Definitions of the Laplace Transform (1A)

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2/9/15

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Improper Integral

Hiding the limiting process

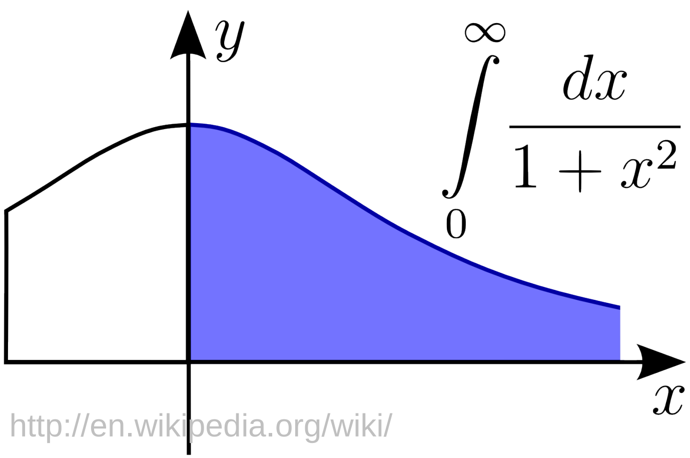
$$\lim_{b \to +\infty} \int_a^b f(x)\, dx \quad I = \int_a^b f(x)\, dx \;\Rightarrow\; \lim_{b \to +\infty} I \;\Rightarrow\; \begin{cases} L & \text{converge} \\ \infty & \text{diverge} \end{cases} \qquad \int_a^{+\infty} f(x)\, dx$$

$$\lim_{a \to -\infty} \int_a^b f(x)\, dx \quad I = \int_a^b f(x)\, dx \;\Rightarrow\; \lim_{a \to -\infty} I \;\Rightarrow\; \begin{cases} L & \text{converge} \\ \infty & \text{diverge} \end{cases} \qquad \int_{-\infty}^{b} f(x)\, dx$$

$$\lim_{c \to b^-} \int_a^c f(x)\, dx \quad I = \int_a^c f(x)\, dx \;\Rightarrow\; \lim_{c \to b^-} I \;\Rightarrow\; \begin{cases} L & \text{converge} \\ \infty & \text{diverge} \end{cases} \qquad \int_a^{b^-} f(x)\, dx$$

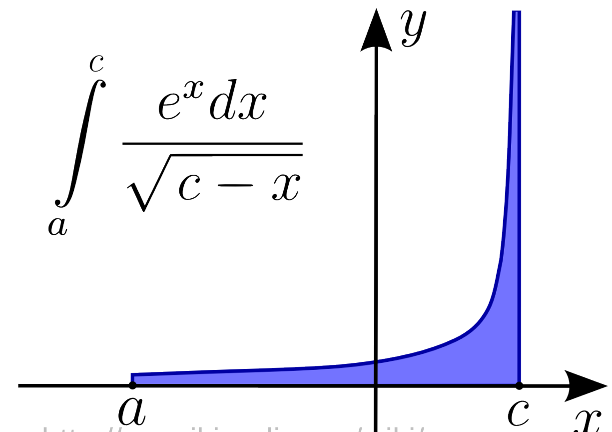
$$\lim_{c \to a^+} \int_c^b f(x)\, dx \quad I = \int_c^b f(x)\, dx \;\Rightarrow\; \lim_{c \to a^+} I \;\Rightarrow\; \begin{cases} L & \text{converge} \\ \infty & \text{diverge} \end{cases} \qquad \int_{a^+}^{b} f(x)\, dx$$

Improper Integral Examples



$$\int_0^{\infty} \frac{dx}{1+x^2}$$

$$\int_a^{c} \frac{e^x\,dx}{\sqrt{c-x}}$$



$$\lim_{b \to +\infty} \int_a^b f(x)\,dx \;\Rightarrow\; \int_a^{+\infty} f(x)\,dx$$

$$\lim_{b \to c^-} \int_a^b f(x)\,dx \;\Rightarrow\; \int_a^{c^-} f(x)\,dx$$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b \to \infty} \left[-\frac{1}{x}\right]_1^b$$

$$= \lim_{b \to \infty} \left(-\frac{1}{b} + \frac{1}{1}\right) = 1 \quad \text{converge}$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a \to 0^+} \left[2\sqrt{x}\right]_a^1$$

$$= \lim_{a \to 0^+} \left(2 - 2\sqrt{b}\right) = 2 \quad \text{converge}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{\sqrt{x}} = \infty \qquad f(0)$$

An Improper Integration

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

Complex Number — Real Number — Real Number

$$s = \sigma + i\omega$$

$\Re\{s\}$ real part

$\Im\{s\}$ imag part

t Integration Variable

The improper integral **converges** if the limit defining it exists.

F(s) : a function of s

For a given function f(t)

