EigenSpaces (4A)

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Eigenvalues and Eigenvectors

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$$

$$\boldsymbol{p}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{A}\,\boldsymbol{p}_1 = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\,\boldsymbol{p}_1$$

$$\boldsymbol{p}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \boldsymbol{A}\,\boldsymbol{p}_2 = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 1\,\boldsymbol{p}_2$$

$$\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$$

eigenvalue eigenvector

EigenValues and EigenVectors

n x n

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

eigenvalue

eigenvector

$$\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$$

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(\boldsymbol{A} - \lambda\,\boldsymbol{I})\,\boldsymbol{x} = \boldsymbol{0}$$

characteristic Equation

$$det(\boldsymbol{A} - \lambda\,\boldsymbol{I}) = 0$$

Solving Homogeneous System

n x n

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

$$\begin{cases} det(\boldsymbol{A} - \lambda \boldsymbol{I}) \neq 0 \\ det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \end{cases}$$

unique solution $\boldsymbol{x} = \boldsymbol{0}$

infinite solution $\boldsymbol{x} \neq \boldsymbol{0}$

$n - rank(\boldsymbol{A})$ arbitrary parameters

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0$$

Characteristic Equation

n x n

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

characteristic Equation

$$det(\boldsymbol{A}-\lambda\boldsymbol{I}) = 0$$

$$\begin{pmatrix} \lambda-a_{22} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda-a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(\lambda\boldsymbol{I}-\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$$

characteristic Equation

$$det(\lambda\boldsymbol{I}-\boldsymbol{A}) = 0$$

$$\left.\begin{aligned} det(\boldsymbol{A}-\lambda\boldsymbol{I}) &= 0 \\ det(\lambda\boldsymbol{I}-\boldsymbol{A}) &= 0 \end{aligned}\right\} \qquad \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0$$

Gauss-Jordon Elimination

$$\left[\, \boldsymbol{A} \;\middle|\; \boldsymbol{b} \,\right]$$

$$\left[\, \lambda_i \boldsymbol{I} - \boldsymbol{A} \;\middle|\; \boldsymbol{0} \,\right]$$ **for each** λ_i

after applying G-J Elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

free variable

one or more zero rows

$$n - rank(\boldsymbol{A})$$

arbitrary parameters

after applying G-J Elimination

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{13} & 0 \\ 0 & 1 & \frac{6}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = -\frac{1}{13} x_3$$

$$x_2 = -\frac{6}{13} x_3$$

let $x_3 = 13$

for the eigenvalue λ_i

the corresponding eigenvector $\boldsymbol{x}_i = \begin{bmatrix} -1 \\ -6 \\ 13 \end{bmatrix}$

Finding EigenVectors

n x n

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

to find a non-zero x

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0$$

$$\lambda_{1,} \quad \lambda_{2,} \quad \cdots \ , \ \lambda_n$$

for the eigenvalue λ_1 $(\boldsymbol{A} - \lambda_1 \boldsymbol{I} \mid \boldsymbol{0}) \longrightarrow$ Guass-Jordan elimination $\longrightarrow \boldsymbol{x}_1$

for the eigenvalue λ_2 $(\boldsymbol{A} - \lambda_2 \boldsymbol{I} \mid \boldsymbol{0}) \longrightarrow$ Guass-Jordan elimination $\longrightarrow \boldsymbol{x}_2$

for the eigenvalue λ_n $(\boldsymbol{A} - \lambda_n \boldsymbol{I} \mid \boldsymbol{0}) \longrightarrow$ Guass-Jordan elimination $\longrightarrow \boldsymbol{x}_n$

Not Uinque EigenVectors

n x n

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

different eigentvectors

$$(\mathbf{A} - \lambda\mathbf{I})k\,\mathbf{x} = \mathbf{0}$$

to find a non-zero x

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0$$

$$\lambda_{1,} \quad \lambda_{2,} \quad \cdots \quad , \quad \lambda_n$$

non-zero constant multiple

Triangular Matrix

n x n

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Upper Triangular

n x n

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Lower Triangular

n x n

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Diagonal

$$\boldsymbol{A} - \lambda \boldsymbol{I}$$

$$\begin{pmatrix} a_{11} - \lambda & & \\ & a_{22} - \lambda & \\ & & \\ & & a_{nn} - \lambda \end{pmatrix}$$

characteristic Equation

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \quad det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$$

$$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) = 0$$

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \cdots , \quad \lambda = a_{nn}$$

