	Re(s) > $-\alpha$	Bracewell 1978, p. 227
exponentially decaying cosine wave	$e^{-\alpha t}\cos(\omega t) \cdot u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$	Re(s) > $-\alpha$	Bracewell 1978, p. 227
natural logarithm	$\ln(t) \cdot u(t)$	$-\frac{1}{s}\,[\ln(s)+\gamma]$	Re(s) > 0	Williams 1973, p. 88
Bessel function of the first kind, of order n	$J_n(\omega t) \cdot u(t)$	$\frac{\left(\sqrt{s^2+\omega^2}-s\right)^n}{\omega^n\sqrt{s^2+\omega^2}}$	Re(s) > 0 (n > −1)	Williams 1973, p. 89
Error function	$\mathrm{erf}(t) \cdot u(t)$	$\frac{e^{s^2/4}\left(1-\mathrm{erf}\left(s/2\right)\right)}{s}$	Re(s) > 0	Williams 1973, p. 89

http://en.wikipedia.org/wiki/Laplace_transform

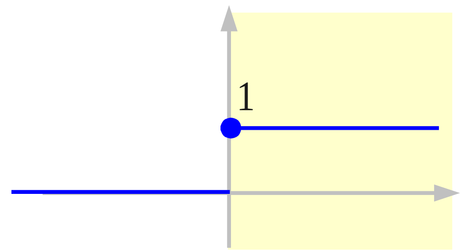
Unit Step Function

1



$\longleftrightarrow \dfrac{1}{s}$

$u(t)$



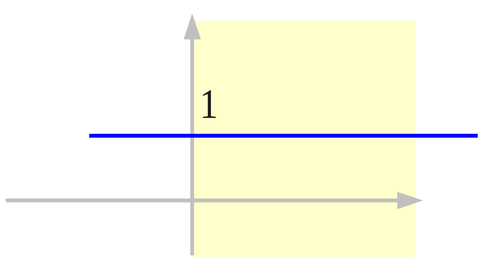
$\longleftrightarrow \dfrac{1}{s}$

$$F(s) = \int_0^\infty f(t)\cdot e^{-st}\,dt = \int_0^\infty 1\cdot e^{-st}\,dt$$

$$= \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s0}\right]$$

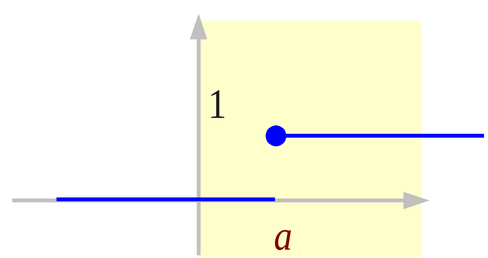
$$s > 0 \quad \Rightarrow \quad F(s) = \frac{1}{s}$$

1



$\longleftrightarrow \dfrac{1}{s}$

$u(t-a)$



$\longleftrightarrow e^{-as}\dfrac{1}{s}$

$$F(s) = \int_0^\infty f(t)\cdot e^{-st}\,dt = \int_0^\infty u(t-a)\cdot e^{-st}\,dt$$

$$= \int_0^a \cancel{u(t-a)}\cdot e^{-st}\,dt + \int_a^\infty u(t-a)\cdot e^{-st}\,dt$$

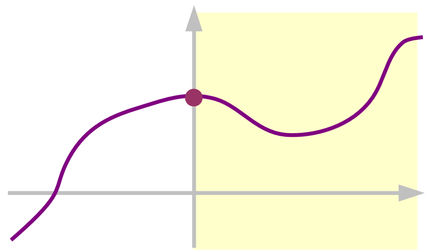
$0<t<a$	$v = t-a \quad dv = dt$ $v+a = t$

$$= \int_a^\infty u(v)\cdot e^{-s(v+a)}\,dv = e^{-as}\int_a^\infty u(v)\cdot e^{-sv}\,dv$$