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

$$g(s,t) = f(t) e^{-st}$$

$$\frac{\partial}{\partial t} G(s,t) = g(s,t)$$

G: an antiderivative of **g** with respect to **t**

$$\int_0^\infty g(s,t)\,dt = \lim_{b\to\infty} \left[G(s,t) \right]_0^b$$

$$= \lim_{b\to\infty} \left[G(s,b) - G(s,0) \right]$$

During integration, complex variable **s** is treated as a constant

In the result, the literal ***t*** vanishes

$$\int_0^\infty g(s,t)\,dt = F(s)$$ *a function of* ***s***

An Integration Function

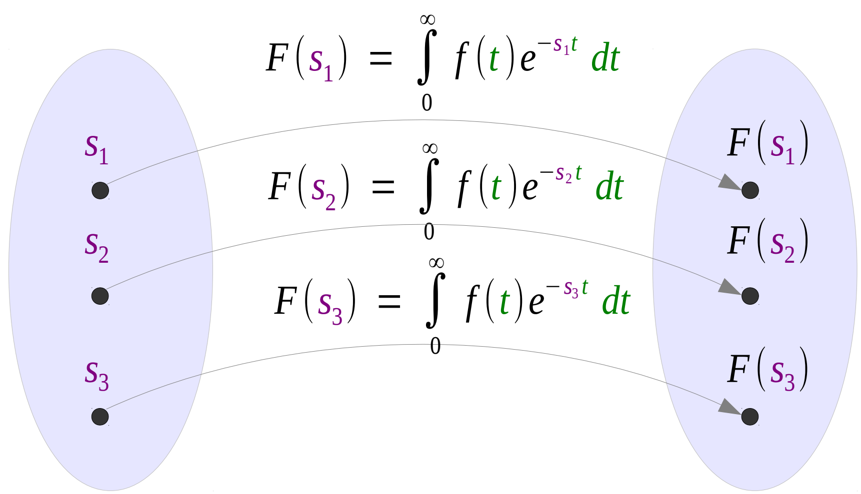
For a given function f(t)

$$F(s) = \int_0^{\infty} f(t) e^{-st} \, dt$$

Complex Number → s; Real Number → σ, ω

$$s = \sigma + i\omega$$

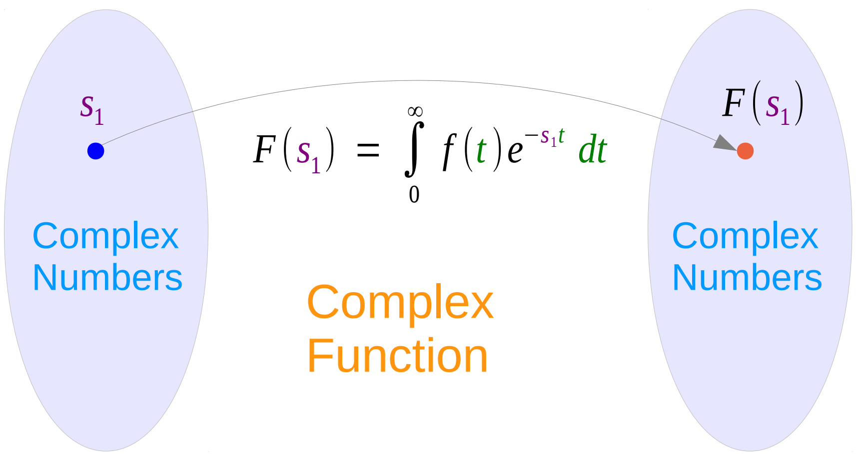
t Real Number Integration Variable



Complex Numbers Complex Numbers

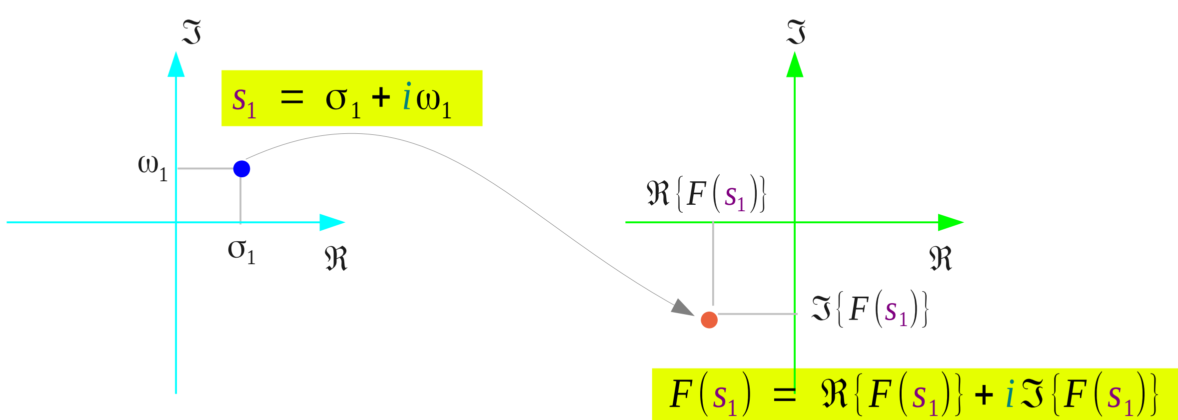
F(s) : a Complex Function

For a given function f(t)