Eigenvectors of Symmetric Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(1-\lambda)^2 = 0 \qquad \lambda = 1 \qquad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \boldsymbol{x} = \lambda \, \boldsymbol{x} \qquad \textit{for every } \boldsymbol{x}$$

2 linearly independent eigenvectors

Eigenvectors of Diagonal Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(1-\lambda)^2 = 0 \qquad \lambda = 1 \qquad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \boldsymbol{x} = \lambda \, \boldsymbol{x} \qquad \textit{for every } \boldsymbol{x}$$

2 linearly independent eigenvectors

Zero EigenValue

n x n

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

characteristic Equation

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

$$det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$$

$$c_n = 0 \Leftrightarrow \lambda = 0 \Leftrightarrow det(-\boldsymbol{A}) = c_n \Leftrightarrow$$ **Non-invertible A**

$$(-1)^n det(\boldsymbol{A}) = c_n$$

$$det(\boldsymbol{A}) = 0$$

Distinct Eigenvalues

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0$$

n distinct roots

$\lambda_{1,} \quad \lambda_{2,} \quad \cdots \quad , \quad \lambda_n$

linear independent

$\boldsymbol{x}_1 , \quad \boldsymbol{x}_2 , \quad \cdots \quad , \quad \boldsymbol{x}_n$

repeated roots exist
k distinct roots (k<n)

$\lambda_{1,} \quad \lambda_{2,} \cdots , \lambda_k$

may not linear independent

$\boldsymbol{x}_1 , \quad \boldsymbol{x}_2 , \cdots , \boldsymbol{x}_k$

$\boldsymbol{x}_1 , \quad \boldsymbol{x}_2 , \quad \cdots \quad , \quad \boldsymbol{x}_n$

Distinct EigenValues

to find a non-zero x

$$det\left(\boldsymbol{A} - \lambda\,\boldsymbol{I}\right) \;=\; 0$$

n x n

$$\left(\boldsymbol{A} - \lambda\,\boldsymbol{I}\right)\boldsymbol{x} \;=\; \mathbf{0}$$

distinct eigenvalues ⬌ **linearly independent eigenvectors**

$$\lambda_1, \lambda_2, \;\cdots\; , \lambda_n$$

$$\boldsymbol{x}_1, \boldsymbol{x}_2, \;\cdots\; \boldsymbol{x}_n$$

Multiplicity of Roots

n x n

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

to find a non-zero x

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

$$\lambda^2 + c_1 \lambda^1 + c_2 = 0$$

$\boldsymbol{x}_1, \quad \boldsymbol{x}_2$ linear independent

$$(\lambda - \alpha_1)(\lambda - \alpha_2) = 0$$

$\boldsymbol{x}_1, \quad \boldsymbol{x}_2$ may not linear independent

$$(\lambda - \lambda_1)^2 = 0$$

$\boldsymbol{x}_1, \quad \boldsymbol{x}_2$ linear independent

$$(\lambda - \alpha_1 - i\beta)(\lambda - \alpha_2 + i\beta) = 0$$

$$\overline{\boldsymbol{x}_1} = \boldsymbol{x}_2$$

$$\boldsymbol{x}_1 = \overline{\boldsymbol{x}_2}$$

Repeated EigenValues

to find a non-zero x

$$det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) = 0$$

n x n

$$\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right)\boldsymbol{x} = \boldsymbol{0}$$

k distinct eigenvalues ⟺ **at least k linearly independent eigenvectors**

$$\lambda_1, \lambda_2, \cdots, \lambda_k$$

$$\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots \boldsymbol{x}_k$$

$$k < n$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad (1-\lambda)^2 = 0 \qquad \lambda = 1 \qquad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (1-\lambda)^2 = 0 \qquad \lambda = 1 \qquad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\boldsymbol{x} = \lambda\,\boldsymbol{x} \qquad \textit{for every } \boldsymbol{x}$$

2 linearly independent eigenvectors

Powers of Matrix

n x n	A
eigenvalue	λ
eigenvector	x

⇒

n x n	A^k
eigenvalue	λ^k
eigenvector	x

$$A^2 x = A\,(A\,x) = A(\lambda\,I)\,x = \lambda\,A\,x = \lambda^2 x$$

$$A^2 x = \lambda^2 x$$

Cayley-Hamilton Theorem

n x n $\boldsymbol{A}$

eigenvalue λ_i

eigenvector $\boldsymbol{x}_i$

$$(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) = 0$$

$$(-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda^1 + c_0 = 0$$

$$((-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda^1 + c_0)\boldsymbol{x} = 0$$

$$(-1)^n\lambda^n\boldsymbol{x} + c_{n-1}\lambda^{n-1}\boldsymbol{x} + \cdots + c_1\lambda^1\boldsymbol{x} + c_0\lambda^0\boldsymbol{x} = 0$$

$$(-1)^n\boldsymbol{A}^n + c_{n-1}\boldsymbol{A}^{n-1} + \cdots + c_1\boldsymbol{A}^1 + c_0\boldsymbol{A}^0 = 0$$

$$(-1)^n\boldsymbol{A}^n + c_{n-1}\boldsymbol{A}^{n-1} + \cdots + c_1\boldsymbol{A}^1 + c_0\boldsymbol{A}^0 = 0$$