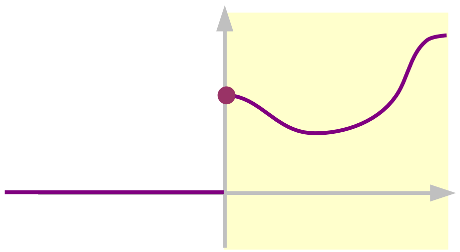
$$= e^{-as}\cdot\frac{1}{s}$$

Unit Step Function

$f(t)$

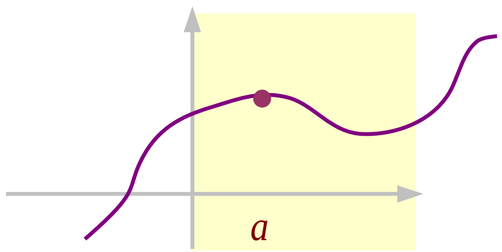


$f(t)u(t)$



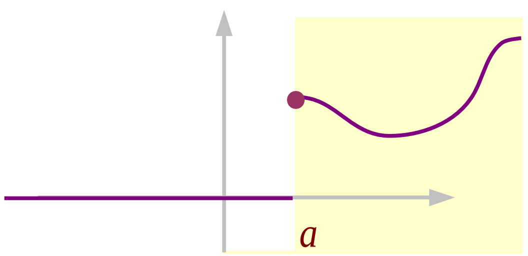
$\Longleftrightarrow \quad F(s)$

$f(t-a) = 1$



$\Longleftrightarrow \quad \cancel{e^{-as}F(s)}$

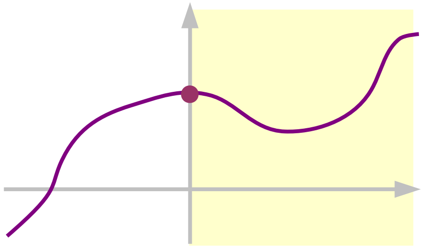
$f(t-a)u(t-a)$



$\Longleftrightarrow \quad e^{-as}F(s)$

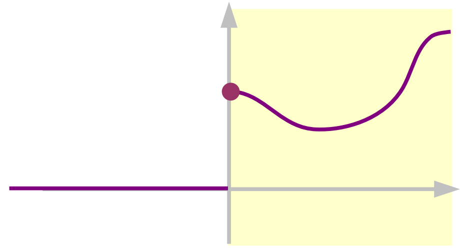
Unit Step Function

$f(t)$



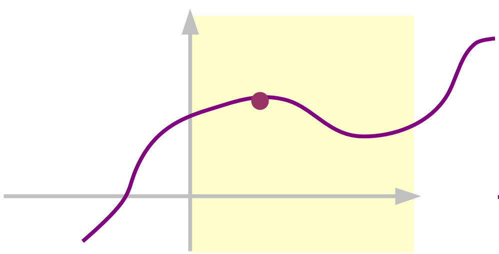
$\Longleftrightarrow \quad F(s)$

$f(t)u(t)$



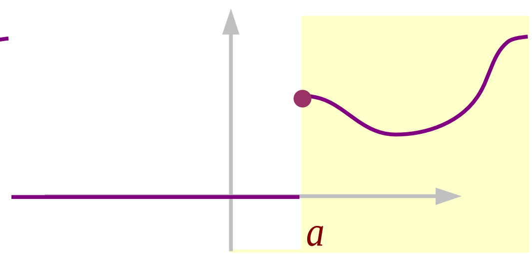
$\Longleftrightarrow \quad F(s)$

$f(t-a) = 1$



$\Longleftrightarrow \quad \cancel{e^{-as}F(s)}$

$f(t-a)u(t-a)$



$\Longleftrightarrow \quad e^{-as}F(s)$

$$
\begin{aligned}
F(s) &= \int_0^{\infty} f(t-a)u(t-a)\cdot e^{-st}\,dt \\
&= \cancel{\int_0^{a} f(t-a)u(t-a)\cdot e^{-st}\,dt} \\
&+ \int_a^{\infty} f(t-a)u(t-a)\cdot e^{-st}\,dt \\
&= \int_a^{\infty} f(\nu)u(\nu)\cdot e^{-s(\nu+a)}\,d\nu \\
&= e^{-as}\int_a^{\infty} f(\nu)\cdot e^{-s\nu}\,d\nu \\
&= e^{-as}\cdot F(s)
\end{aligned}
$$

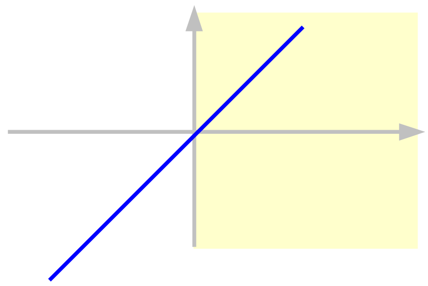
$0 < t < a$

$\nu = t - a$

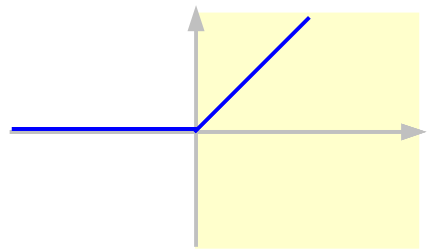
$d\nu = dt$

Transforms of f(t) and f(t)u(t)

t

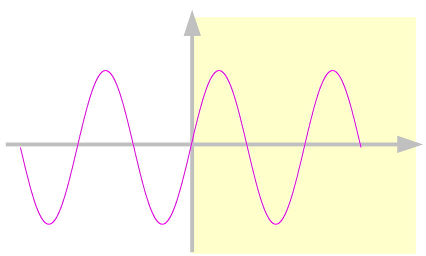


$tu(t)$

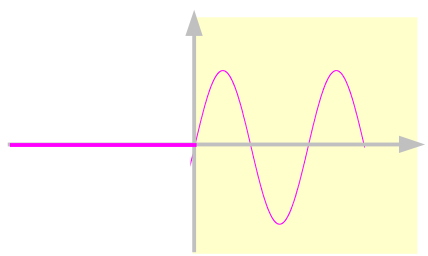


$\Longleftrightarrow \quad \dfrac{1}{s}$

$\sin(\omega t)$

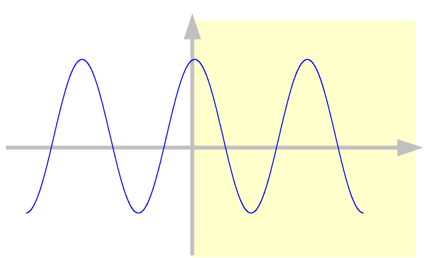


$\sin(\omega t)u(t)$

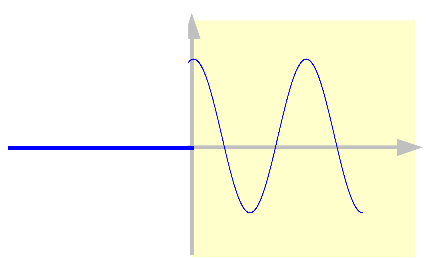


$\Longleftrightarrow \quad \dfrac{\omega}{s^2+\omega^2}$

$\cos(\omega t)$



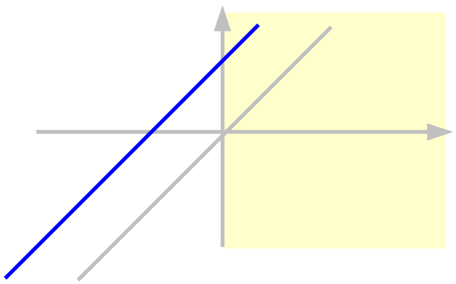
$\cos(\omega t)u(t)$



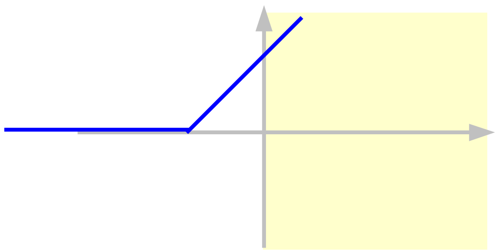
$\Longleftrightarrow \quad \dfrac{s}{s^2+\omega^2}$