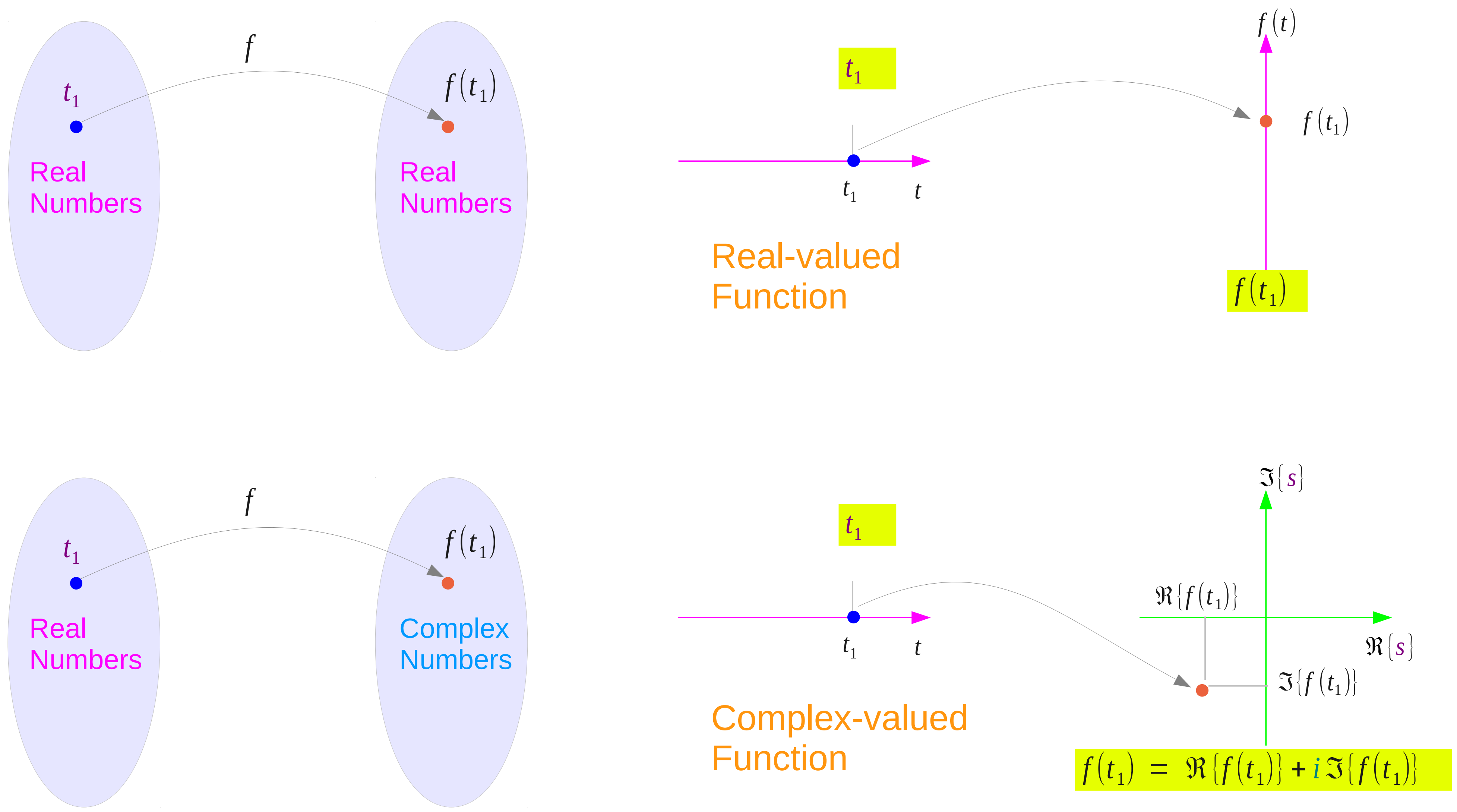


Complex Number   Real Number

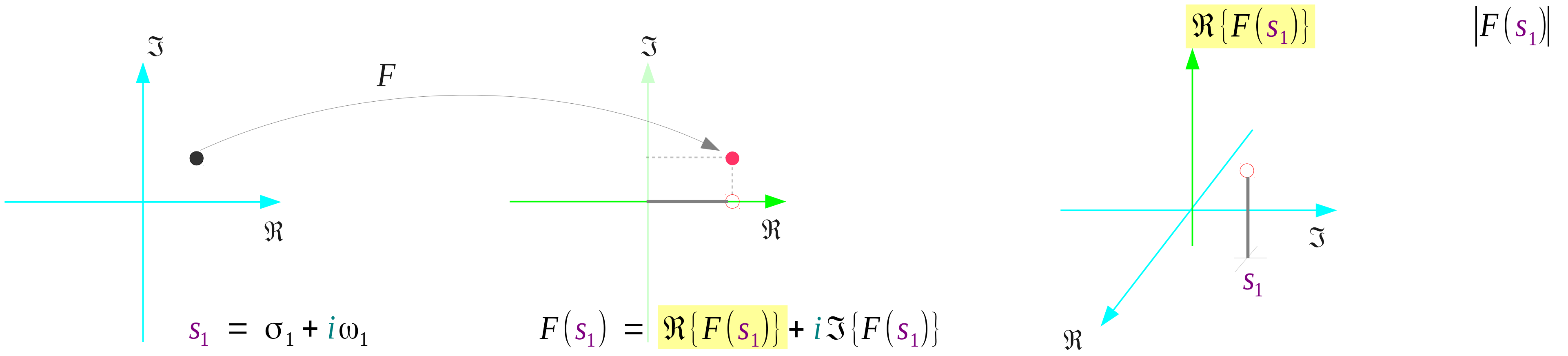
$$s = \sigma + i\omega$$



f(t) : a real-valued or complex-valued function

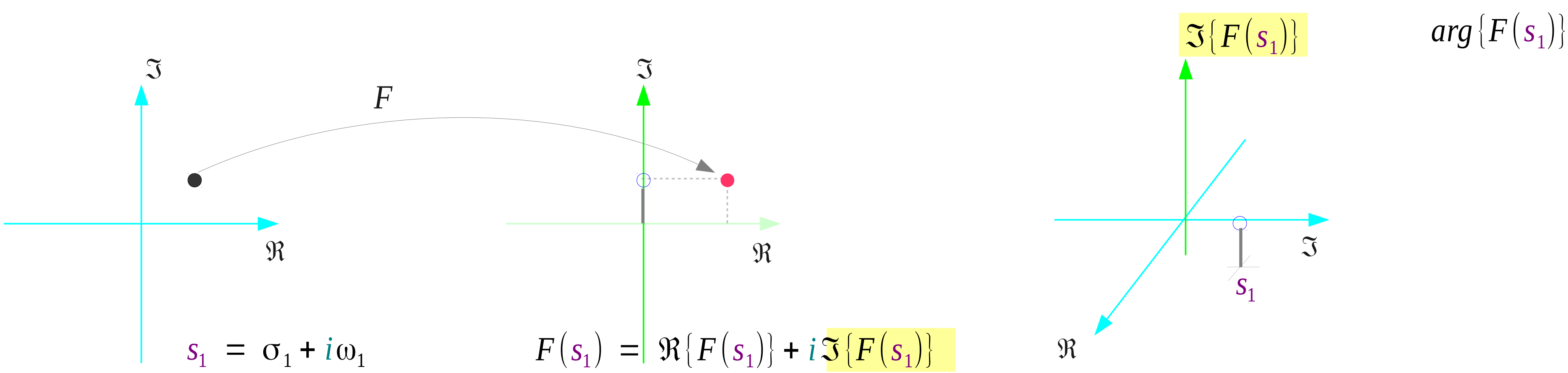


Complex Function Plot



$s_1 = \sigma_1 + i\omega_1$

$F(s_1) = \Re\{F(s_1)\} + i\Im\{F(s_1)\}$



$s_1 = \sigma_1 + i\omega_1$

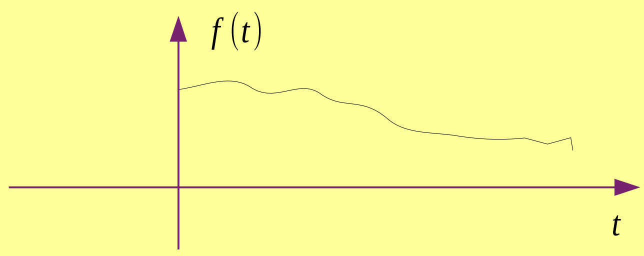
$F(s_1) = \Re\{F(s_1)\} + i\Im\{F(s_1)\}$

Two Functions: f(t) & F(s)

