# Reduction of Orders (2A)

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## *Reduction of Orders*

# Finding another solution $y_2$ from the known $y_1$

***Second Order EQ***

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$y_1 = f(x)$ ***known solution***

$y_2 = u(x)f(x)$ ***another solution to be found***

We know one solution $\quad y_1(x) = e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$

Suppose the other solution $\quad y_2(x) = u(x)\,y_1(x) = u(x)\,e^{m_1 x}$

Condition for $y_2(t)$ to be a solution → Find u(x)

# Conditions for $y_2$ to be another solution

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = 0$$

$$a\,y'' + b\,y' + c\,y = 0$$

$$\begin{cases} y_2 = u\,y_1 \\ y_2' = u'y_1 + u\,y_1' \\ y_2'' = u''y_1 + 2u'y_1' + u\,y_1'' \end{cases}$$

$$a\,y_2'' + b\,y_2' + c\,y_2 = 0 \;\Rightarrow\; a[u''y_1 + 2u'y_1' + \underline{u\,y_1''}] + b[u'y_1 + \underline{u\,y_1'}] + \underline{c\,u\,y_1} = 0$$

$$a\,y_1'' + b\,y_1' + c\,y_1 = 0 \;\Rightarrow\; u[\underline{a\,y_1'' + b\,y_1' + c\,y_1}] + a[u''y_1 + 2u'y_1'] + b[u'y_1] = 0$$

Condition for $y_2(t)$ to be a solution

$$y_2(x) = u(x)\,y_1(x) \;\Rightarrow\; a u''y_1 + u'[2a\,y_1' + b\,y_1] = 0$$

# Reduction of Order

We know one solution $y_1(x)$

Suppose the other solution $y_2(x) = u(x)\, y_1(x)$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = 0$$

$$a\,y'' + b\,y' + c\,y = 0$$

$$a\,y''_2 + b\,y'_2 + c\,y_2 = 0 \quad\Rightarrow\quad a\,u''y_1 + u'[2a\,y_1' + b\,y_1] = 0$$

$$w(x) = u'(x) \quad\Rightarrow\quad a\,w'y_1 + w[2a\,y_1' + b\,y_1] = 0$$

$$w'y_1 = -w[2y_1' + \frac{b}{a}y_1]$$

$$\frac{w'}{w} = -2\frac{y_1'}{y_1} - \frac{b}{a}$$

$$\frac{1}{w}\frac{dw}{dx} = -2\frac{1}{y_1}\frac{dy_1}{dx} - \frac{b}{a}$$

$$\int \frac{1}{w}\frac{dw}{dx}dx = -\int 2\frac{1}{y_1}\frac{dy_1}{dx}dx - \int \frac{b}{a}dx$$

$$\ln|w| = -\ln|y_1|^2 - \frac{b}{a}x + c$$

$$\ln|w| + \ln|y_1|^2 = -\frac{b}{a}x + c$$

$$\ln|w\,y_1^2| = -\frac{b}{a}x + c$$

$$|w\,y_1^2| = C\,e^{-(b/a)x}$$

$$w\,y_1^2 = c_1 e^{-(b/a)x}$$

$$w = c_1 e^{-(b/a)x}/y_1^2$$

$$u' = c_1 e^{-(b/a)x}/y_1^2$$

$$u = c_1 \int \frac{e^{-(b/a)x}}{y_1^2}dx + c_2$$

# Another Solution $y_2$

We know one solution $y_1(x)$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Suppose the other solution $y_2(x) = u(x) y_1(x)$

$$a y'' + b y' + c y = 0$$

$$a y_2'' + b y_2' + c y_2 = 0 \quad \Rightarrow \quad a u'' y_1 + u'[2a y_1' + b y_1] = 0$$

$$u = c_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2 \quad \Rightarrow \quad y_2 = c_1 y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2 y_1 \qquad (c_1 = 1, \ c_2 = 0)$$

$$\Rightarrow \quad y_2 = y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx$$

# General Solutions for the repeated roots case

$$y_2 = y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$$\Rightarrow \quad b^2 - 4ac = 0 \quad \Rightarrow$$

$$m_1 = -b/2a$$

$$m_2 = -b/2a$$

$$e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

$$y_1(x) = e^{-\frac{b}{2a}x}$$

$$y_1^2 = e^{-\frac{b}{a}x}$$

$$y_2 = e^{-\frac{b}{2a}x} \int \frac{e^{-(b/a)x}}{e^{-(b/a)x}} dx = e^{-\frac{b}{2a}x} \int 1 \, dx$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$y_1(x) = e^{-\frac{b}{2a}x}$$

$$y_2(x) = x e^{-\frac{b}{2a}x}$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

# How can $xy_1$ be another solution to the known $y_1$

We know one solution $y_1(x)$

Suppose the other solution $y_2(x) = x\,y_1(x)$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = 0$$

$$a\,y'' + b\,y' + c\,y = 0$$

$$(xy_1)' = y_1 + xy_1'$$

$$(xy_1)'' = y_1' + y_1' + x\,y_1'' = 2\,y_1' + x\,y_1''$$

$$a(x\,y_1)'' + b(x\,y_1)' + c(x\,y_1)$$

$$= a(2\,y_1' + x\,y_1'') + b(y_1 + xy_1') + c(x\,y_1)$$

$$= x[a\,y_1'' + b\,y_1' + c\,y_1] + 2a\,y_1' + b\,y_1$$

$$= 2a\,y_1' + b\,y_1 = 0$$

Condition for $xy_1$ to be a solution

$$y_2(x) = x\,y_1(x) \quad \Rightarrow \quad [2a\,y_1' + b\,y_1] = 0$$

$$y_2(x) = u(x)\,y_1(x) \quad \Rightarrow \quad a\,u''\,y_1 + u'[2a\,y_1' + b\,y_1] = 0$$