Transforms of (t±1) and (t±1)u(t±1)

$t+1$



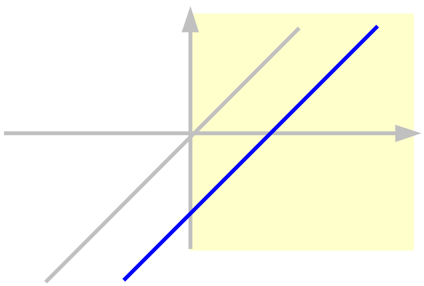
$(t+1)u(t+1)$



$f_1(t) = t+1 \quad \equiv \quad f_2(t) = (t+1)u(t+1) \qquad (t \geq 0)$

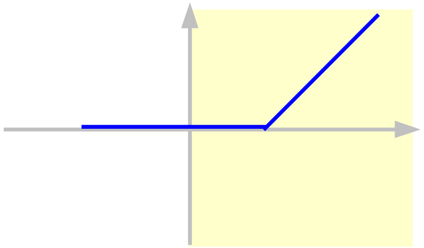
$$\Longleftrightarrow \quad \frac{1}{s^2} + \frac{1}{s}$$

$t-1$



$$\Longleftrightarrow \quad \frac{1}{s^2} - \frac{1}{s}$$

$(t-1)u(t-1)$



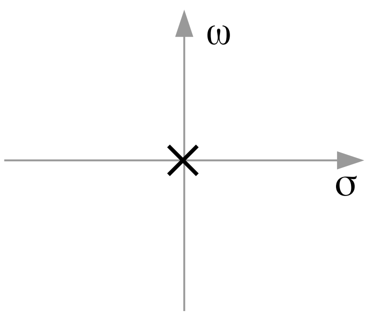
$$\Longleftrightarrow \quad e^{-s}\frac{1}{s^2}$$

Translation in the s-domain

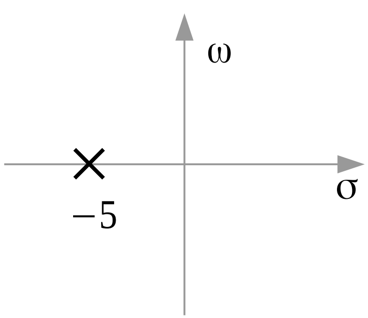
$$e^{+at}f(t) \Longleftrightarrow F(s-a)$$

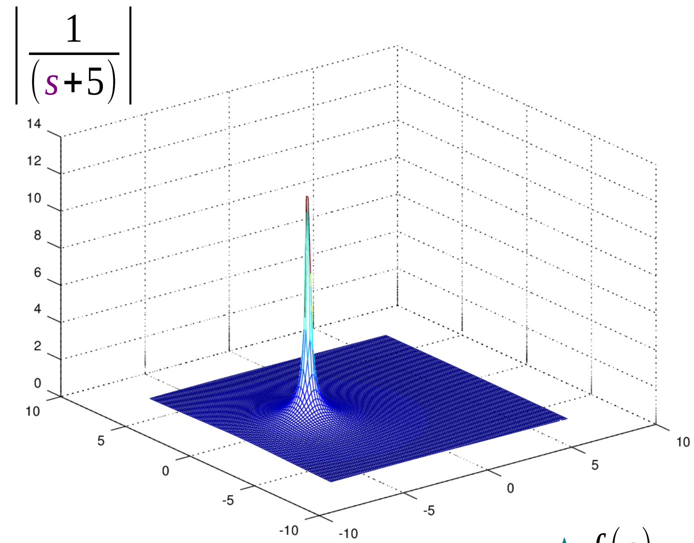
$$F(s-a) = \int_0^{\infty} f(t)\cdot e^{-(s-a)t}\,dt = \int_0^{\infty} \left[e^{+at}f(t)\right]e^{-st}\,dt$$