For a given function f(t)
there exits a unique F(s)

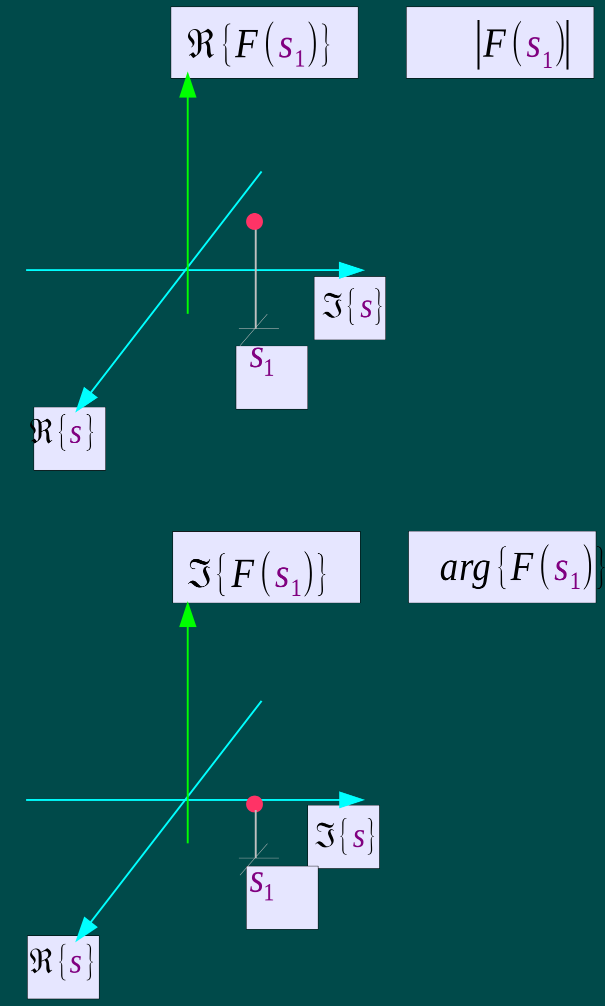
$$f(t) \Longleftrightarrow F(s)$$

t-domain function **f(t)**



Real number domain function f(t)

s-domain function **F(s)**



Complex number domain function F(s)

Laplace Transform

$f_a(x) \Rightarrow \boxed{L} \Rightarrow F_A(s)$

$f_b(t) \Rightarrow \boxed{L} \Rightarrow F_B(s)$

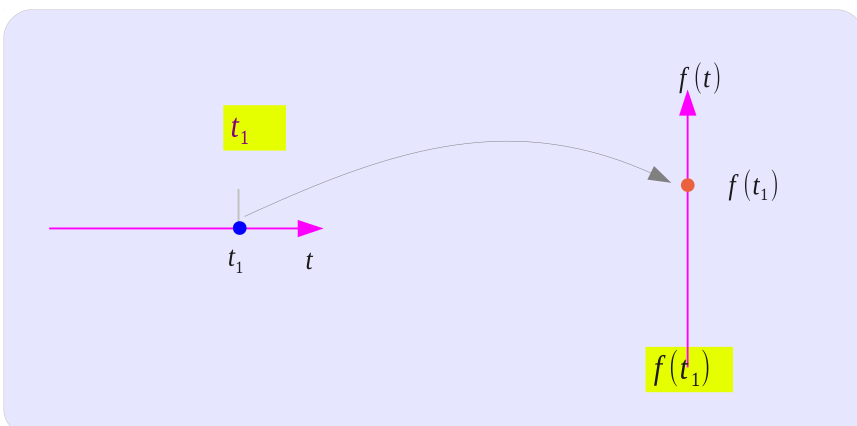
$f_c(t) \Rightarrow \boxed{L} \Rightarrow F_C(s)$