Cayley-Hamilton Theorem

n x n	$\boldsymbol{A}$
eigenvalue	λ_i
eigenvector	$\boldsymbol{x}_i$

$$(-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda^1 + c_0 = 0$$

$$(-1)^n \boldsymbol{A}^n + c_{n-1} \boldsymbol{A}^{n-1} + \cdots + c_1 \boldsymbol{A} + c_0 \boldsymbol{I} = 0$$

$$\boldsymbol{A}^n = \text{linear combination of} \{\boldsymbol{A}^{n-1}, \ \boldsymbol{A}^{n-2}, \ \cdots, \ \boldsymbol{A}^1, \ \boldsymbol{I}\}$$

$$\boldsymbol{A}^{n+1} = \text{linear combination of} \{\boldsymbol{A}^{n-1}, \ \boldsymbol{A}^{n-2}, \ \cdots, \ \boldsymbol{A}^1, \ \boldsymbol{I}\}$$

$$\boldsymbol{A}^{n+2} = \text{linear combination of} \{\boldsymbol{A}^{n-1}, \ \boldsymbol{A}^{n-2}, \ \cdots, \ \boldsymbol{A}^1, \ \boldsymbol{I}\}$$

$$\boldsymbol{A}^m = k_{n-1} \boldsymbol{A}^{n-1} + \cdots + k_1 \boldsymbol{A}^1 + k_0 \boldsymbol{I}$$

Cayley-Hamilton Theorem

n x n $\boldsymbol{A}$

eigenvalue λ_i

eigenvector $\boldsymbol{x_i}$

$$(-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda^1 + c_0 \ = \ 0$$

$$(-1)^n \boldsymbol{A}^n + c_{n-1} \boldsymbol{A}^{n-1} + \cdots + c_1 \boldsymbol{A} + c_0 \boldsymbol{I} \ = \ 0$$

$$(-1)^n \boldsymbol{A}^n + c_{n-1} \boldsymbol{A}^{n-1} + \cdots + c_1 \boldsymbol{A} \ = \ -c_0 \boldsymbol{I}$$

$$(-1)^n \boldsymbol{A}^{n-1} + c_{n-1} \boldsymbol{A}^{n-2} + \cdots + c_1 \boldsymbol{I} \ = \ -c_0 \boldsymbol{A}^{-1}$$

$$-\left(\frac{(-1)^n}{c_0} \boldsymbol{A}^{n-1} + \frac{c_{n-1}}{c_0} \boldsymbol{A}^{n-2} + \cdots + \frac{c_1}{c_0} \boldsymbol{I} \right) \ = \ \boldsymbol{A}^{-1}$$

Types of Matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\boldsymbol{A} = \boldsymbol{A}^T$$

Symmetric Matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{-1} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^T$$

Orthogonal Matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{-1} = \begin{bmatrix} a_1^* & b_1^* & c_1^* \\ a_2^* & b_2^* & c_2^* \\ a_3^* & b_3^* & c_3^* \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \overline{\boldsymbol{A}^T}$$

Hermitian Matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{-1} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \boldsymbol{A}$$

Unitary Matrix

Orthonormal Set

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{-1} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \qquad \boldsymbol{A}^{-1} = \boldsymbol{A}^{T}$$

Orthogonal Matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{a}^T \\ \boldsymbol{b}^T \\ \boldsymbol{c}^T \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}^T\boldsymbol{a} & \boldsymbol{a}^T\boldsymbol{b} & \boldsymbol{a}^T\boldsymbol{c} \\ \boldsymbol{b}^T\boldsymbol{a} & \boldsymbol{b}^T\boldsymbol{b} & \boldsymbol{b}^T\boldsymbol{c} \\ \boldsymbol{c}^T\boldsymbol{a} & \boldsymbol{c}^T\boldsymbol{b} & \boldsymbol{c}^T\boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{x}_i^T \boldsymbol{x}_j = 0 \qquad i \neq j \qquad i, j = 1, 2, \cdots, n$$

$$\boldsymbol{x}_i^T \boldsymbol{x}_i = 1 \qquad i = 1, 2, \cdots, n$$

Orthogonal EigenVectors

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\boldsymbol{A} = \boldsymbol{A}^T$$