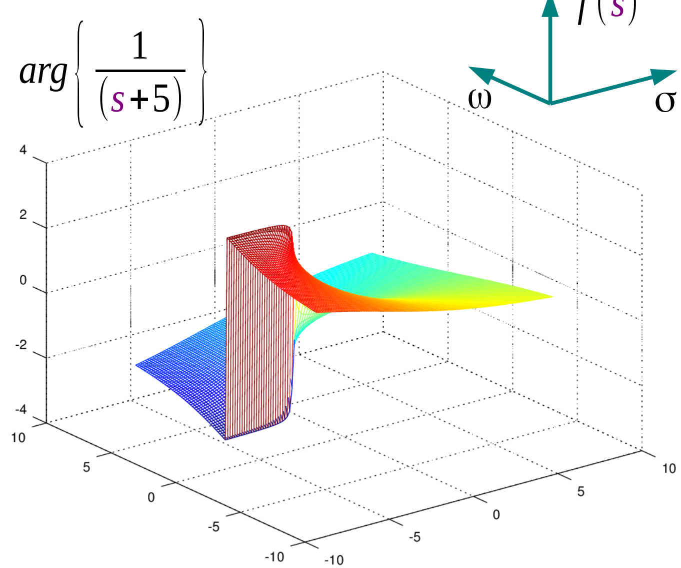
$$u(t) \Longleftrightarrow$$



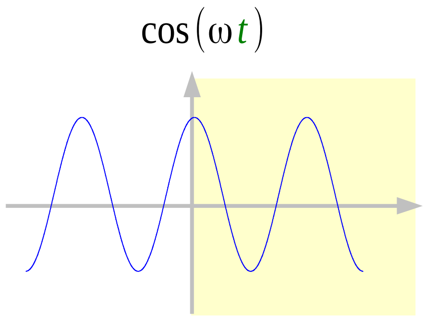
$$e^{+5t}u(t) \Longleftrightarrow$$

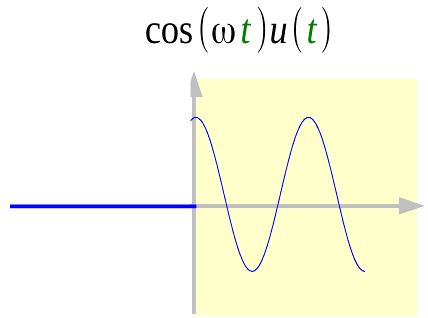


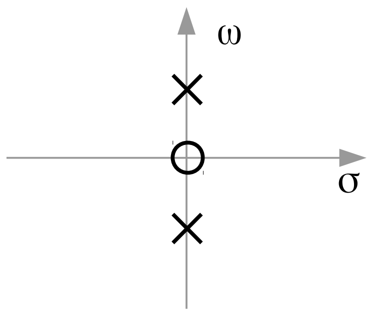




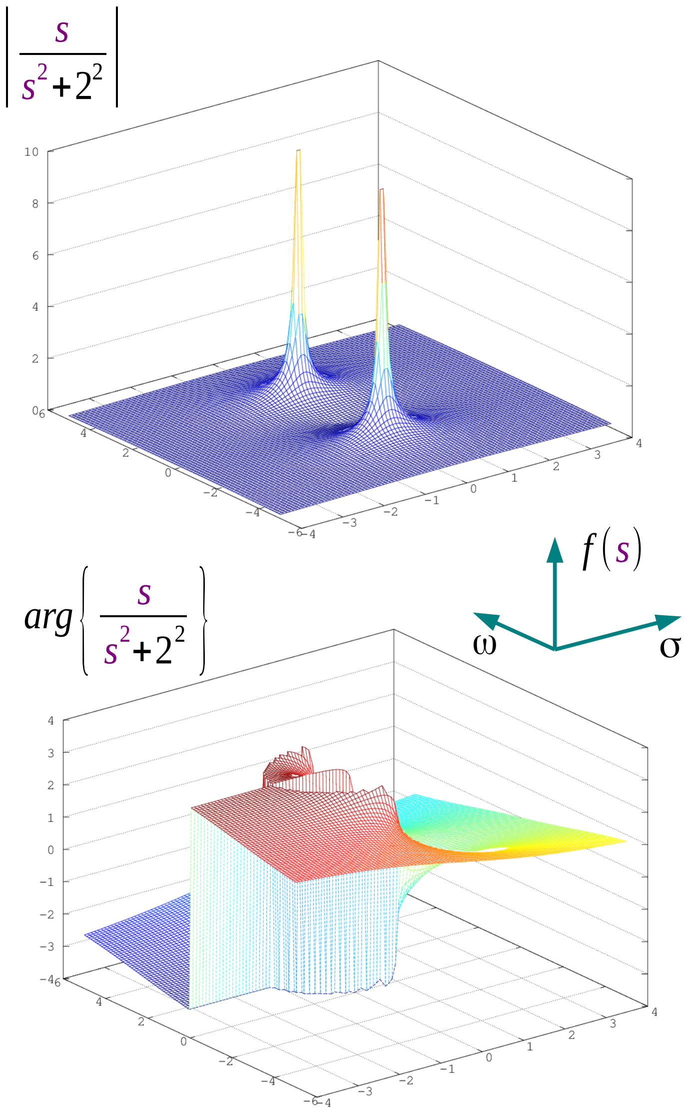
cos(ωt)



