

First Order ODE's (1B)

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First Order ODE examples (I)

$$\frac{dy}{dx} + y = 0$$

$$y(x) = ce^{-x};$$

$$\frac{dy}{dx} + xy = 0$$

$$y(x) = ce^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2 y = 0$$

$$y(x) = ce^{-\frac{x^3}{3}};$$

$$y' + y = 0$$

$$\frac{dy}{dx} + y = 0$$

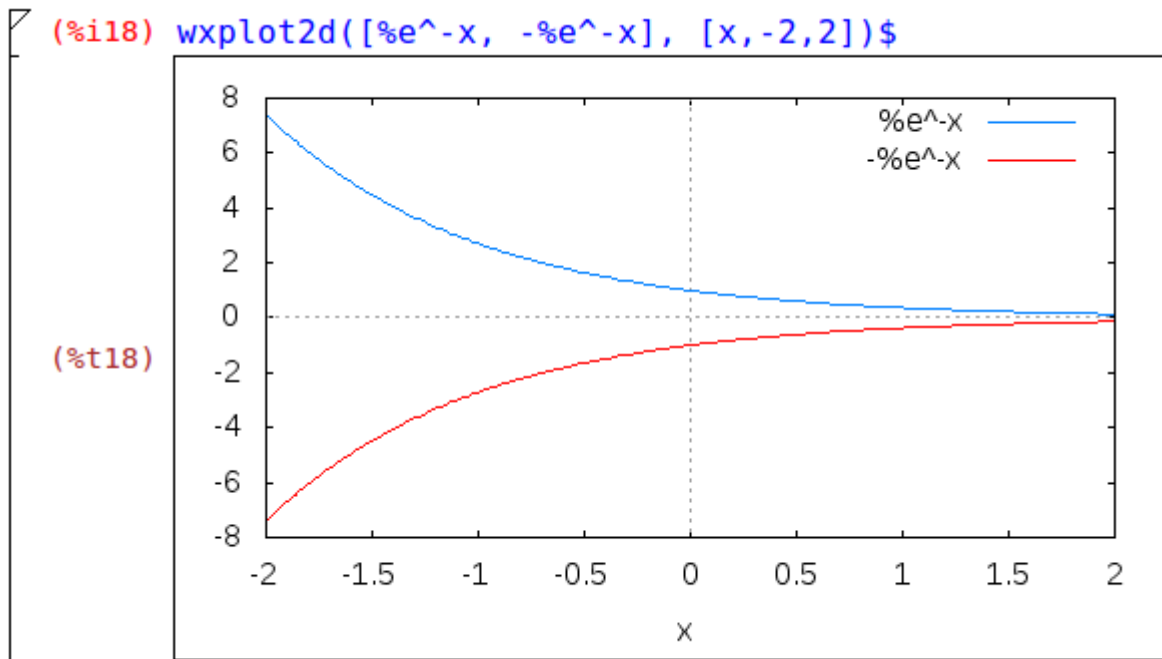
$$y(x) = ce^{-x};$$

$$\frac{dy}{dx} + xy = 0$$

$$y(x) = ce^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2y = 0$$

$$y(x) = ce^{-\frac{x^3}{3}};$$



$$y = e^{-x};$$

$$y' = -e^{-x};$$

$$y' + xy = 0$$

$$\frac{dy}{dx} + y = 0$$

$$y(x) = ce^{-x};$$