Symmetric Matrix

symmetric Matrix A

distinct eigenvalues → **orthogonal eigenvectors**

$\lambda_1, \quad \lambda_2$

$\boldsymbol{p}_1, \quad \boldsymbol{p}_2$

$$\boldsymbol{A}\,\boldsymbol{p}_1 = \lambda_1\,\boldsymbol{p}_1$$

$$\boldsymbol{p}_1^T\,\boldsymbol{A} = \lambda_1\,\boldsymbol{p}_1^T$$

$$\boldsymbol{p}_1^T\,\boldsymbol{A}\,\boldsymbol{p}_2 = \lambda_1\,\boldsymbol{p}_1^T\,\boldsymbol{p}_2$$

$$\boldsymbol{A}\,\boldsymbol{p}_2 = \lambda_2\,\boldsymbol{p}_2$$

$$\boldsymbol{p}_1^T\,\boldsymbol{A}\,\boldsymbol{p}_2 = \lambda_2\,\boldsymbol{p}_1^T\,\boldsymbol{p}_2$$

$$\begin{aligned} &\boldsymbol{p}_1^T\,\boldsymbol{A}\,\boldsymbol{p}_2 - \boldsymbol{p}_1^T\,\boldsymbol{A}\,\boldsymbol{p}_2 \\ &= \lambda_1\,\boldsymbol{p}_1^T\,\boldsymbol{p}_2 - \lambda_2\,\boldsymbol{p}_1^T\,\boldsymbol{p}_2 \\ &= (\lambda_1 - \lambda_2)\;\boldsymbol{p}_1^T\,\boldsymbol{p}_2 = 0 \\ &\qquad\qquad \boldsymbol{p}_1^T\,\boldsymbol{p}_2 = 0 \end{aligned}$$

Orthogonal Matrix

$$\begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \end{bmatrix} = \boldsymbol{A}$$

$$\begin{bmatrix} \boldsymbol{p}_1^T \\ \boldsymbol{p}_2^T \\ \boldsymbol{p}_3^T \end{bmatrix} = \boldsymbol{A}^T$$

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^T$$

Orthogonal Matrix

$$\begin{bmatrix} \boldsymbol{p}_1^T \\ \boldsymbol{p}_2^T \\ \boldsymbol{p}_3^T \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}_1^T\boldsymbol{p}_1 & \boldsymbol{p}_1^T\boldsymbol{p}_2 & \boldsymbol{p}_1^T\boldsymbol{p}_3 \\ \boldsymbol{p}_2^T\boldsymbol{p}_1 & \boldsymbol{p}_2^T\boldsymbol{p}_2 & \boldsymbol{p}_2^T\boldsymbol{p}_3 \\ \boldsymbol{p}_3^T\boldsymbol{p}_1 & \boldsymbol{p}_3^T\boldsymbol{p}_2 & \boldsymbol{p}_3^T\boldsymbol{p}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{p}_i^T \boldsymbol{p}_j = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

$$i,\ j = 1,\ 2,\ \cdots,\ n$$

Gram-Shmidt Process

Diagonalizable

n x n **n x n**

$\boldsymbol{A}$ → $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$

$$\det(\boldsymbol{B}) = \det(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}) = \det(\boldsymbol{P}^{-1})\det(\boldsymbol{A})\det(\boldsymbol{P})$$

$$= \frac{1}{\det(\boldsymbol{P})}\det(\boldsymbol{A})\det(\boldsymbol{P}) = \det(\boldsymbol{A})$$

$$rank(\boldsymbol{B}) = rank(\boldsymbol{A})$$

$$nullity(\boldsymbol{B}) = nullity(\boldsymbol{A})$$

$$(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0 \qquad (\lambda\boldsymbol{I} - \boldsymbol{B}) = 0$$