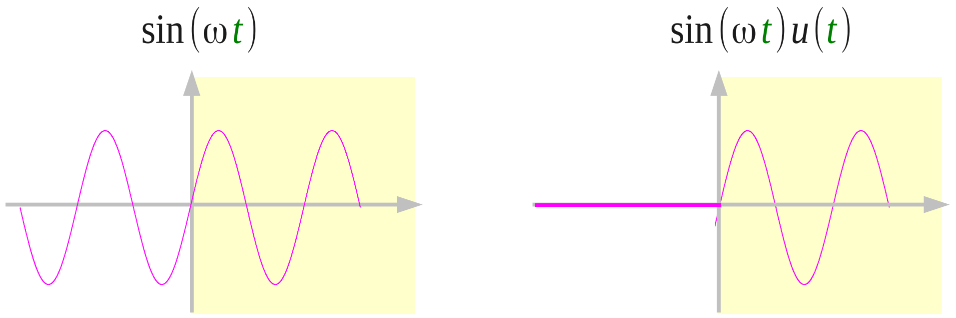


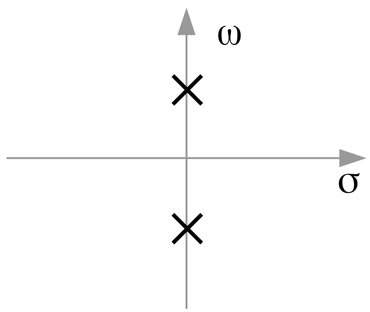


$$\frac{s}{s^2+\omega^2}$$

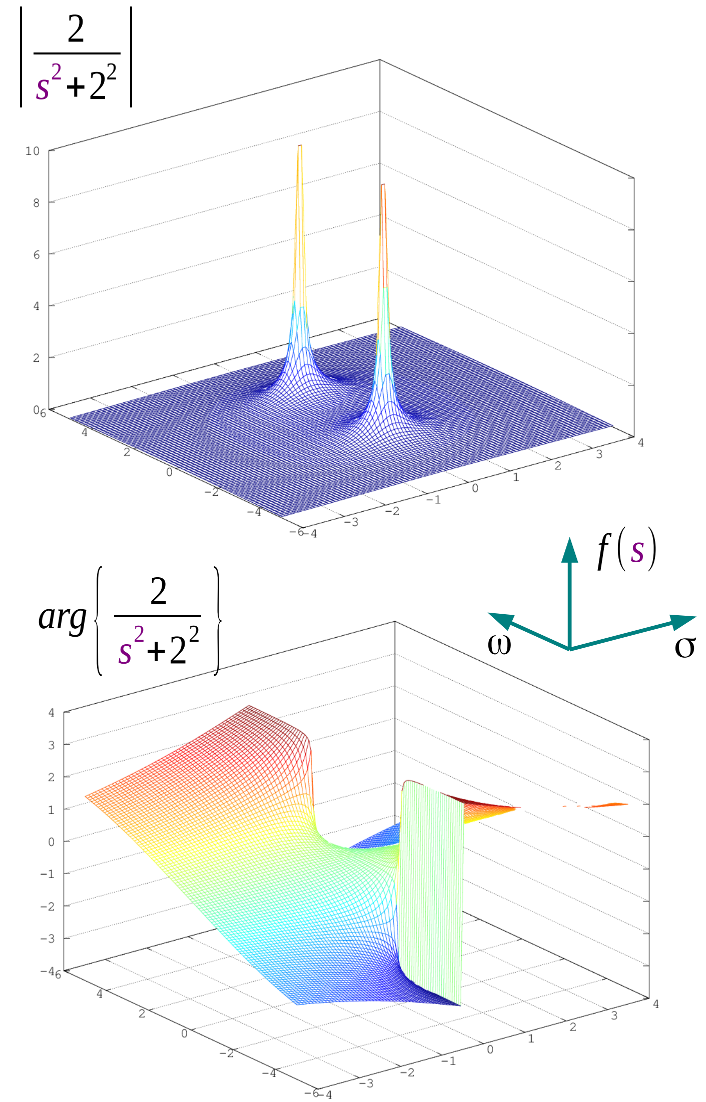


sin(ωt)



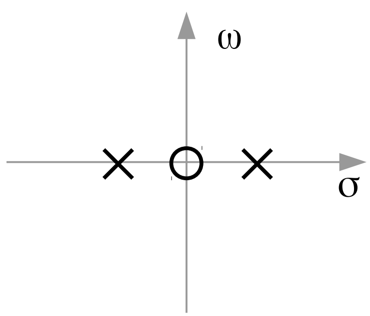


$$\frac{\omega}{s^2+\omega^2}$$



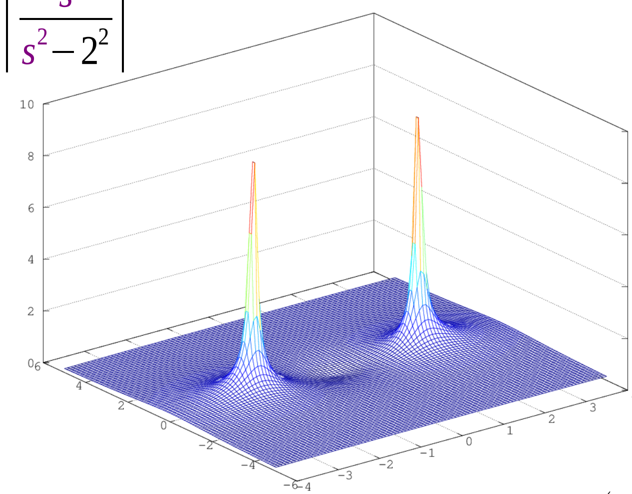
cosh(ωt)

$\cosh(\omega t)$ $\cosh(\omega t) u(t)$

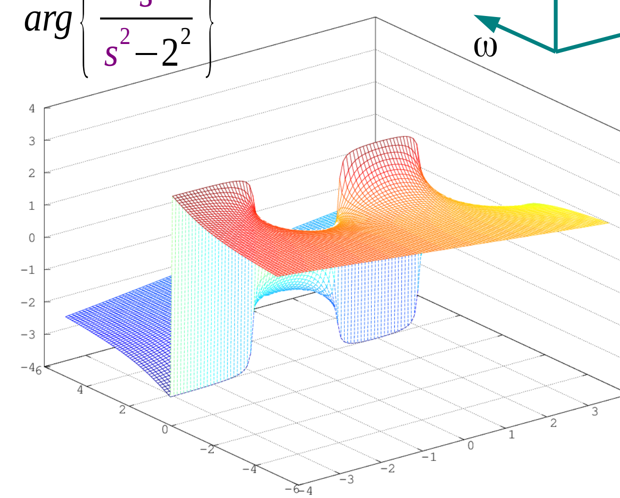


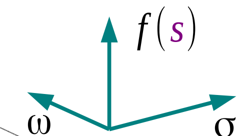
$$\frac{s}{s^2 - \omega^2}$$

$$\left| \frac{s}{s^2 - 2^2} \right|$$



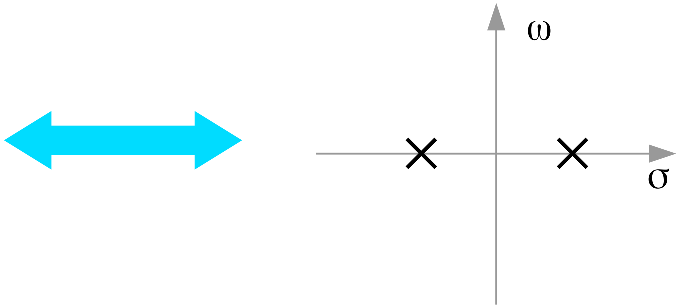
$$arg\left\{ \frac{s}{s^2 - 2^2} \right\}$$





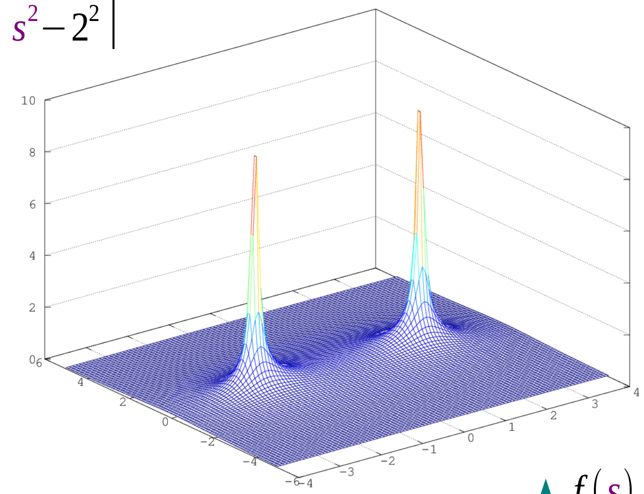
sinh(ωt)