$$\frac{dy}{dx} + xy = 0$$

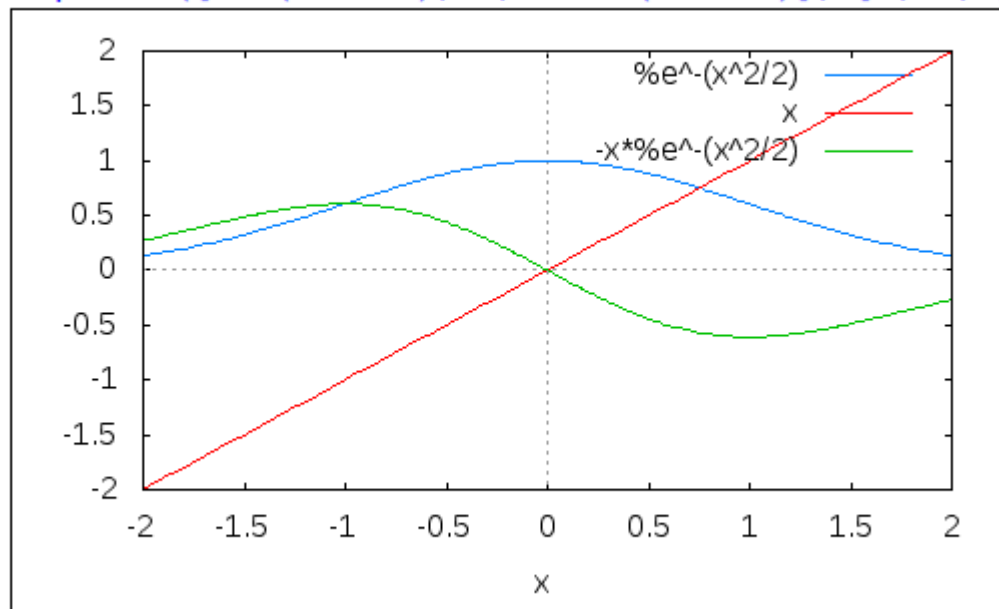
$$y(x) = ce^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2y = 0$$

$$y(x) = ce^{-\frac{x^3}{3}};$$

```
(%i19) wxplot2d([%e^(-x^2/2),+x,-x*%e^(-x^2/2)], [x,-2,2])$
```

```
(%t19)
```



$$y = e^{-\frac{x^2}{2}}$$

$$y'(x) = e^{-\frac{x^2}{2}} \frac{d}{dx} \left\{ -\frac{x^2}{2} \right\}$$

$$y'(x) = -xe^{-\frac{x^2}{2}}$$

$$y'(x) = -\frac{x}{e^{x^2/2}}$$

$$y' + x^2 y = 0$$

$$\frac{dy}{dx} + y = 0$$

$$y(x) = ce^{-x};$$

$$\frac{dy}{dx} + xy = 0$$

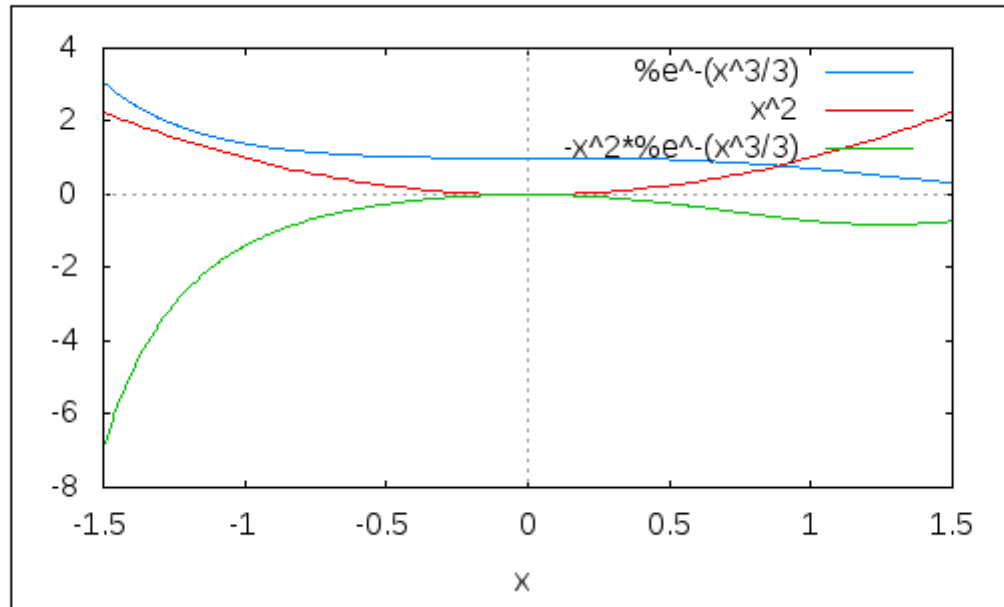
$$y(x) = ce^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2 y = 0$$