# Example: Finding $y_2 \leftarrow y_1$ (1)

$$1 \cdot y'' + 0 \cdot y' - 4 \cdot y = 0$$

$$y_1(x) = e^{-2x}$$

$$Find \quad y_2(x)$$

$$y_2 = u y_1$$

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + u' y_1' + u' y_1' + u y_1''$$

$$y_2 = u e^{-2x}$$

$$y_2' = u' e^{-2x} - u 2 e^{-2x}$$

$$y_2'' = u'' e^{-2x} - u' 2 e^{-2x} - u' 2 e^{-2x} + u 4 e^{-2x}$$

$$y_2'' - 4 \cdot y_2 = 0 \quad \Rightarrow \quad \left(u'' e^{-2x} - u' 2 e^{-2x} - u' 2 e^{2x} + u 4 e^{-2x}\right) - 4 u e^{-2x} = 0$$

$$\left(u'' e^{-2x} - 4 u' e^{-2x}\right) + u\left(4 e^{-2x} - 4 e^{-2x}\right) = 0 \quad \Rightarrow \quad \left(u'' e^{-2x} - 4 u' e^{-2x}\right) = 0$$

$$1 \cdot y'' - 4 \cdot y = 0 \qquad a u'' y_1 + 2 u' a y_1' = 0$$

Condition for $y_2(t)$ to be a solution

$$y_2(x) = u(x) y_1(x) \quad \Rightarrow \quad a u'' y_1 + u'[2 a y_1' + b y_1] = 0$$

# Example: Finding $y_2 \leftarrow y_1$ (2)

$$1 \cdot y'' + 0 \cdot y' - 4 \cdot y = 0$$

$$y_1(x) = e^{-2x}$$

$$\text{Find } \; y_2(x)$$

$$(u''e^{-2x} - 4u'e^{-2x}) + u(4e^{-2x} - 4e^{-2x}) = 0 \;\Rightarrow\; (u''e^{-2x} - 4u'e^{-2x}) = 0$$

$$1 \cdot y'' - 4 \cdot y = 0 \qquad a u'' y_1 + 2u' a y_1' = 0$$

$$(u'' - 4u')e^{-2x} = 0 \;\Rightarrow\; u'' - 4u' = 0$$

$$w = u'$$

$$u'' - 4u' = 0$$

$$w' - 4w = 0$$

$$\frac{w'}{w} = 4$$

$$\int \frac{1}{w}\frac{dw}{dx}\,dx = \int 4\,dx + c$$

$$\int \frac{1}{w}\,dw = \int 4\,dx + c$$

$$\ln|w| = 4x + c$$

$$|w| = e^{4x+c}$$

$$w = c_1 e^{4x}$$

$$u' = c_1 e^{4x}$$

$$u = \frac{1}{4} c_1 e^{4x} + c_2$$

# Example: Finding $y_2 \leftarrow y_1$ (3)

$$1 \cdot y'' + 0 \cdot y' - 4 \cdot y = 0$$

$$y_1(x) = e^{-2x}$$

$$\text{Find } \; y_2(x)$$

$$u' = c_1 e^{4x} \quad \Rightarrow \quad u = \frac{1}{4} c_1 e^{4x} + c_2$$

$$y = k_1 y_1 + k_2 y_2$$

$$= k_1 e^{-2x} + k_2 \left( \frac{1}{4} c_1 e^{+4x} + c_2 \right) e^{-2x}$$

$$= k_1 e^{-2x} + k_2 \frac{1}{4} c_1 e^{+2x} + k_2 c_2 e^{-2x}$$

$$= k_1 e^{-2x} + k_2 c_2 e^{-2x} + k_2 \frac{1}{4} c_1 e^{+2x}$$

$$y = (k_1 + k_2 c_2) e^{-2x} + \left( k_2 \frac{1}{4} c_1 \right) e^{+2x}$$

$$(k_1) e^{-2x} + \left( k_2 \frac{1}{4} c_1 \right) e^{+2x} \qquad c_1 \neq 0 \quad c_2 = 0$$

$$(k_1 + k_2 c_2) e^{-2x} \qquad c_1 = 0 \quad c_2 \neq 0$$

$$k_1 e^{-2x} + k_2 e^{+2x} \qquad c_1 = 4 \quad c_2 = 0$$

# Example: Finding $y_2 \leftarrow y_1$ (4)

$$1 \cdot y'' + 0 \cdot y' - 4 \cdot y = 0 \qquad y_1(x) = e^{-2x} \qquad Find \ \ y_2(x)$$

$$y = (k_1 + k_2 c_2) e^{-2x} + \left(k_2 \frac{1}{4} c_1\right) e^{+2x}$$

$$(k_1) e^{-2x} + \left(k_2 \frac{1}{4} c_1\right) e^{+2x} \qquad c_1 \neq 0 \quad c_2 = 0$$

$$(k_1 + k_2 c_2) e^{-2x} \qquad c_1 = 0 \quad c_2 \neq 0$$

$$k_1 e^{-2x} + k_2 e^{+2x} \qquad c_1 = 4 \quad c_2 = 0$$

$$C_1 e^{-2x} + C_2 e^{+2x}$$

$$W(e^{-2x}, e^{+2x}) = \begin{vmatrix} e^{-2x} & e^{+2x} \\ -2e^{-2x} & +2e^{+2x} \end{vmatrix}$$

$$= 2 - (-2) = 4$$

$$W(e^{-2x}, e^{+2x}) \neq 0$$

linearly independent ⇒

Fundamental Set of Solutions

$$\{e^{-2x}, e^{+2x}\}$$

## References

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