$\sinh(\omega t)$ $\sinh(\omega t)u(t)$



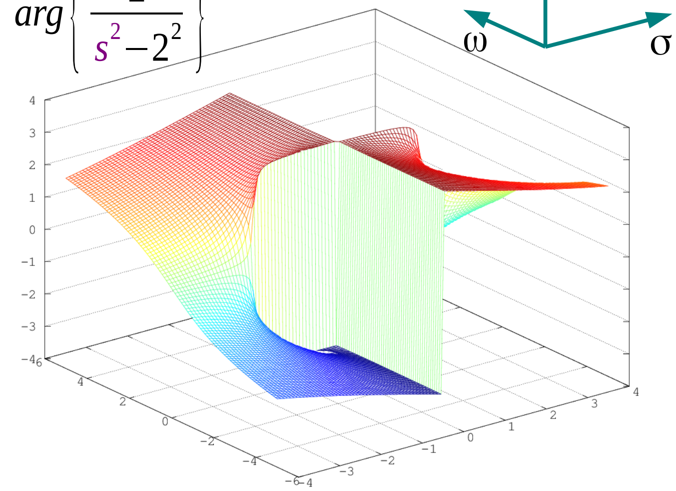
$$\frac{\omega}{s^2-\omega^2}$$

$\left|\frac{2}{s^2-2^2}\right|$

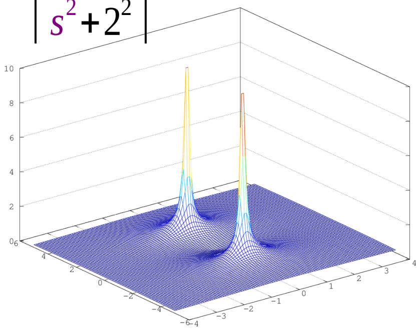


$f(s)$ ω σ

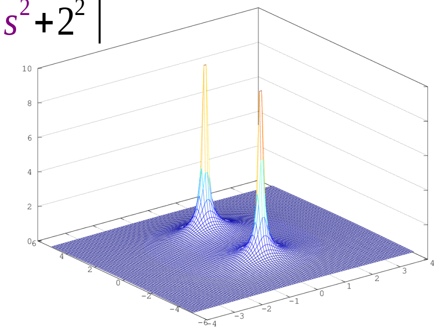
$arg\left\{\frac{2}{s^2-2^2}\right\}$



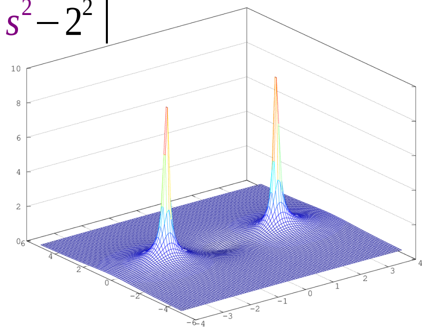
$$\left|\frac{s}{s^2+2^2}\right|$$



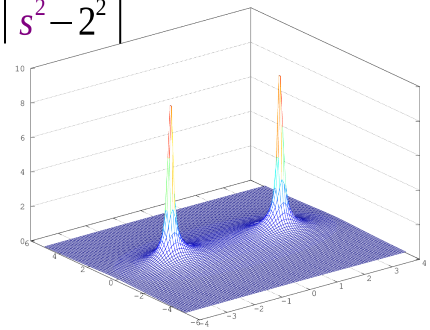
$$\left|\frac{2}{s^2+2^2}\right|$$



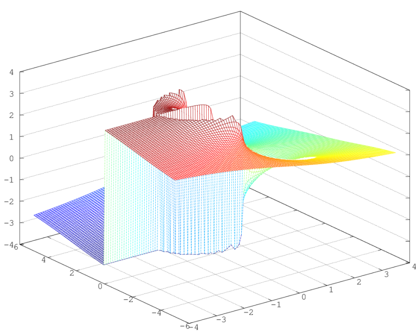
$$\left|\frac{s}{s^2-2^2}\right|$$



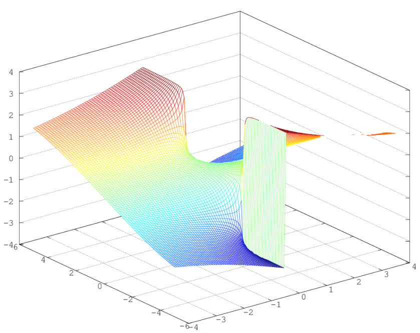
$$\left|\frac{2}{s^2-2^2}\right|$$



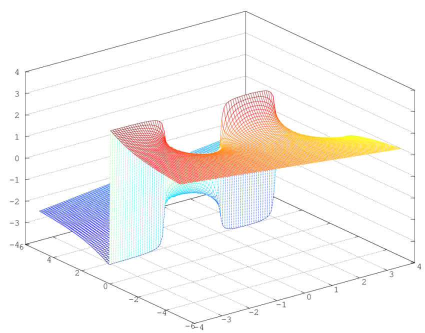
$$arg\left\{\frac{s}{s^2+2^2}\right\}$$



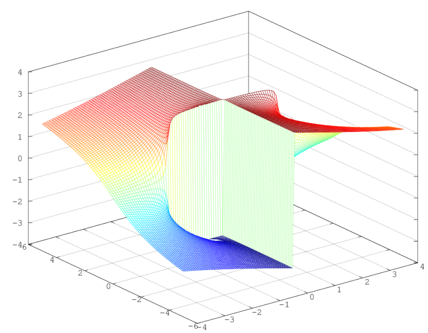
$$arg\left\{\frac{2}{s^2+2^2}\right\}$$



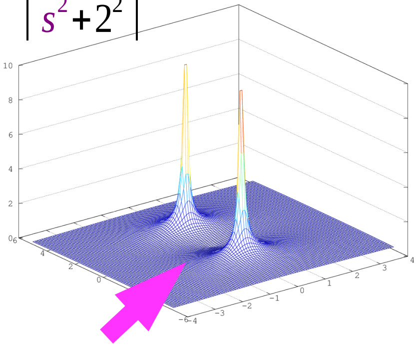