$$y(x) = ce^{-\frac{x^3}{3}};$$

```
(%i28) wxplot2d([%e^(-x^3/3),x^2,-x^2*%e^(-x^3/3)], [x,-1.5,1.5])$
```

```
(%t28)
```



$$y = e^{-\frac{x^3}{3}}$$

$$y' = e^{-\frac{x^3}{3}} \frac{d}{dx} \left\{ -\frac{x^3}{3} \right\}$$

$$y' = -x^2 e^{-\frac{x^3}{3}}$$

$$y' = -\frac{x^2}{e^{x^3/3}}$$

First Order ODE examples (II)

$$\frac{dy}{dx} + y = 1$$

$$e^x \frac{dy}{dx} + e^x y = e^x$$

$$\frac{d}{dx}[e^x y] = e^x$$

$$e^x y = \int e^x dx + c$$

$$y = 1 + ce^{-x}$$

$$\frac{dy}{dx} + xy = x$$

$$e^{x^2/2} \frac{dy}{dx} + e^{x^2/2} xy = xe^{x^2/2}$$

$$\frac{d}{dx}[e^{x^2/2} y] = xe^{x^2/2}$$

$$e^{x^2/2} y = \int xe^{x^2/2} dx + c$$

$$e^{x^2/2} y = \int \left\{ \frac{d}{dx} e^{x^2/2} \right\} dx + c$$

$$e^{x^2/2} y = e^{x^2/2} + c$$

$$y = 1 + ce^{-x^2/2}$$

$$\frac{dy}{dx} + x^2 y = x^2$$

$$e^{x^3/3} \frac{dy}{dx} + e^{x^3/3} x^2 y = x^2 e^{x^3/3}$$

