Residue Integrals (4C)

- Inverse Laplace Transform

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Laplace Transform → Fourier Transform

Laplace Transform

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt$$

$$= \int_0^{\infty} \{f(t) e^{-xt}\}\, e^{-iyt}\, dt$$

$$= F(x, y)$$

Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s) e^{st}\, ds$$

$$f(t) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(x + iy) e^{xt} e^{iyt}\, ds$$

$$f(t) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(x, y) e^{xt} e^{iyt}\, ds$$

$s = x + i\,y$,

$f(t) = 0 \quad t < 0$

$g(t) = \{f(t) e^{-xt}\}$,

for a given x

Fourier Transform

$$F(x, y) = \int_0^{\infty} g(t)\, e^{-iyt}\, dt$$

Inverse Fourier Transform

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x, y) e^{iyt}\, dy$$

$ds = dx + i\,dy = i\,dy$, *for a given* x

Fourier Transform → Laplace Transform

Fourier Transform

$$F(x,y) = \int_0^{\infty} g(t)\, e^{-iyt}\, dt$$

Inverse Fourier Transform

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y) e^{iyt} dy$$

$$s = x + iy\,,$$

$$f(t) = 0 \quad t < 0$$

$$f(t)e^{-xt} = \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} F(x,y) e^{iyt} dy$$

$$f(t) = \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} F(x,y) e^{+xt} e^{iyt} dy$$

$$g(t) = \{f(t)e^{-xt}\}\,,$$

for a given $x, (x > \alpha)$

Laplace Transform

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt$$

Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s) e^{st} ds$$

$ds = dx + i\,dy = i\,dy\,,$ *for a given* x

Exponential Order

Laplace Transform

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

for $s > 0$ $\quad \Re(s) > 0$
the integral converges
if f(t) does not grow too rapidly

the growth rate of a function f(t)

Exponential Order α

a function f has exponential order α

$\triangleq$

there exist constants $M > 0$ and α
such that for some $t > t_0$

$$|f(t)| \le M\, e^{\alpha t}, \qquad t > t_0$$

$\int_0^\infty |f(t)| e^{-\sigma t}\, dt < \infty$ for some σ ➡

$$\int_0^\infty |f(t) e^{-st}|\, dt = \int_0^\infty |f(t) e^{-xt} e^{-iyt}|\, dt = \int_0^\infty |f(t) e^{-xt}|\, dt < \int_0^\infty |f(t)| e^{-\sigma t}\, dt < \infty$$ for $s > \sigma$ $\quad \Re(s) > \sigma$

$f(t)$ exponential order σ ➡ $F(s) = \int_0^\infty f(t) e^{-st}\, dt$ absolutely converges for $s > \sigma$

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Convergence of the Laplace Transform

Laplace Transform

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

$$= \int_0^\infty \{f(t) e^{-xt}\} e^{-iyt}\, dt$$

- f(t) continuous on [0, ∞)
- f(t) = 0 for t < 0
- f(t) has **exponential order α**
- f'(t) piecewise continuous on [0, ∞)

⇒ F(s) converges absolutely for Re(s) > α

$$\int_0^\infty |f(t) e^{-st}|\, dt < \infty$$

$$\int_0^\infty |f(t) e^{-st}|\, dt = \int_0^\infty |f(t)|\, e^{-xt}\, dt < \infty$$

$$\left(|e^{-st}| = |e^{-xt}|\,|e^{-iyt}| = e^{-xt} \right)$$

$$\{f(t) e^{-xt}\} = g(t)$$

absolutely integrable for $x > \alpha$

⇒ Use Fourier Inversion

Fourier Transform

$g(t) = f(t)e^{-xt}$ absolutely integrable for $x > \alpha$

$$F(x,y) = \int_0^{\infty} \{f(t)e^{-xt}\}\, e^{-iyt}\, dt$$

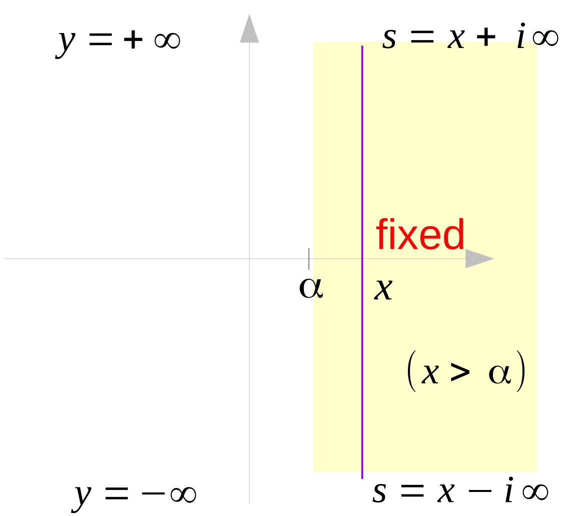
$$F(x,y) = \int_0^{\infty} g(t)\, e^{-iyt}\, dt$$

Fourier Transform $g(t) = f(t)e^{-xt}$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y)e^{iyt}\, dy$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y)e^{xt}e^{iyt}\, dy$$

Inverse Fourier Transform



$$s = x + iy$$
$$ds = i\,dy$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y)e^{(x+iy)t}\, dy$$

$$= \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F(s)e^{st}\, ds$$

$$= \lim_{y\to\infty} \frac{1}{2\pi i}\int_{x-iy}^{x+iy} F(s)e^{st}\, ds$$

$x - iy$

Fourier-Mellin Inversion Formula

$$F(x,y) = \int_0^\infty \underline{\{f(t)e^{-xt}\}}\, e^{-iyt}\, dt$$

$$F(x,y) = \int_0^\infty \underline{g(t)}\, e^{-iyt}\, dt$$

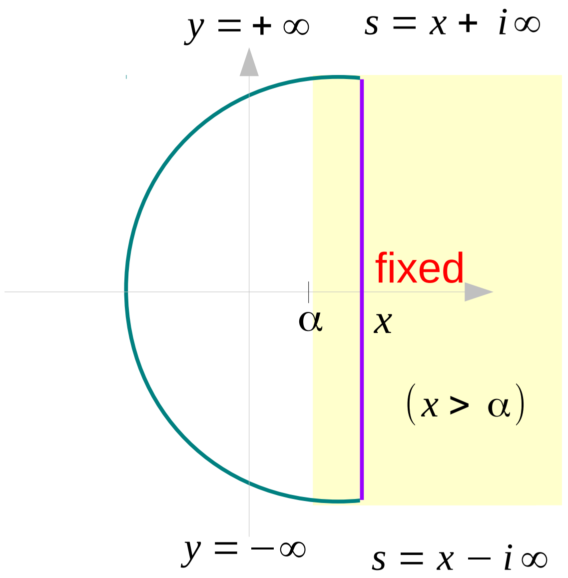
Fourier Transform $g(t) = f(t)e^{-xt}$

$$\underline{g(t)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y)e^{iyt}\, dy$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x,y)e^{xt}e^{iyt}\, dy$$