$$arg\left\{\frac{s}{s^2-2^2}\right\}$$



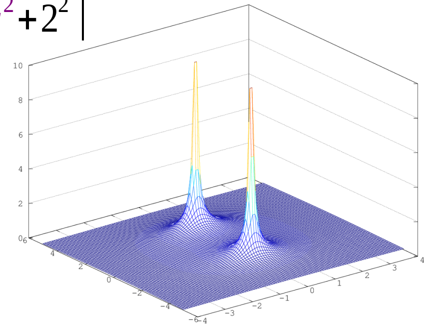
$$arg\left\{\frac{2}{s^2-2^2}\right\}$$



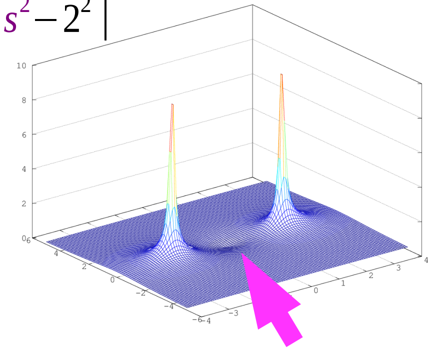
$$\left|\frac{s}{s^2+2^2}\right|$$



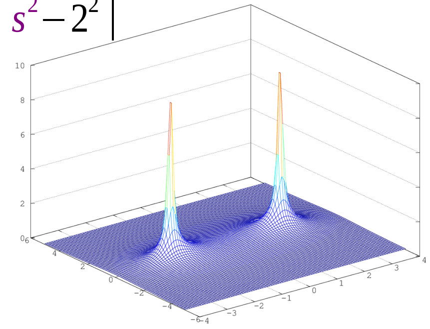
$$\left|\frac{2}{s^2+2^2}\right|$$



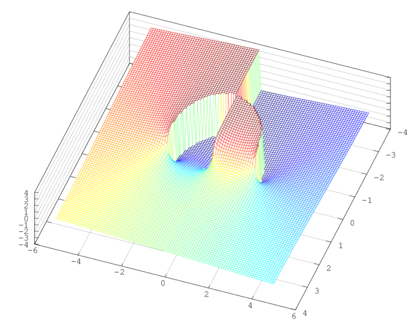
$$\left|\frac{s}{s^2-2^2}\right|$$



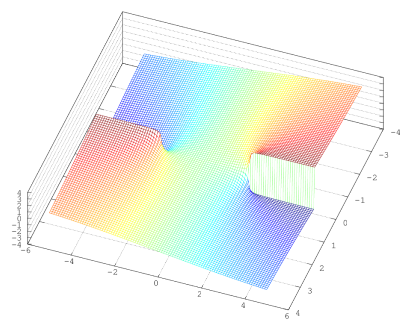
$$\left|\frac{2}{s^2-2^2}\right|$$



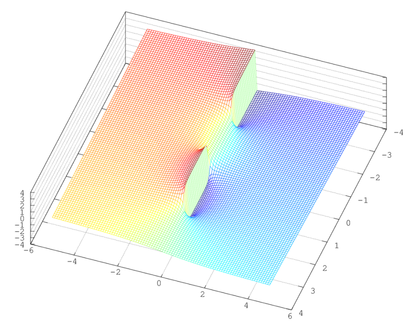
$$arg\left\{\frac{s}{s^2+2^2}\right\}$$



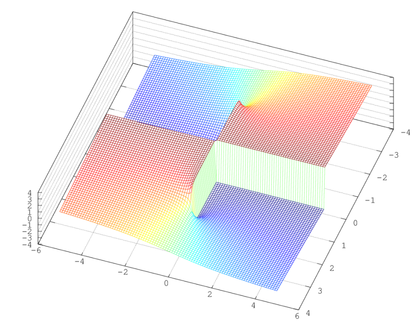
$$arg\left\{\frac{2}{s^2+2^2}\right\}$$



$$arg\left\{\frac{s}{s^2-2^2}\right\}$$



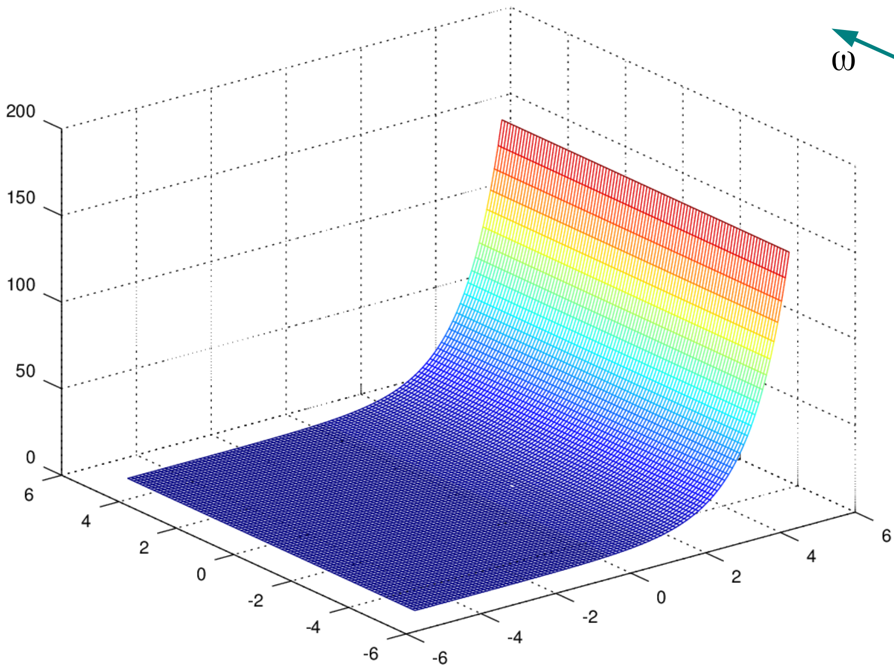
$$arg\left\{\frac{2}{s^2-2^2}\right\}$$

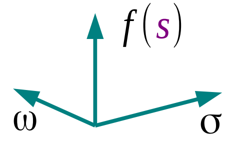