$$\frac{d}{dx}[e^{x^3/3} y] = x^2 e^{x^3/3}$$

$$e^{x^3/3} y = \int x^2 e^{x^3/3} dx + c$$

$$e^{x^3/3} y = \int \left\{ \frac{d}{dx} e^{x^3/3} \right\} dx + c$$

$$e^{x^3/3} y = e^{x^3/3} + c$$

$$y = 1 + ce^{-x^3/3}$$

$$y' + y = 1$$

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + ce^{-x}$$

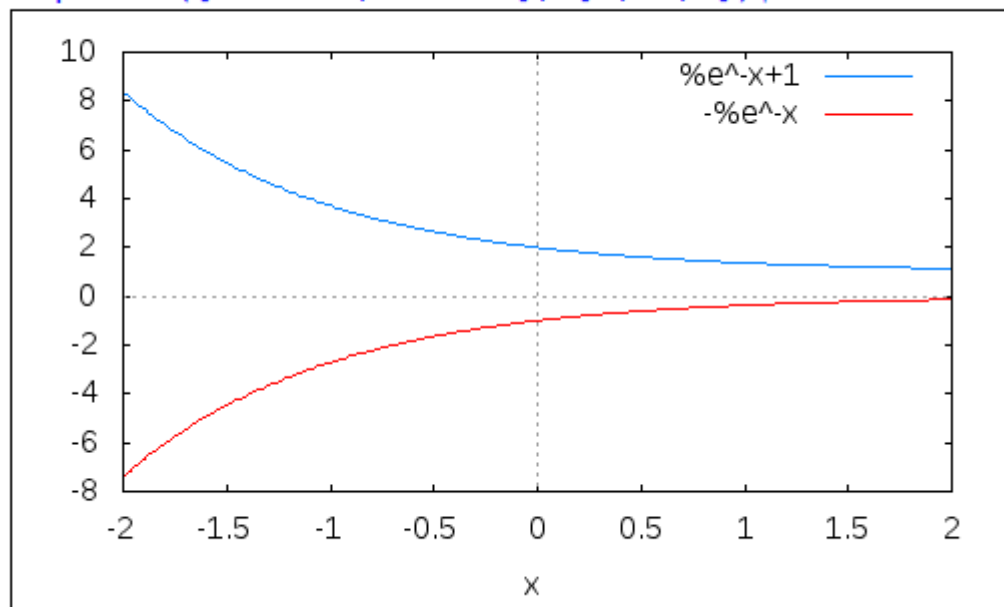
$$\frac{dy}{dx} + xy = x$$

$$y = 1 + ce^{-x^2/2}$$

$$\frac{dy}{dx} + x^2y = x^2$$

$$y = 1 + ce^{-x^3/3}$$

```
(%i30) wxplot2d([1+%e^-x, -%e^-x], [x,-2,2])$
```



$$y = 1 + e^{-x}$$
$$y' = -e^{-x}$$

$$y' + xy = x$$

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + ce^{-x}$$

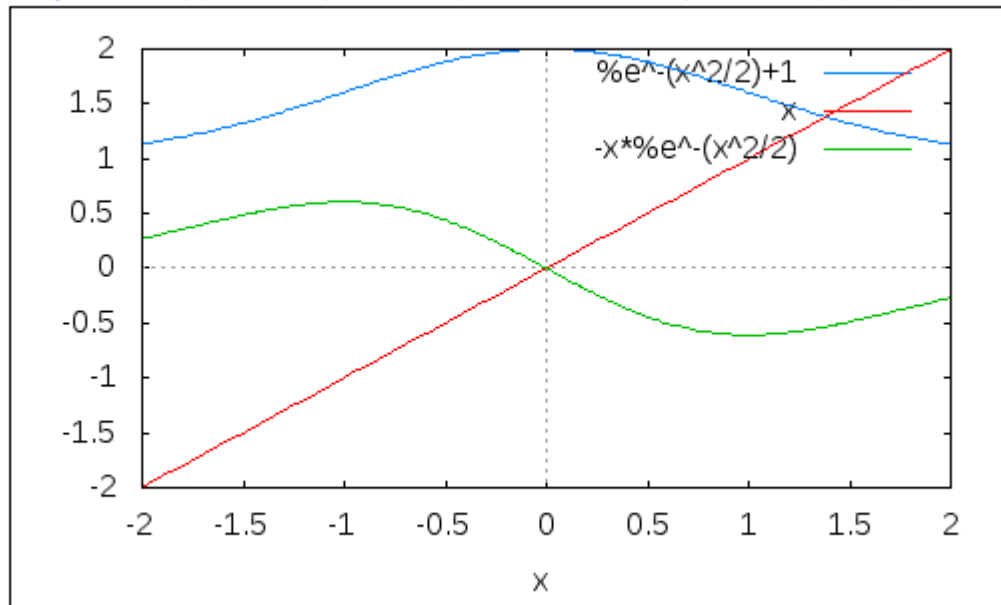
$$\frac{dy}{dx} + xy = x$$

$$y = 1 + ce^{-x^2/2}$$

$$\frac{dy}{dx} + x^2y = x^2$$

$$y = 1 + ce^{x^3/3}$$

```
(%i29) wxplot2d([1+%e^(-x^2/2),+x,-x*%e^(-x^2/2)], [x,-2,2])$
```