Inverse Fourier Transform

Vertical line at x : Bromwich line



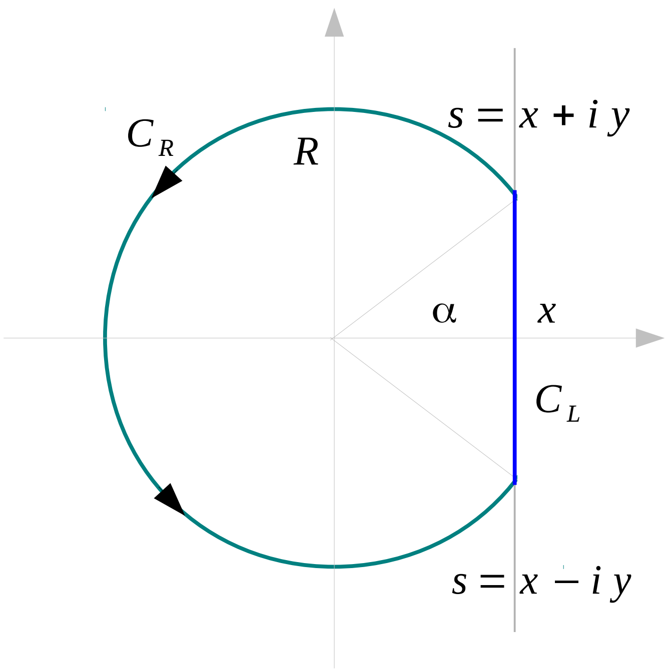
$$s = x + iy$$
$$ds = i\,dy$$

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F(s)e^{st}\, ds$$
$$= \lim_{y\to\infty} \frac{1}{2\pi i}\int_{x-iy}^{x+iy} F(s)e^{st}\, ds$$

Complex Inversion Formula

(Fourier-Mellin Inversion Formula)

Bromwich Contour Integration



$$\frac{1}{2\pi i}\int_C F(s)e^{st}\,ds$$

$$= \frac{1}{2\pi i}\int_{C_R} F(s)e^{st}\,ds + \frac{1}{2\pi i}\int_{C_L} F(s)e^{st}\,ds$$

F(s) is analytic for Re(s) = x > α

F(s)all singularities must lie to the left of **Bromwich line**

Assume F(s) is analytic for Re(s) = x < α except for having finitely many poles

$z_1, z_2, \cdots, z_n$

contour integration on C_R

$$\frac{1}{2\pi i}\int_{C_R} F(s)e^{st}\,ds$$

$R \to \infty$

0

contour integration on C_L

$$\frac{1}{2\pi i}\int_{x-iy}^{x+iy} F(s)e^{st}\,ds$$

$$\sum_{k=1}^{n} Res(z_k)$$

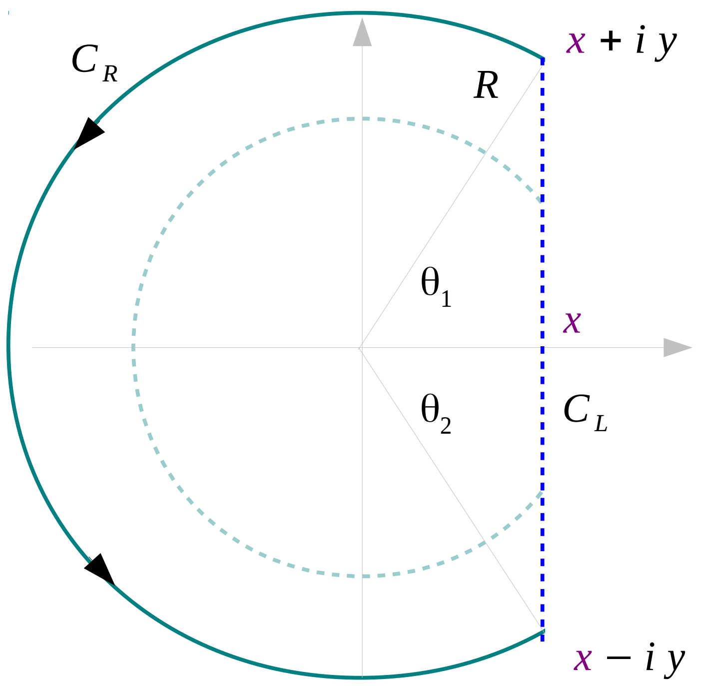
$$\frac{1}{2\pi i}\int_C F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

$$\frac{1}{2\pi i}\int_{C_R} F(s)e^{st}\,ds = 0$$

for a given x

$$\frac{1}{2\pi i}\int_{x-iy}^{x+iy} F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

Growth Restriction Conditions on F(s)