$1 \Rightarrow \boxed{\int_0^\infty f_1(t) e^{-st}\, dt = F_1(s)} \Rightarrow \frac{1}{s}$

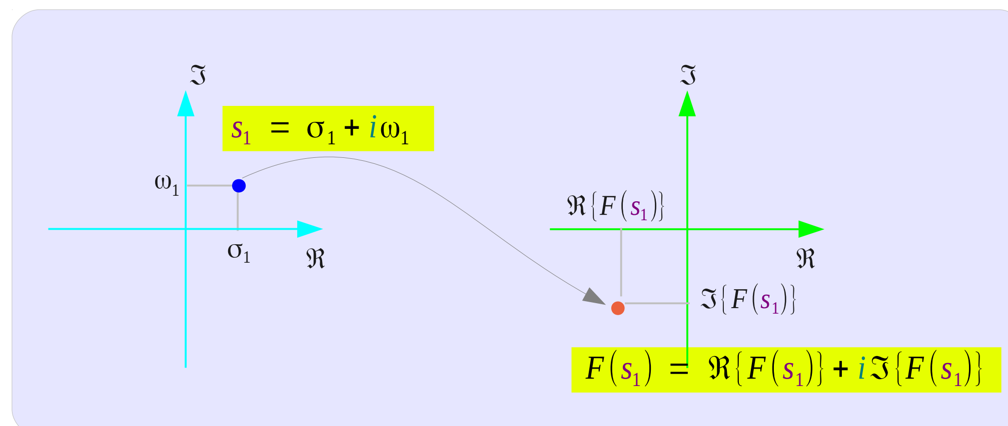
$e^{-at} \Rightarrow \boxed{\int_0^\infty f_2(t) e^{-st}\, dt = F_2(s)} \Rightarrow \frac{1}{s+a}$

$\cos(kt) \Rightarrow \boxed{\int_0^\infty f_3(t) e^{-st}\, dt = F_3(s)} \Rightarrow \frac{s}{s^2+k^2}$

$f_a(x)$ Real-valued Function



$F_A(s)$ Complex Function



Laplace transforms of 1 and exp(−at)

$1 \rightarrow \boxed{\mathcal{L}} \rightarrow \frac{1}{s}$

$$F(s) = \int_0^\infty 1 \cdot e^{-st}\, dt = \lim_{b\to\infty} \left[-\frac{1}{s} e^{-st} \right]_0^b = \lim_{b\to\infty} \left[-\frac{1}{s} e^{-sb} + \frac{1}{s} e^{-s0} \right]$$

$$-s < 0 \Rightarrow \lim_{b\to\infty} e^{-sb} = 0 \qquad s > 0 \Rightarrow F(s) = \frac{1}{s}$$

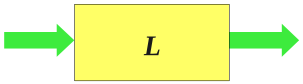
$e^{-at} \rightarrow \boxed{\mathcal{L}} \rightarrow \frac{1}{s+a}$

$$F(s) = \int_0^\infty e^{-at} \cdot e^{-st}\, dt = \lim_{b\to\infty} \left[-\frac{1}{(s+a)} e^{-(s+a)t} \right]_0^b = \lim_{b\to\infty} \left[-\frac{1}{(s+a)} e^{-(s+a)b} + \frac{1}{(s+a)} e^{-(s+a)0} \right]$$

$$-(s+a) < 0 \Rightarrow \lim_{b\to\infty} e^{-(s+a)b} = 0 \qquad s > -a \Rightarrow F(s) = \frac{1}{(s+a)}$$

Laplace transforms of exp(+at) and exp(–at)

e^{-at}

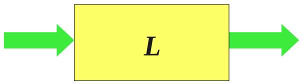


$\dfrac{1}{s+a}$

$$F(s) = \int_0^\infty e^{-at} \cdot e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{(s+a)}e^{-(s+a)t}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{(s+a)}e^{-(s+a)b} + \frac{1}{(s+a)}e^{-(s+a)0}\right]$$

$$-(s+a) < 0 \Rightarrow \lim_{b\to\infty} e^{-(s+a)b} = 0 \qquad s > -a \Rightarrow F(s) = \frac{1}{(s+a)}$$

e^{+at}



$\dfrac{1}{s-a}$

$$F(s) = \int_0^\infty e^{+at} \cdot e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{(s-a)}e^{-(s-a)t}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{(s-a)}e^{-(s-a)b} + \frac{1}{(s-a)}e^{-(s-a)0}\right]$$

$$-(s-a) < 0 \Rightarrow \lim_{b\to\infty} e^{-(s-a)b} = 0 \qquad s > +a \Rightarrow F(s) = \frac{1}{(s-a)}$$

Laplace transforms of cosh(kt) and sinh(kt)

$$\cosh(kt) \xrightarrow{\ \mathcal{L}\ } \frac{s}{s^2-k^2}$$

$$\cosh(kt) = \frac{(e^{+kt} + e^{-kt})}{2}$$

$$F(s) = \int_0^{\infty} \frac{(e^{+kt}+e^{-kt})}{2} \cdot e^{-st}\, dt = \frac{1}{2}\int_0^{\infty} e^{+kt} \cdot e^{-st}\, dt + \frac{1}{2}\int_0^{\infty} e^{-kt} \cdot e^{-st}\, dt$$