$$y = 1 + e^{-x^2/2}$$

$$y' = -x e^{-x^2/2}$$

$$y' + x^2y = x^2$$

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + ce^{-x}$$

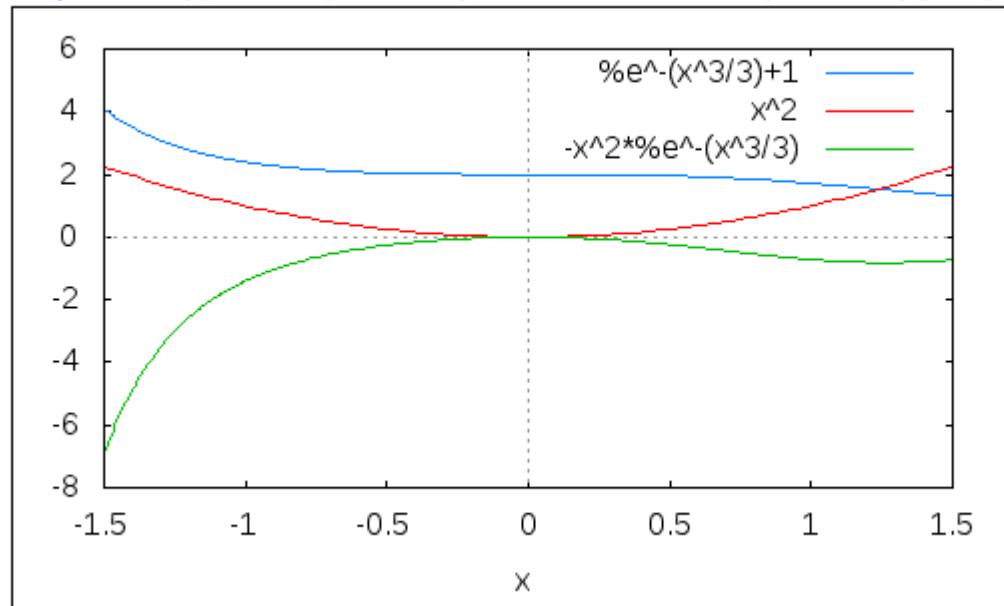
$$\frac{dy}{dx} + xy = x$$

$$y = 1 + ce^{-x^2/2}$$

$$\frac{dy}{dx} + x^2y = x^2$$

$$y = 1 + ce^{-x^3/3}$$

```
(%i31) wxplot2d([1+%e^(-x^3/3),+x^2,-x^2*%e^(-x^3/3)], [x,-1.5,1.5])$
```



$$y = 1 + e^{-x^3/3}$$

$$y' = -x^2 e^{-x^3/3}$$

First Order ODE examples (III)