For s on C_R, some $p > 0$ all $R > R_0$

$$|F(s)| \leq \frac{M}{|s|^p}$$

$$\Rightarrow \lim_{R \to \infty} \int_{C_R} F(s) e^{st} \, ds = 0 \quad (t > 0)$$

$$|F(s)| \leq \frac{M}{|s|^p}$$ Growth Restriction

$$\Rightarrow |F(s)| \to 0 \quad as \quad |s| \to \infty$$

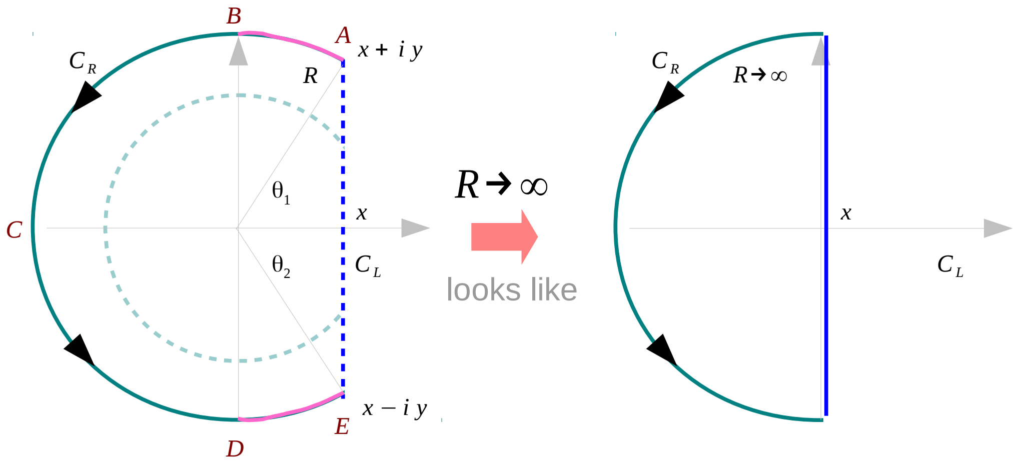
a function f has exponential order α

$\triangleq$

there exist constants $M > 0$ and α such that for some $t > t_0$

$$|f(t)| \leq M e^{\alpha t}, \qquad t > t_0$$

Contour Integration on C_R (x > 0)



$$\left| \int_{C_R} F(s) e^{st} \, ds \right| \le \int_{AB} |F(s)| \, |e^{st}| \, |ds|$$

$$\le \frac{2M}{R^{p-1}} \int_{\theta_1}^{\pi/2} e^{Rt(-\sin\varphi)} \, d\varphi$$

$$\le \frac{2M}{R^{p-1}} \int_{\theta_1}^{\pi/2} c \, d\varphi$$

$$\le \frac{2M}{R^{p}} \, l(\widetilde{AB}) \to 0 \qquad R \to \infty$$

$$r\theta = l$$

$$\theta = \frac{l}{r}$$

$\widetilde{AB}, \ \widetilde{DE} \qquad 0 \le \Re(s) \le x$

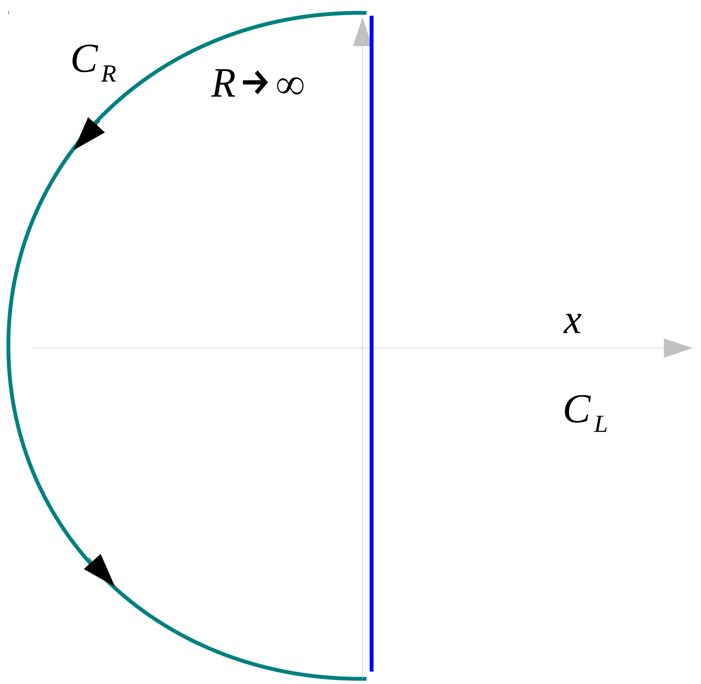
$|e^{ts}| \le e^{tx} = c$ for a fixed $t > 0$

$$\left| \int_{\widetilde{AB}} F(s) e^{st} \, ds \right| \to 0 \qquad R \to \infty$$

$$\left| \int_{\widetilde{DE}} F(s) e^{st} \, ds \right| \to 0 \qquad R \to \infty$$

$$\lim_{R \to \infty} \int_{C_R} F(s) e^{st} \, ds = 0$$

Contour Integration on C_R (x < 0)



$$s = Re^{i\theta} = R(\cos\theta + i\sin\theta)$$

$$e^{st} = e^{Rt(\cos\theta + i\sin\theta)} = e^{Rt\cos\theta} e^{iRt\sin\theta}$$

$$\left|e^{st}\right| = e^{Rt\cos\theta}$$

$$|F(s)| \le \frac{M}{|s|^p}$$ Growth Restriction

$$s = Re^{i\theta}$$

$$ds = iRe^{i\theta} d\theta$$

$$|ds| = R\, d\theta$$

$$\theta_1 \le \theta \le \theta_2$$

$R \to \infty$

$$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

$$\left|\int_{C_R} F(s)e^{st}\, ds\right| \le \int_{C_R} |F(s)|\left|e^{st}\right||ds|$$

$$\le \int_{\pi/2}^{3\pi/2} \frac{M}{R^p} e^{Rt\cos\theta} R\, d\theta$$

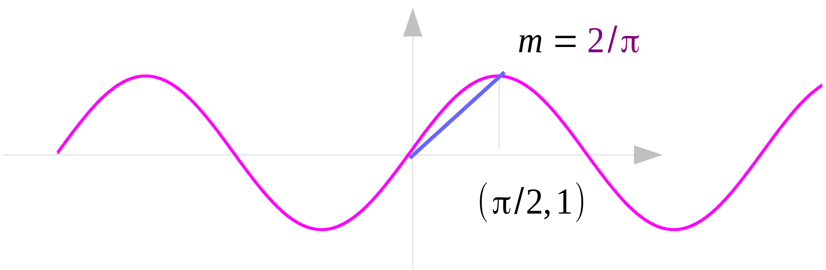
$$\le \frac{M}{R^{p-1}} \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta} d\theta$$