$s > +k \Rightarrow \frac{1}{(s-k)}$ $\quad$ $s > -k \Rightarrow \frac{1}{(s+k)}$

$$F(s) = \frac{1}{2}\left(\frac{1}{(s-k)} + \frac{1}{(s+k)}\right) = \frac{s}{(s^2-k^2)}$$

$$\sinh(kt) \xrightarrow{\ \mathcal{L}\ } \frac{k}{s^2-k^2}$$

$$\sinh(kt) = \frac{(e^{+kt} - e^{-kt})}{2}$$

$$F(s) = \int_0^{\infty} \frac{(e^{+kt}-e^{-kt})}{2} \cdot e^{-st}\, dt = \frac{1}{2}\int_0^{\infty} e^{+kt} \cdot e^{-st}\, dt - \frac{1}{2}\int_0^{\infty} e^{-kt} \cdot e^{-st}\, dt$$

$s > +k \Rightarrow \frac{1}{(s-k)}$ $\quad$ $s > -k \Rightarrow \frac{1}{(s+k)}$

$$F(s) = \frac{1}{2}\left(\frac{1}{(s-k)} - \frac{1}{(s+k)}\right) = \frac{k}{(s^2-k^2)}$$

Laplace transforms of cos(kt) and sin(kt)

$$\cos(kt) \;\rightarrow\; \boxed{L} \;\rightarrow\; \frac{s}{s^2+k^2}$$

$$\cos(kt) = \frac{\left(e^{+jkt} + e^{-jkt}\right)}{2}$$

$$F(s) = \int_0^\infty \frac{\left(e^{+jkt}+e^{-jkt}\right)}{2}\cdot e^{-st}\,dt = \frac{1}{2}\int_0^\infty e^{+jkt}\cdot e^{-st}\,dt + \frac{1}{2}\int_0^\infty e^{-jkt}\cdot e^{-st}\,dt$$

$s > 0$ $\rightarrow$ $\frac{1}{(s-j\omega)}$ $\qquad$ $s > 0$ $\rightarrow$ $\frac{1}{(s+j\omega)}$

$$F(s) = \frac{1}{2}\left(\frac{1}{(s-j\omega)} + \frac{1}{(s+j\omega)}\right) = \frac{s}{(s^2+k^2)}$$

$$\sin(kt) \;\rightarrow\; \boxed{L} \;\rightarrow\; \frac{k}{s^2+k^2}$$

$$\sin(kt) = \frac{\left(e^{+jkt} - e^{-jkt}\right)}{2j}$$

$$F(s) = \int_0^\infty \frac{\left(e^{+jkt}-e^{-jkt}\right)}{2j}\cdot e^{-st}\,dt = \frac{1}{2j}\int_0^\infty e^{+jkt}\cdot e^{-st}\,dt - \frac{1}{2j}\int_0^\infty e^{-jkt}\cdot e^{-st}\,dt$$

$s > 0$ $\rightarrow$ $\frac{1}{(s-j\omega)}$ $\qquad$ $s > 0$ $\rightarrow$ $\frac{1}{(s+j\omega)}$

$$F(s) = \frac{1}{2}\left(\frac{1}{(s-j\omega)} - \frac{1}{(s+j\omega)}\right) = \frac{k}{(s^2+k^2)}$$

Laplace transform of cos(kt)

$$\cos(kt) \rightarrow \boxed{L} \rightarrow \frac{s}{s^2+k^2}$$

$$F(s) = \int_0^\infty \cos(kt)\cdot e^{-st}\,dt = \lim_{b\to\infty}\left\{\left[\frac{1}{k}\sin(kt)e^{-st}\right]_0^b - \int_0^\infty -\frac{s}{k}\sin(kt)\cdot e^{-st}\,dt\right\}$$

$$\frac{s}{k}\int_0^\infty \sin(kt)\cdot e^{-st}\,dt = \lim_{b\to\infty}\frac{s}{k}\left\{\left[-\frac{1}{k}\cos(kt)e^{-st}\right]_0^b - \int_0^\infty \frac{s}{k}\cos(kt)\cdot e^{-st}\,dt\right\}$$

★ $$\lim_{b\to\infty}\left\{\left[\frac{1}{k}\sin(kt)e^{-st}\right]_0^b\right\} = \lim_{b\to\infty}\left\{\frac{1}{k}\sin(kb)e^{-sb} - \frac{1}{k}\sin(k0)e^{-s0}\right\} = 0$$

⬢ $$\lim_{b\to\infty}\left\{\left[-\frac{1}{k}\cos(kt)e^{-st}\right]_0^b\right\} = \lim_{b\to\infty}\left\{-\frac{1}{k}\cos(kb)e^{-sb} + \frac{1}{k}\cos(k0)e^{-s0}\right\} = \frac{1}{k}$$

$$F(s) = \int_0^\infty \cos(kt)\cdot e^{-st}\,dt = \lim_{b\to\infty}\frac{s}{k}\left\{\frac{1}{k} - \frac{s}{k}\int_0^\infty \cos(kt)\cdot e^{-st}\,dt\right\}$$

$$F(s) = \frac{s}{k^2} - \frac{s^2}{k^2}F(s) \Rightarrow \left(1+\frac{s^2}{k^2}\right)F(s) = \frac{s}{k^2} \Rightarrow \left(\frac{s^2+k^2}{k^2}\right)F(s) = \frac{s}{k^2}$$

$$F(s) = \frac{s}{(s^2+k^2)}$$

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Laplace transform of sin(kt)