Similarity Transform

n x n   n x n

A → $D = P^{-1} A P$ **: Diagonal Matrix**

A: diagonalizable ⟺ **n linear independent eigenvectors**

A: diagonalizable → $D = P^{-1} A P$ $P D = A P$

$$P = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{pmatrix}$$

$$P D = \begin{pmatrix} \lambda_1 p_1 & \lambda_2 p_2 & & \lambda_n p_n \end{pmatrix}$$

$$A P = \begin{pmatrix} A p_1 & A p_2 & & A p_n \end{pmatrix}$$

Col & Row Vectors

$$\begin{array}{c}\boldsymbol{A}\\ \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\end{array} \quad \begin{array}{c}\boldsymbol{X}\\ \left[\begin{array}{ccc} x_1 & y_2 & z_3 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \end{array}\right]\end{array} \qquad \begin{bmatrix} a_1 x_1 + a_2 x_2 + a_3 x_3 & a_1 y_1 + a_2 y_2 + a_3 y_3 & a_1 z_1 + a_2 z_2 + a_3 z_3 \\ b_1 x_1 + b_2 x_2 + b_3 x_3 & b_1 y_1 + b_2 y_2 + b_3 y_3 & b_1 z_1 + b_2 z_2 + b_3 z_3 \\ c_1 x_1 + c_2 x_2 + c_3 x_3 & c_1 y_1 + c_2 y_2 + c_3 y_3 & c_1 z_1 + c_2 z_2 + c_3 z_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \boldsymbol{A}\,\boldsymbol{x} \qquad \begin{bmatrix} a_1 x_1 + a_2 x_2 + a_3 x_3 \\ b_1 x_1 + b_2 x_2 + b_3 x_3 \\ c_1 x_1 + c_2 x_2 + c_3 x_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \boldsymbol{A}\,\boldsymbol{y} \qquad \begin{bmatrix} a_1 y_1 + a_2 y_2 + a_3 y_3 \\ b_1 y_1 + b_2 y_2 + b_3 y_3 \\ c_1 y_1 + c_2 y_2 + c_3 y_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad \boldsymbol{A}\,\boldsymbol{z} \qquad \begin{bmatrix} a_1 z_1 + a_2 z_2 + a_3 z_3 \\ b_1 z_1 + b_2 z_2 + b_3 z_3 \\ c_1 z_1 + c_2 z_2 + c_3 z_3 \end{bmatrix}$$

$$\boldsymbol{X} = \left[\,\boldsymbol{x} \mid \boldsymbol{y} \mid \boldsymbol{z}\,\right]$$

$$\boldsymbol{A}\,\boldsymbol{X} = \left[\,\boldsymbol{A}\,\boldsymbol{x} \mid \boldsymbol{A}\,\boldsymbol{y} \mid \boldsymbol{A}\,\boldsymbol{z}\,\right]$$

Col & Row Vectors

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & z_3 \end{bmatrix} \qquad \begin{bmatrix} a_1 x_1 & a_2 y_2 & a_3 z_3 \\ b_1 x_1 & b_2 y_2 & b_3 z_3 \\ c_1 x_1 & c_2 y_2 & c_3 z_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a_1 x_1 \\ b_1 x_1 \\ c_1 x_1 \end{bmatrix} = x_1 \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\ y_2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a_2 y_2 \\ b_2 y_2 \\ c_2 y_2 \end{bmatrix} = y_2 \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ z_3 \end{bmatrix} \qquad \begin{bmatrix} a_3 z_3 \\ b_3 z_3 \\ c_3 z_3 \end{bmatrix} = z_3 \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$$

$$\boldsymbol{X} = \left[\boldsymbol{x} \mid \boldsymbol{y} \mid \boldsymbol{z} \right]$$

$$\boldsymbol{A}\,\boldsymbol{X} = \left[\boldsymbol{A}\,\boldsymbol{x} \mid \boldsymbol{A}\,\boldsymbol{y} \mid \boldsymbol{A}\,\boldsymbol{z} \right]$$

A nxn Matrix A (1)

1. **A** is invertible
2. **Ax** = **0** has only the trivial solution
3. The RREF(**A**) = $\mathbf{I}_n$
4. **A** can be written as a product of elementary matrix
5. **Ax** = **b** is consistent for every n x 1 **b**
6. **Ax** = **b** has exactly one solution for every n x 1 **b**
7. det(**A**) ≠ 0
8. The column vectors are linearly independent
9. The row vectors are linearly independent
10. The column vectors span R^n
11. The row vectors span R^n
12. The column vectors form a basis for R^n
13. The row vectors form a basis for R^n
14. rank(**A**) = n
15. nullity(**A**) = 0
16. The orthogonal complement of the null space is R^n
17. The orthogonal complement of the row space is {**0**}

A nxn Matrix A (2)

18. The range of T_A is R^n

19. T_A is one-to-one

20. λ=0 is not the eigenvalue of **A**

References

[1] http://en.wikipedia.org/
[2] Anton, et al., Elementary Linear Algebra, 10th ed, Wiley, 2011
[3] Anton, et al., Contemporary Linear Algebra,