$\varphi = \theta - \pi/2$

$$= \frac{M}{R^{p-1}} \int_0^{\pi} e^{Rt(-\sin\varphi)} d\varphi$$

$$= \frac{2M}{R^{p-1}} \int_0^{\pi/2} e^{Rt(-\sin\varphi)} d\varphi$$

y=sin(x) and y=mx



$$\sin\varphi \;\le\; \frac{2}{\pi}\varphi \qquad \left(0 \le \varphi \le \frac{\pi}{2}\right)$$

$$\left|\int_{C_R} F(s)e^{st}\,ds\right| \;\le\; \int_{C_R} |F(s)|\,|e^{st}|\,|ds|$$

$$\le\; \frac{2M}{R^{p-1}} \int_0^{\pi/2} e^{Rt(-\sin\varphi)}\,d\varphi$$

$$\le\; \frac{2M}{R^{p-1}} \int_0^{\pi/2} e^{\frac{-2Rt\varphi}{\pi}}\,d\varphi$$

$$\le\; \frac{2M}{R^{p-1}} \left[-\frac{\pi}{2Rt}\, e^{\frac{-2Rt\varphi}{\pi}}\right]^{\pi/2}$$

$$=\; \frac{M\pi}{R^{p}t}\left(1 - e^{-Rt}\right) \;\rightarrow\; 0 \qquad R \rightarrow \infty$$

Convergence of the Laplace Transform

Laplace Transform

- f(t) continuous on [0, ∞)
- f(t) = 0 for t < 0
- f(t) has **exponential order α**
- f'(t) piecewise continuous on [0, ∞)

⇒ F(s) converges absolutely

for Re(s) > α

$$\int_0^\infty \left| f(t) e^{-st} \right| dt < \infty$$

converging

⇒ $$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

$$= \int_0^\infty \{f(t) e^{-xt}\}\, e^{-iyt}\, dt$$

Inverse Laplace Transform

- f(t) continuous on [0, ∞)
- f(t) has **exponential order α** on [0, ∞)
- f'(t) piecewise continuous on [0, ∞)
- F(s) = L{f(t)} for Re(s) > α
- $|F(s)| \le \dfrac{M}{|s|^p}$ some p > 0

  for all |s| sufficiently large
- F(s) is analytic except for finitely many poles at $z_1, z_2, \ldots, z_n$

for a given x

⇒ $$f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} F(s) e^{st}\, ds = \sum_{k=1}^{n} Res(z_k)$$

$\mathcal{L}^{-1}\{1/s(s-a)\}$

$$F(s) = \frac{1}{s(s-a)}$$

$$|F(s)| \le \frac{2}{|s|^2} \qquad |s| \ge 2|a|$$

$$|s| \ge 2|a|$$

$$\big||s| - |a|\big| \le |s-a| \le |s| - |a|$$

$$|a| \le |s| - |a| \le |s-a|$$

$$\frac{1}{|s-a|} \le \frac{1}{|a|}$$

$$\frac{1}{|s|} \le \frac{1}{2|a|}$$

$$|F(s)| = \left|\frac{1}{s}\right|\left|\frac{1}{s-a}\right| \le \frac{1}{2|a|^2} = \frac{2}{(2|a|)^2}$$

$|s| \ge 2|a|$ for a given x

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

$$\widetilde{AB},\ \widetilde{DE} \qquad 0 \le \Re(s) \le x$$

$|e^{ts}| \le e^{tx} = c$ for a fixed $t > 0$

$$\left|\int_{\widetilde{AB}} F(s)e^{st}\,ds\right| \rightarrow 0 \qquad R \rightarrow \infty$$

$$\left|\int_{\widetilde{DE}} F(s)e^{st}\,ds\right| \rightarrow 0 \qquad R \rightarrow \infty$$

$$Res(0) = \lim_{s\to 0} s e^{ts} F(s) = \lim_{s\to 0} \frac{e^{ts}}{(s-a)} = -\frac{1}{a}$$

$$Res(a) = \lim_{s\to a} (s-a) e^{ts} F(s) = \lim_{s\to 0} \frac{e^{ts}}{s} = \frac{e^{at}}{a}$$

$$f(t) = \frac{1}{a}(e^{at} - 1)$$

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$\mathcal{L}^{-1}\{s/(s^2 - a^2)\}$

$$F(s) = \frac{s}{s^2 - a^2} = L^{-1}(\cosh(a t))$$

$$|F(s)| = \left|\frac{s}{s^2 - a^2}\right| \le \frac{|s|}{|s|^2 - |a|^2}$$

$|s| \ge 2|a|$

$$|s|^2 \ge 4|a|^2$$

$$|a|^2 \le \frac{|s|^2}{4}$$

$$-|a|^2 \ge -\frac{|s|^2}{4}$$

$$|s|^2 - |a|^2 \ge \frac{3}{4}|s|^2$$

$$|F(s)| \le \frac{|s|}{3/4|s|^2} = \frac{4/3}{|s|} \qquad |s| \ge 2|a|$$

$|s| \ge 2|a|$ for a given x

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

$\widetilde{AB},\ \widetilde{DE} \qquad 0 \le \Re(s) \le x$

$\left|e^{ts}\right| \le e^{tx} = c$ for a fixed $t > 0$

$$\left|\int_{\widetilde{AB}} F(s)e^{st}\,ds\right| \to 0 \qquad R \to \infty$$

$$\left|\int_{\widetilde{DE}} F(s)e^{st}\,ds\right| \to 0 \qquad R \to \infty$$

$$Res(a) = \lim_{s\to 0}(s-a)e^{ts}F(s) = \lim_{s\to 0}\frac{s\,e^{ts}}{(s+a)} = \frac{e^{at}}{2}$$

$$Res(-a) = \lim_{s\to a}(s+a)e^{ts}F(s) = \lim_{s\to 0}\frac{s\,e^{ts}}{(s-a)} = -\frac{e^{-at}}{2}$$

$$f(t) = \frac{e^{+at} - e^{-at}}{2} = \cosh(at)$$

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$\mathcal{L}^{-1}\{1/s(s^2+a^2)^2\}$

$$F(s) = \frac{1}{s(s^2+a^2)^2}$$

$$(s^2+a^2) \geq s^2 \qquad \frac{1}{(s^2+a^2)} \leq \frac{1}{s^2}$$

$$|F(s)| = \left|\frac{1}{s(s^2+a^2)^2}\right| \leq \frac{1}{|s|^5}$$

$$|F(s)| \leq \frac{M}{|s|^5}$$ for a suitably large $|s|$

$$F(s) = \frac{1}{s(s^2+a^2)^2} = \frac{1}{s(s-ia)^2(s+ia)^2}$$

for a suitably large x

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

$$Res(0) = \lim_{s\to 0}\{s e^{ts}F(s)\} = \frac{e^{t0}}{(0-ia)^2(0+ia)^2} = \frac{1}{a^4}$$