$\sin(kt) \rightarrow$ L $\rightarrow \dfrac{k}{s^2+k^2}$

$$F(s) = \int_0^\infty \sin(kt)\cdot e^{-st}\,dt = \lim_{b\to\infty}\left\{\left[-\frac{1}{k}\cos(kt)e^{-st}\right]_0^b - \int_0^\infty +\frac{s}{k}\cos(kt)\cdot e^{-st}\,dt\right\}$$

$$\frac{s}{k}\int_0^\infty \cos(kt)\cdot e^{-st}\,dt = \lim_{b\to\infty}\frac{s}{k}\left\{\left[+\frac{1}{k}\sin(kt)e^{-st}\right]_0^b - \int_0^\infty \frac{s}{k}\sin(kt)\cdot e^{-st}\,dt\right\}$$

$$\lim_{b\to\infty}\left\{\left[-\frac{1}{k}\cos(kt)e^{-st}\right]_0^b\right\} = \lim_{b\to\infty}\left\{-\frac{1}{k}\cos(kb)e^{-sb} + \frac{1}{k}\cos(k0)e^{-s0}\right\} = \frac{1}{k}$$

$$\lim_{b\to\infty}\left\{\left[+\frac{1}{k}\sin(kt)e^{-st}\right]_0^b\right\} = \lim_{b\to\infty}\left\{+\frac{1}{k}\sin(kb)e^{-sb} - \frac{1}{k}\sin(k0)e^{-s0}\right\} = 0$$

$$F(s) = \int_0^\infty \sin(kt)\cdot e^{-st}\,dt = \lim_{b\to\infty}\left\{\frac{1}{k} - \frac{s^2}{k^2}\int_0^\infty \sin(kt)\cdot e^{-st}\,dt\right\}$$

$$F(s) = \frac{1}{k} - \frac{s^2}{k^2}F(s) \;\Rightarrow\; \left(1+\frac{s^2}{k^2}\right)F(s) = \frac{1}{k} \;\Rightarrow\; \left(\frac{s^2+k^2}{k^2}\right)F(s) = \frac{1}{k}$$

$$F(s) = \frac{k}{(s^2+k^2)}$$

Integration by parts

$$f(x)g(x) \rightarrow \boxed{\frac{d}{dx}} \rightarrow f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}(f\,g) = \frac{df}{dx}g + f\frac{dg}{dx}$$

$$f(x)g(x) \leftarrow \boxed{\int \cdot\, dx} \leftarrow f'(x)g(x) + f(x)g'(x)$$

$$f\,g = \int f'g\,dx + \int f\,g'\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

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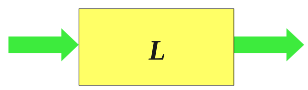
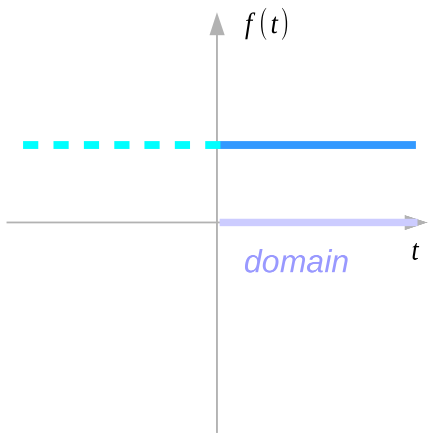
Region of Convergence

1  $\frac{1}{s}$

$$\int_0^\infty e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s0}\right] = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-sb} + \frac{1}{s}\right]$$

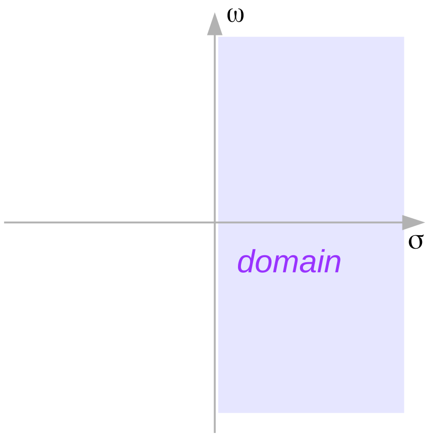
$$-s < 0 \quad \Leftrightarrow \quad -\Re\{s\} < 0$$

$$-(\sigma + i\omega) < 0 \quad \Leftrightarrow \quad -\sigma < 0$$



$$\lim_{b\to\infty} e^{-sb} = \lim_{b\to\infty} e^{-(\sigma+i\omega)b} = \lim_{b\to\infty} e^{-b\sigma} e^{+ib\omega} = 0$$

$$\left|e^{+ib\omega}\right| = 1$$



right-sided function $t > 0$

right-sided ROC $\sigma > 0$

$$-(s+a) < 0 \Rightarrow \lim_{b\to\infty} e^{-(s+a)b} = 0 \qquad s > -a \Rightarrow F(s) = \frac{1}{(s+a)}$$

Converging Improper Integrals

$$\int_{t_1}^{t_2} e^{+kt}\,dt = \left[\frac{1}{k}e^{kt}\right]_{t_1}^{t_2} = \frac{1}{k}\cdot\left(e^{kt_2} - e^{kt_1}\right)$$

$$\int_{0}^{\infty} e^{+kt}\,dt = \left[\frac{1}{k}e^{kt}\right]_{0}^{\infty} = \frac{1}{k}\cdot\left(e^{k\cdot\infty} - e^{k\cdot 0}\right)$$

$$k = \sigma + j\omega$$

$$\begin{cases} -\dfrac{1}{k} & \Leftarrow \quad \Re\{k\} < 0 \;\Rightarrow\; e^{k\cdot\infty} \to 0 \\[2ex] +\dfrac{1}{k}\cdot\left(e^{j\omega} - 1\right) & \Leftarrow \quad \Re\{k\} = 0 \;\Rightarrow\; e^{k\cdot\infty} \to e^{j\omega} \end{cases}$$

Exponential Order

Laplace Transform

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt$$

for $s > 0$ $\quad \Re(s) > 0$
the integral converges
if f(t) does not grow too rapidly

the growth rate of a function f(t)