$$\frac{dy}{dx} + y = 1$$

$$e^x \frac{dy}{dx} + e^x y = e^x$$

$$\frac{d}{dx}[e^x y] = e^x$$

$$e^x y = \int e^x dx + c$$

$$y = 1 + ce^{-x}$$

$$\frac{dy}{dx} + y = x$$

$$e^x \frac{dy}{dx} + e^x y = xe^x$$

$$\frac{d}{dx}[e^x y] = xe^x$$

$$e^x y = \int xe^x dx + c$$

$$e^x y = xe^x - e^x + c$$

$$y = (x-1) + ce^{-x}$$

$$\frac{d}{dx}[xe^x] = e^x + xe^x$$

$$xe^x = \int e^x dx + \int xe^x dx$$

$$\frac{dy}{dx} + y = x^2$$

$$e^x \frac{dy}{dx} + e^x y = x^2 e^x$$

$$\frac{d}{dx}[e^x y] = x^2 e^x$$

$$e^x y = \int x^2 e^x dx + c$$

$$e^x y = x^2 e^x - 2(xe^x - e^x) + c$$

$$y = x^2 - 2x + 2 + ce^{-x}$$

$$\frac{d}{dx}[x^2 e^x] = 2xe^x + x^2 e^x$$

$$x^2 e^x = 2 \int xe^x dx + \int x^2 e^x dx$$

$$y' + y = 1$$

$$\frac{dy}{dx} + y = 1$$

$$\frac{dy}{dx} + y = x$$

$$\frac{dy}{dx} + y = x^2$$

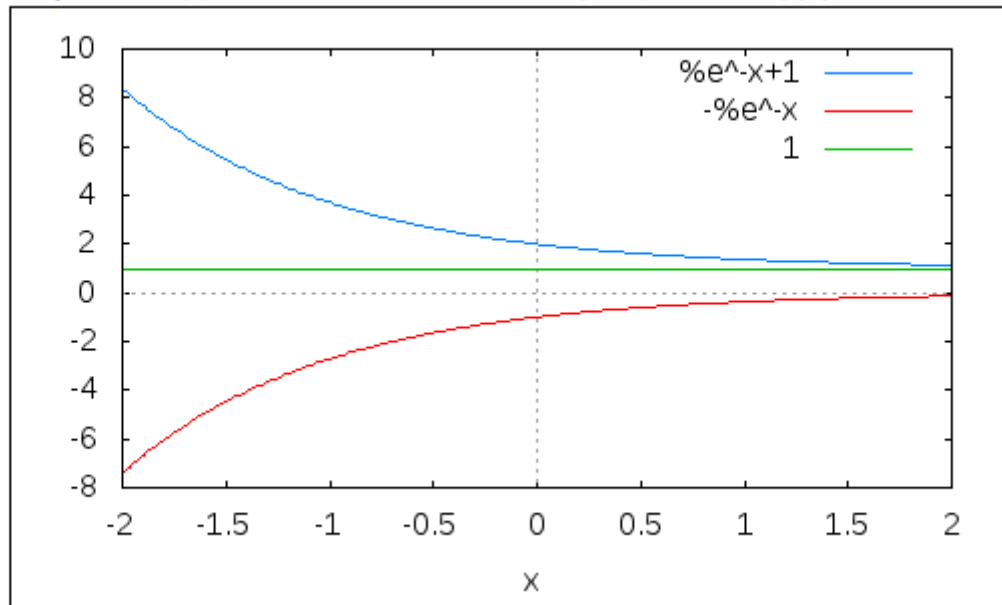
$$y = 1 + ce^{-x}$$

$$y = (x-1) + ce^{-x}$$

$$y = x^2 - 2x + 2 + ce^{-x}$$

```
(%i34) wxplot2d([1+%e^-x, -%e^-x, 1], [x,-2,2])$
```

```
(%t34)
```



$$y = 1 + e^{-x}$$

$$y' = -e^{-x}$$

$$y' + y = x$$

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + ce^{-x}$$

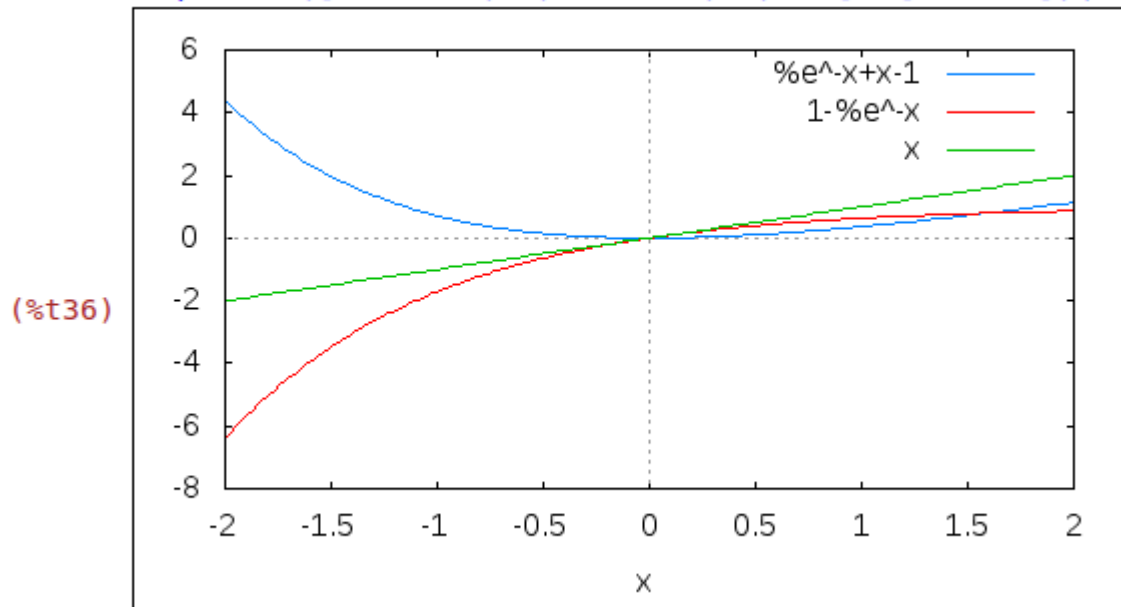
$$\frac{dy}{dx} + y = x$$

$$y = (x-1) + ce^{-x}$$

$$\frac{dy}{dx} + y = x^2$$

$$y = x^2 - 2x + 2 + ce^{-x}$$

```
(%i36) wxplot2d([x-1+%e^(-x),1-%e^(-x), x], [x,-2,2])$
```