$$Res(\pm ia) = \lim_{s\to\pm ia}\frac{d}{ds}\{(s\mp ia)^2 e^{ts}F(s)\}$$

$$= \lim_{s\to\pm ia}\frac{d}{ds}\left\{\frac{e^{ts}}{s(s\pm ia)^2}\right\} = \lim_{s\to\pm ia}\frac{d}{ds}\left\{\frac{e^{ts}}{s^3-a^2s\pm 2ias^2}\right\}$$

$$= \lim_{s\to\pm ia}\left\{\frac{e^{ts}[t(s^3-a^2s\pm 2ias^2)-(3s^2-a^2\pm 4ias)]}{(s^3-a^2s\pm 2ias^2)^2}\right\}$$

$$= \lim_{s\to\pm ia}\left\{\frac{e^{\pm iat}[t(\mp ia^3\mp ia^3\mp 2ia^3)-(-3a^2-a^2-4a^2)]}{(\mp ia^3\mp ia^3\mp 2ia^3)^2}\right\}$$

$$= \left\{\frac{e^{\pm iat}[it(\mp 4a^3)+(8a^2)]}{-(\mp 4a^3)^2}\right\} = \left\{\pm\frac{it}{4a^3}e^{\pm iat} - \frac{1}{2a^4}e^{\pm iat}\right\}$$

$$f(t) = \left\{+\frac{it}{4a^3}e^{+iat} - \frac{1}{2a^4}e^{+iat}\right\} + \left\{-\frac{it}{4a^3}e^{-iat} - \frac{1}{2a^4}e^{-iat}\right\} + \frac{1}{a^4}$$

$$= \frac{1}{a^3}\left\{1 - \frac{a}{2}t\sin at - \cos at\right\}$$

Inverse Laplace Transform

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0} = \frac{a_n + a_{n-1}s^{-1} + \cdots + a_0 s^{-n}}{s^{m-n}\left(b_m + b_{m-1}s^{-1} + \cdots + b_0 s^{-m}\right)}$$

$$\left| a_n + \frac{a_{n-1}}{s} + \cdots + \frac{a_0}{s^n} \right| \le |a_n| + |a_{n-1}| + \cdots + |a_0| = c_1$$

$$\left| b_m + \frac{b_{m-1}}{s} + \cdots + \frac{b_0}{s^m} \right| \ge |b_m| - \frac{|b_{m-1}|}{|s|} - \cdots - \frac{|b_0|}{|s|^m}$$

$$\ge \frac{|b_m|}{2} = c_2$$

$$|s| \ge \frac{|b_{m-1}|}{|b_m|} \qquad \frac{1}{|s|} \le \frac{|b_m|}{|b_m - 1|}$$

$$|s| \ge \frac{m}{2}\frac{|b_{m-1}|}{|b_m|} \qquad \frac{|b_m - 1|}{|s|} \le 2\frac{|b_m|}{m}$$

$$|s| \ge \frac{m}{2}\frac{|b_{m-1}|}{|b_m|}, \quad \frac{m}{2}\frac{|b_{m-2}|}{|b_m|} \quad \cdots, \quad \frac{m}{2}\frac{|b_0|}{|b_m|}$$

$$|F(s)| \le \frac{c_1/c_2}{|s|^{m-n}}$$ for a suitably large $|s|$

$$Res(z_0) = \lim_{z \to z_0} (z - z_0)\frac{P(z)}{Q(z)}$$

$$= \lim_{z \to z_0} \left\{ \frac{P(z)}{\dfrac{Q(z) - Q(z_0)}{z - z_0}} \right\} = \frac{P(z_0)}{Q'(z_0)}$$

$$Res(z_k) = \frac{e^{z_k t} P(z_k)}{Q'(z_k)}$$

for a suitably large x

$$f(t) = \frac{1}{2\pi i}\int_{x - i\infty}^{x + i\infty} F(s) e^{st}\, ds = \sum_{k=1}^{n} Res(z_k)$$

$$f(t) = \sum_{z_k} Res(z_k) = \sum_{z_k} \frac{e^{z_k t} P(z_k)}{Q'(z_k)}$$

Infinitely Many Poles

Assume F(s) has
infinitely many poles to the left of the line

$\{z_k\}_{k=1}^{\infty}$ $\Re\{s\} = x_0 > 0$

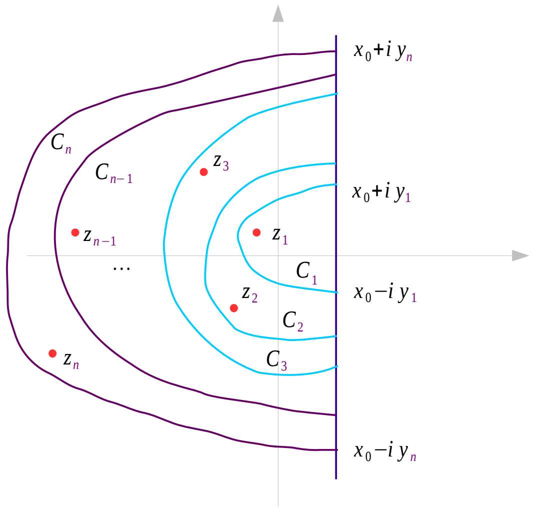
each pole has the condition

$$|z_1| \le |z_2| \le \cdots \le |z_k| \le \cdots$$

$$|z_k| \to \infty \quad as \quad k \to \infty$$

Consider the contour
which encloses the first n poles

$$\Gamma_n = C_n \cup [x_0 - iy_n,\ x_0 + iy_n]$$



Cauchy Residue Theorem

$$f(t) = \frac{1}{2\pi i}\int_{\Gamma_n} F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

$$= \frac{1}{2\pi i}\int_{x_0-iy_n}^{x_0+iy_n} F(s)e^{st}\,ds + \frac{1}{2\pi i}\int_{C_n} F(s)e^{st}\,ds$$