Exponential Order α

a function f has exponential order α

$\triangleq$

there exist constants $M > 0$ and α
such that for some $t > t_0$

$$|f(t)| \le M e^{\alpha t}, \qquad t > t_0$$

right-sided

$\int_0^{\infty} |f(t)| e^{-\sigma t}\, dt < \infty$ for some σ ➡

$$\int_0^{\infty} \left| f(t) e^{-st} \right| dt = \int_0^{\infty} \left| f(t) e^{-xt} e^{-iyt} \right| dt = \int_0^{\infty} \left| f(t) e^{-xt} \right| dt < \int_0^{\infty} |f(t)| e^{-\sigma t}\, dt < \infty$$

for $s > \sigma$ $\quad \Re(s) > \sigma$

$f(t)$ exponential order σ ➡ $F(s) = \int_0^{\infty} f(t) e^{-st}\, dt$ absolutely converges for $s > \sigma$

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Convergence of the Laplace Transform

Laplace Transform

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt$$

$$= \int_0^{\infty} \{f(t) e^{-xt}\}\, e^{-iyt}\, dt$$

$$\int_0^{\infty} |f(t) e^{-st}|\, dt = \int_0^{\infty} |f(t)|\, e^{-xt}\, dt < \infty$$

$$\left(|e^{-st}| = |e^{-xt}|\,|e^{-iyt}| = e^{-xt} \right)$$

- f(t) continuous on [0, ∞)
- f(t) = 0 for t < 0
- f(t) has **exponential order α**
- f'(t) piecewise continuous on [0, ∞)

⇒

F(s) converges absolutely

for Re(s) > α

$$\int_0^{\infty} |f(t) e^{-st}|\, dt < \infty$$

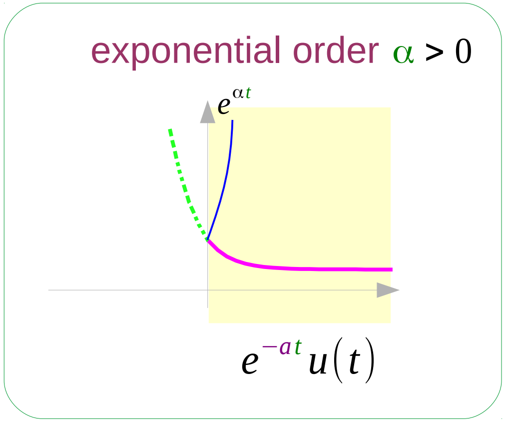
right-sided function $t > 0$

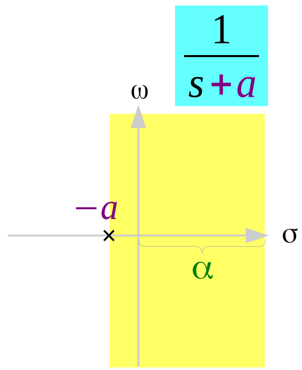
right-sided ROC $\sigma > 0$

Exponential Order and ROC

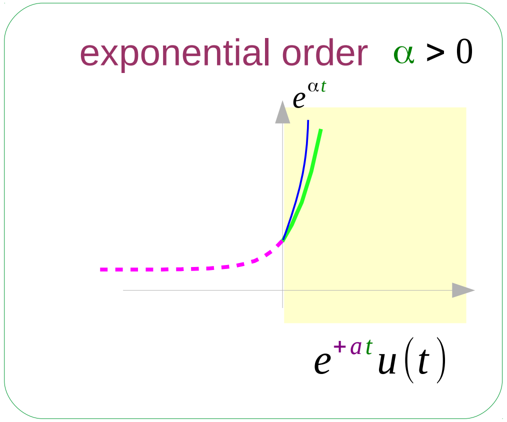
$a > 0 \quad \alpha > 0 \quad \beta < 0$

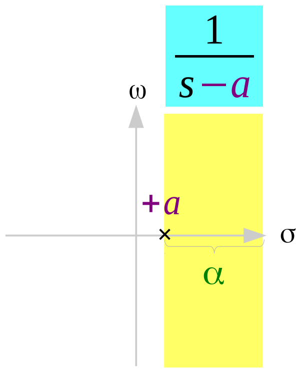
Right-sided function



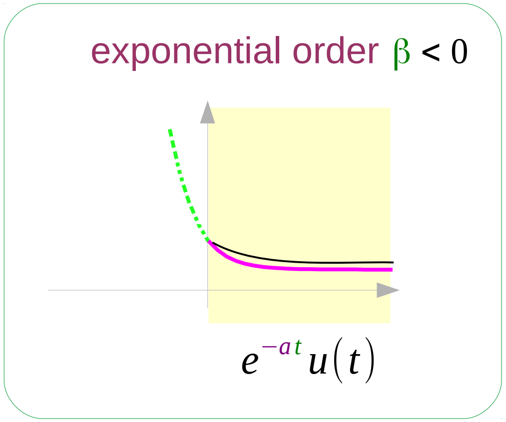


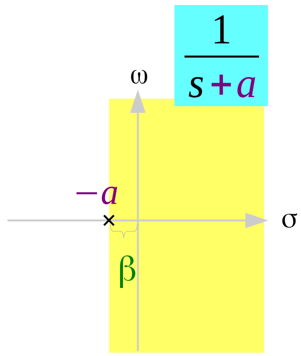
Right-sided function





Right-sided function





Forward and Inverse Laplace Transform

Forward Laplace Transform

$$f(t) \Rightarrow F(s)$$

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt$$

Inverse Laplace Transform

$$f(t) \Leftarrow F(s)$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{+st}\, ds$$

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