Plot of e^s

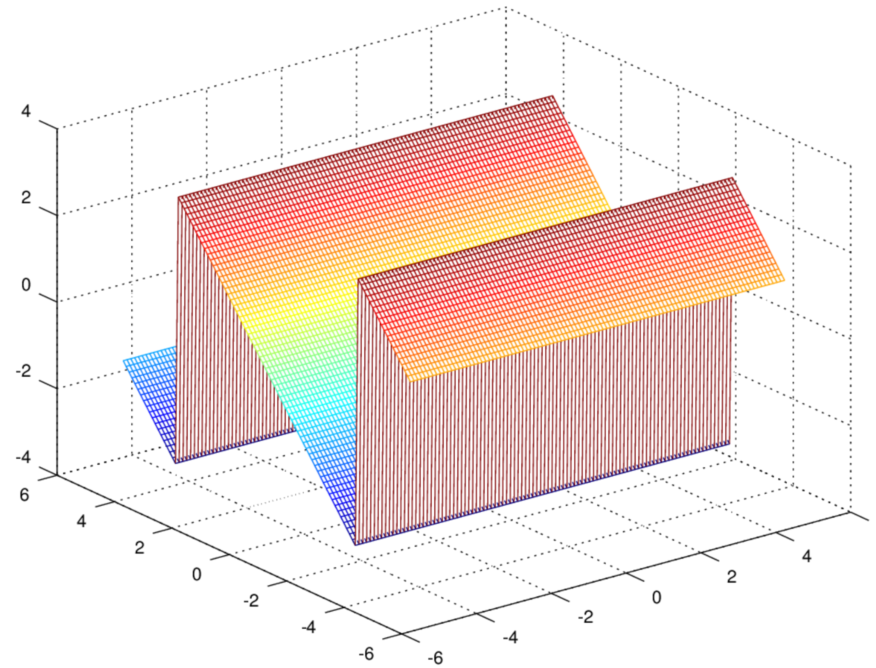
$$e^{s} = e^{\sigma + i\omega} = e^{\sigma} e^{i\omega}$$

$$\left| e^{s} \right| = e^{\sigma} \left| e^{i\omega} \right| = e^{\sigma}$$





$$arg\{ e^{s} \} = 0 + arg\{ e^{i\omega} \} = \omega$$



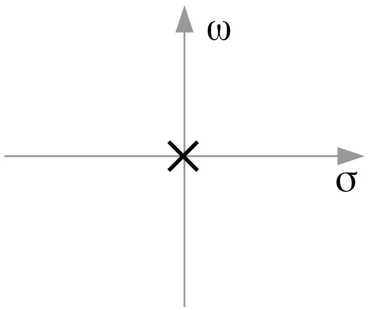
Translation in the t-domain

$$f(t-a)u(t-a) \iff e^{-as}F(s)$$

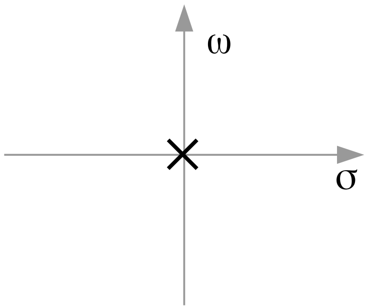
$$\int_0^\infty f(t-a)u(t-a)\cdot e^{-st}\,dt$$

$$= \int_0^a f(t-a)\cancel{u(t-a)}\cdot e^{-st}\,dt + \int_a^\infty f(t-a)u(t-a)\cdot e^{-st}\,dt$$

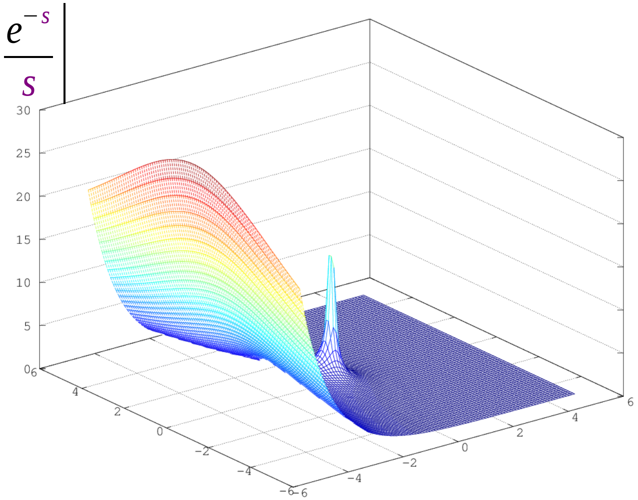
$$u(t) \iff$$



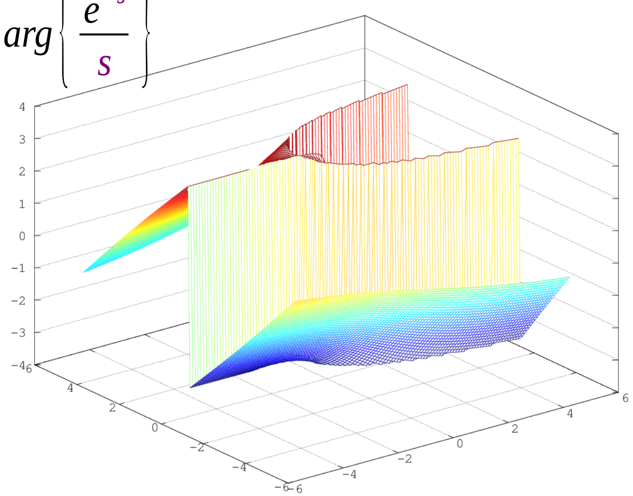
$$u(t-2) \iff$$



$$\left|\frac{e^{-s}}{s}\right|$$



$$arg\left\{\frac{e^{-s}}{s}\right\}$$



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