If we can show

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{C_n} F(s)e^{st}\,ds = 0$$

$$|y_n| \to \infty$$

then we have

$$f(t) = \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} F(s)e^{st}\,ds = \sum_{k=1}^{\infty} Res(z_k)$$

$\mathcal{L}^{-1}\{1/s(1 + e^{as})\}$ (a>0)

$$F(s) = \frac{1}{s(1 + e^{as})}$$

$$s = 0 \qquad e^{as} = -1 = e^{(2n-1)\pi i}$$

$$s = \left(\frac{2n-1}{a}\right)\pi i \qquad n = 0, \pm 1, \pm 2, \cdots$$

$$G(s) = 1 + e^{as} \qquad \frac{d}{ds}G(s) = a e^{as}$$

$$G(s_n) = 0 \qquad G'(s_n) = -a < 0$$

All are simple pole

$$Res(0) = \lim_{s \to 0} s e^{ts} F(s) = \lim_{s \to 0} \frac{1}{(1+e^{as})} = \frac{1}{2}$$

$$Res(s_n) = \lim_{s \to 0} (s-s_n) e^{ts} F(s)$$

$$= \lim_{s \to 0} (s-s_n) \frac{P(s_n)}{Q'(s_n)}$$

$$P(s) = e^{as}$$
$$Q(s) = s G(s)$$

$$= \frac{e^{ts_n}}{G(s_n)+sG'(s_n)} = \frac{e^{ts_n}}{(1 + e^{as_n})+a s_n e^{as_n}}$$

$$= \frac{e^{ts_n}}{a s_n e^{as_n}} = -\frac{e^{ts_n}}{a s_n} \qquad as = (2n-1)\pi i$$

$$Res(s_n) = -\frac{e^{t\left(\frac{2n-1}{a}\right)\pi i}}{(2n-1)\pi i}$$

$$\sum Res = \frac{1}{2} - \sum_{n=-\infty}^{\infty} \frac{e^{t\left(\frac{2n-1}{a}\right)\pi i}}{(2n-1)\pi i}$$

$$= \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\left\{e^{t\left(\frac{2n-1}{a}\right)\pi i} - e^{t\left(\frac{2n-1}{a}\right)\pi i}\right\}}{2i(2n-1)}$$

$$f(t) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\left\{\left(\frac{2n-1}{a}\right)\pi t\right\}$$

Cauchy Residue Theorem

$\mathcal{L}^{-1}\{1/s(s^2 + a^2)^2\}$

$$F(s) = \frac{1}{s(s^2+a^2)^2}$$

$$(s^2+a^2) \geq s^2 \qquad \frac{1}{(s^2+a^2)} \leq \frac{1}{s^2}$$

$$|F(s)| = \left|\frac{1}{s(s^2+a^2)^2}\right| \leq \frac{1}{|s|^5}$$

$$|F(s)| \leq \frac{M}{|s|^5}$$ for a suitably large $|s|$

$$F(s) = \frac{1}{s(s^2+a^2)^2} = \frac{1}{s(s-ia)^2(s+ia)^2}$$

for a suitably large x

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

$$Res(0) = \lim_{s\to 0}\{s e^{ts} F(s)\} = \frac{e^{t0}}{(0-ia)^2(0+ia)^2} = \frac{1}{a^4}$$

$$Res(\pm ia) = \lim_{s\to\pm ia}\frac{d}{ds}\{(s\mp ia)^2 e^{ts}F(s)\}$$

$$= \lim_{s\to\pm ia}\frac{d}{ds}\left\{\frac{e^{ts}}{s(s\pm ia)^2}\right\} = \lim_{s\to\pm ia}\frac{d}{ds}\left\{\frac{e^{ts}}{s^3-a^2s\pm 2ias^2}\right\}$$

$$= \lim_{s\to\pm ia}\left\{\frac{e^{ts}[t(s^3-a^2s\pm 2ias^2)-(3s^2-a^2\pm 4ias)]}{(s^3-a^2s\pm 2ias^2)^2}\right\}$$

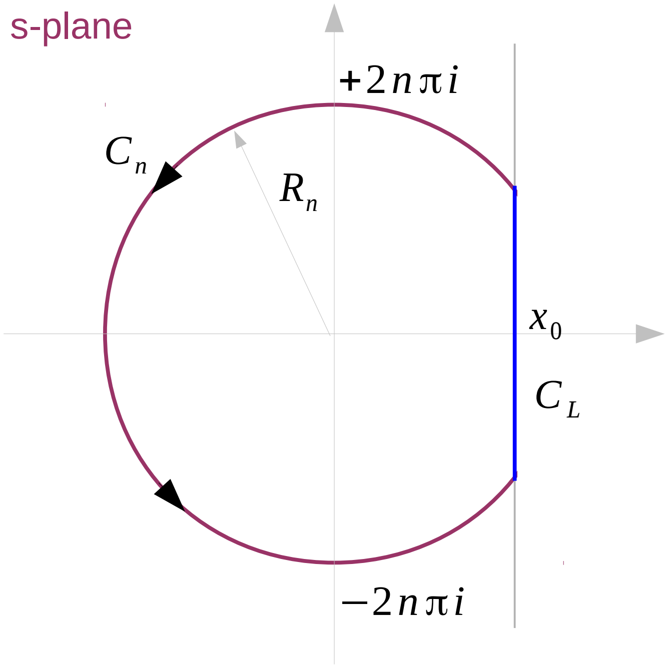
$$= \lim_{s\to\pm ia}\left\{\frac{e^{\pm iat}[t(\mp ia^3\mp ia^3\mp 2ia^3)-(-3a^2-a^2-4a^2)]}{(\mp ia^3\mp ia^3\mp 2ia^3)^2}\right\}$$

$$= \left\{\frac{e^{\pm iat}[it(\mp 4a^3)+(8a^2)]}{-(\mp 4a^3)^2}\right\} = \left\{\pm\frac{it}{4a^3}e^{\pm iat} - \frac{1}{2a^4}e^{\pm iat}\right\}$$

$$f(t) = \left\{+\frac{it}{4a^3}e^{+iat} - \frac{1}{2a^4}e^{+iat}\right\} + \left\{-\frac{it}{4a^3}e^{-iat} - \frac{1}{2a^4}e^{-iat}\right\} + \frac{1}{a^4}$$

$$= \frac{1}{a^3}\left\{1 - \frac{a}{2}t\sin at - \cos at\right\}$$

$\mathcal{L}^{-1}\{1/s(1 + e^{as})\}$ (a>0)



points on the contour C_n

$$s = R_n e^{i\theta} = \frac{2n\pi}{a} e^{i\theta} \qquad \left(R_n = \frac{2n\pi}{a}\right)$$

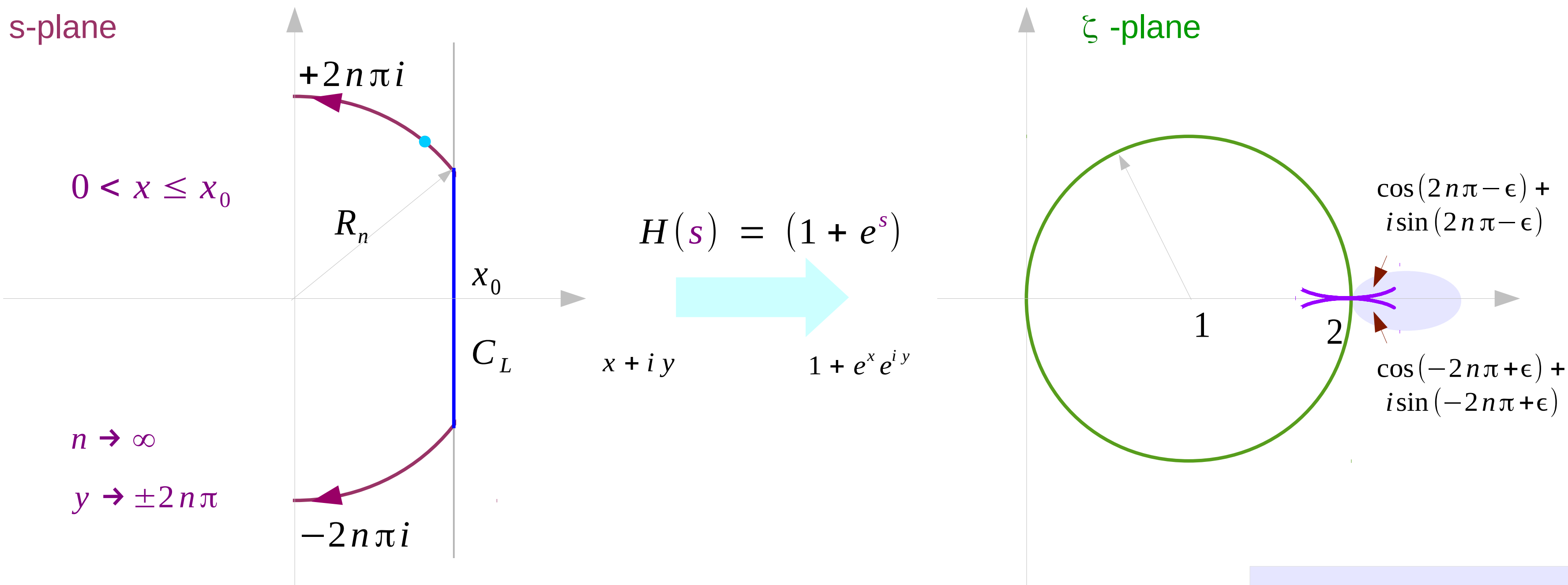
assume $a = 1$

$$s = R_n e^{i\theta} = 2n\pi e^{i\theta} \qquad \left(R_n = 2n\pi\right)$$

$s = +2n\pi i$ ← $\theta = +\pi/2$

$s = -2n\pi i$ ← $\theta = -\pi/2$

$\mathcal{L}^{-1}\{1/s(1 + e^{as})\}$ (a>0)



$$s = x + iy \qquad\qquad \zeta = 1 + e^x e^{iy}$$

$0 < x \le x_0$

$y \to \pm 2n\pi$ as $n \to \infty$

$0 < x \le x_0$

$-2n\pi \le y \le +2n\pi$

→

$1 < e^x \le e^{x_0}$

$e^{iy} = 1$

→

$|1 + e^s| > 2$

$$\cos(2n\pi) - i\sin(2n\pi) = 1$$

$$\cos(2n\pi) + i\sin(2n\pi) = 1$$

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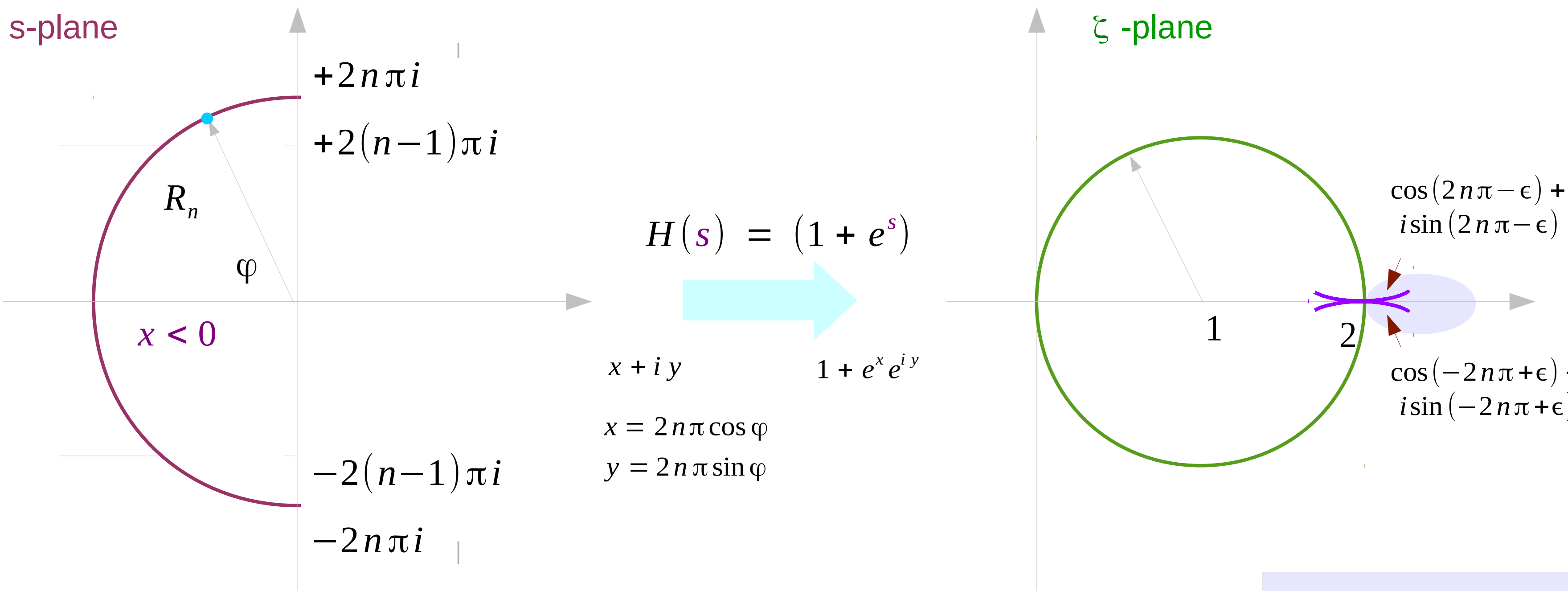
$\boldsymbol{L}^{-1}\{1/s(1 + e^{as})\}$ (a>0)



$s = x + iy$

$\zeta = 1 + e^x e^{iy}$

$0 < x \leq x_0$

$y \rightarrow \pm 2n\pi$ as $n \rightarrow \infty$

$x < 0$

$y = +2n\pi$

$\wr$

$y = +2(n-1)\pi$

→

$e^x < 1$

$e^{iy} = 1$

$e^{2n\pi\cos\varphi}\cos(2n\pi\sin\varphi)$

$e^{iy} = 1$

→

$|1 + e^s| >$

$e^{2n\pi\cos\varphi}\cos(2n\pi\sin\varphi)$

$> b > 0$

$\mathcal{L}^{-1}\{1/s(1 + e^{as})\}$ (a>0)

$x = 2n\pi\cos\varphi$

$y = 2n\pi\sin\varphi$

$s = x + iy$ $\qquad$ $\zeta = 1 + e^{x}e^{iy}$

	$x < 0$	→	$e^{x} < 1$
$\sin\varphi = 1$	$y = +2n\pi$		$e^{iy} = +1$
$\sin\varphi = \frac{(n-1/4)}{n}$	$y = +2n\pi - \frac{1}{2}\pi$	→	$e^{iy} = +i$
$\sin\varphi = \frac{(n-1/2)}{n}$	$y = +2n\pi - \pi$		$e^{iy} = -1$
$\sin\varphi = \frac{(n-3/4)}{n}$	$y = +2n\pi - \frac{3}{2}\pi$		$e^{iy} = -i$
$\sin\varphi = \frac{(n-1)}{n}$	$y = +2n\pi - 2\pi$		$e^{iy} = +1$

$\mathcal{L}^{-1}\{1/s(1 + e^{as})\}$ (a>0)

$x = 2n\pi\cos\varphi$

$y = 2n\pi\sin\varphi$

$s = x + iy$

$\zeta = 1 + e^{x}e^{iy}$

$0 < x \le x_0$

$y \to \pm 2n\pi \quad$ as $n \to \infty$

$x < 0$

$\varphi = +\pi/2$: $y = +2n\pi$

$y = +(2n-1)\pi$

$\varphi = -\pi/2$: $y = +(2n-2)\pi$

→

$e^{x} < 1$

$e^{iy} = +1$

$e^{iy} = -i$

$e^{iy} = +1$

→

$|1 + e^{s}| >$

$e^{2n\pi\cos\varphi}\cos(2n\pi\sin\varphi)$

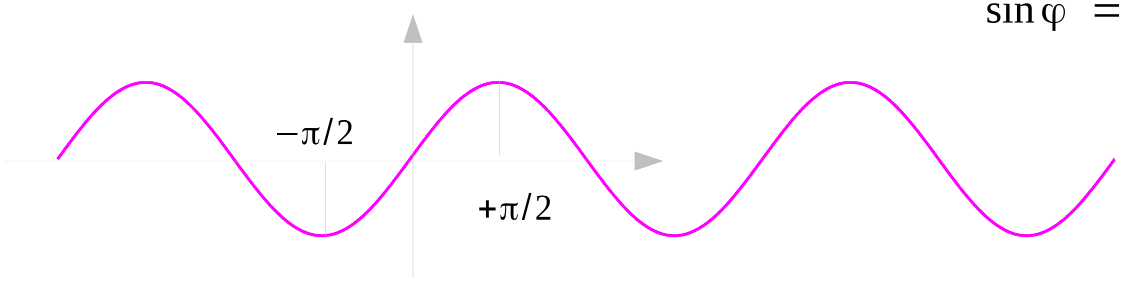
$> b > 0$

$e^{2n\pi\cos\varphi}\cos(2n\pi\sin\varphi)$

$m = 2/\pi$

$y = +2(n-1)\pi = 2n\pi\sin\varphi$

$$\sin\varphi = \frac{(n-1)}{n} \qquad \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{999}{1000}, \cdots$$



$\mathcal{L}^{-1}\{1/s(s^2 + a^2)^2\}$

$\{z_k\}_{k=1}^{\infty}$ $\Re\{s\} = x_0 > 0$

$|z_1| \leq |z_2| \leq \cdots$

$\Gamma_n = C_n \cup [x_0 - iy_n,\, x_0 + iy_n]$

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

$$\sum_{k=1}^{n} Res(z_k) = \frac{1}{2\pi i}\int_{x-iy_n}^{x+iy_n} F(s)e^{st}\,ds + \frac{1}{2\pi i}\int_{C_n} F(s)e^{st}\,ds$$

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{C_n} F(s)e^{st}\,ds = 0$$

Another consequence is that if $f(z) = \sum a_n z^n$ is holomorphic in $|z| < R$ and $0 < r < R$ then the coefficients a_n satisfy **Cauchy's inequality**[1]

$$|a_n| \leq r^{-n} \sup_{|z|=r} |f(z)|.$$

$$\sum_{k=1}^{n} Res(z_k) = \frac{1}{2\pi i}\int_{x-iy_n}^{x+iy_n} F(s)e^{st}\,ds = \sum_{k=1}^{n} Res(z_k)$$

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