$$y = (x-1) + e^{-x}$$

$$y' = 1 - e^{-x}$$

$$y' + y = x^2$$

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + ce^{-x}$$

$$\frac{dy}{dx} + y = x$$

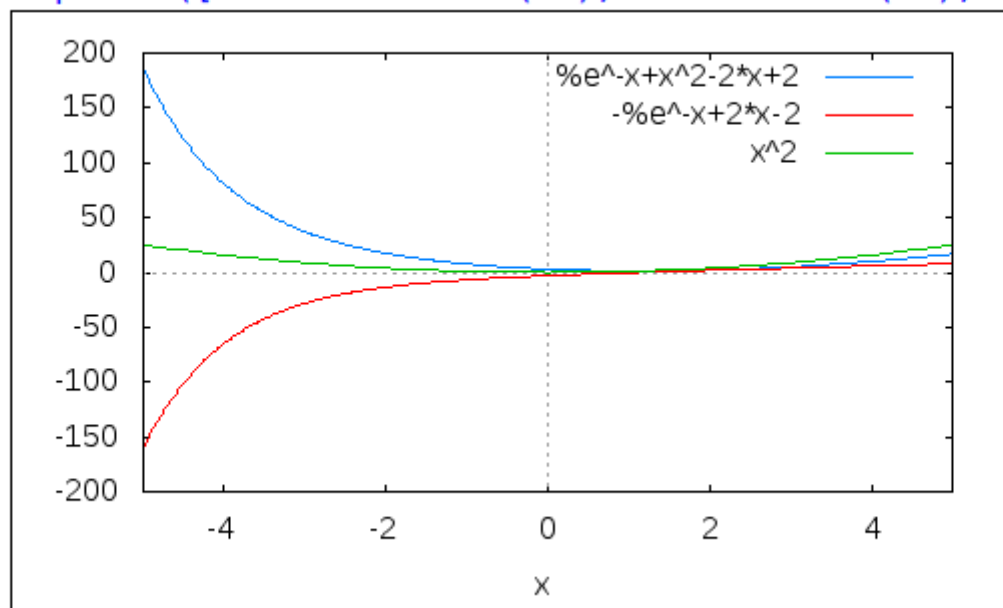
$$y = (x-1) + ce^{-x}$$

$$\frac{dy}{dx} + y = x^2$$

$$y = x^2 - 2x + 2 + ce^{-x}$$

```
(%i44) wxplot2d([x^2 -2*x +2+%e^(-x), +2*x -2 -%e^(-x), x^2 ], [x,-5, 5])$
```

```
(%t44)
```



$$y = x^2 - 2x + 2 + e^{-x}$$

$$y' = 2x - 2 - e^{-x}$$

Integration by parts

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

The diagram shows the integration by parts formula with color-coded terms and arrows. The integrand $f(x)g'(x)$ is split into $f(x)$ (green) and $g'(x)$ (pink). The result $f(x)g(x) - \int f'(x)g(x) dx$ is split into $f(x)g(x)$ (green and blue) and $-\int f'(x)g(x) dx$ (green and blue). A pink arrow points from $f(x)$ in the integrand to $f(x)$ in the result. A pink arrow points from $g'(x)$ in the integrand to $g(x)$ in the result. A green arrow points from $f(x)$ in the result to $f'(x)$ in the integrand. A green arrow points from $g(x)$ in the result to $g(x)$ in the integrand.

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + c_1 = (x-1)e^x + c_1$$

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx = x^2 e^x - 2x e^x + 2e^x + c_2 = (x^2 - 2x + 2)e^x + c_2$$

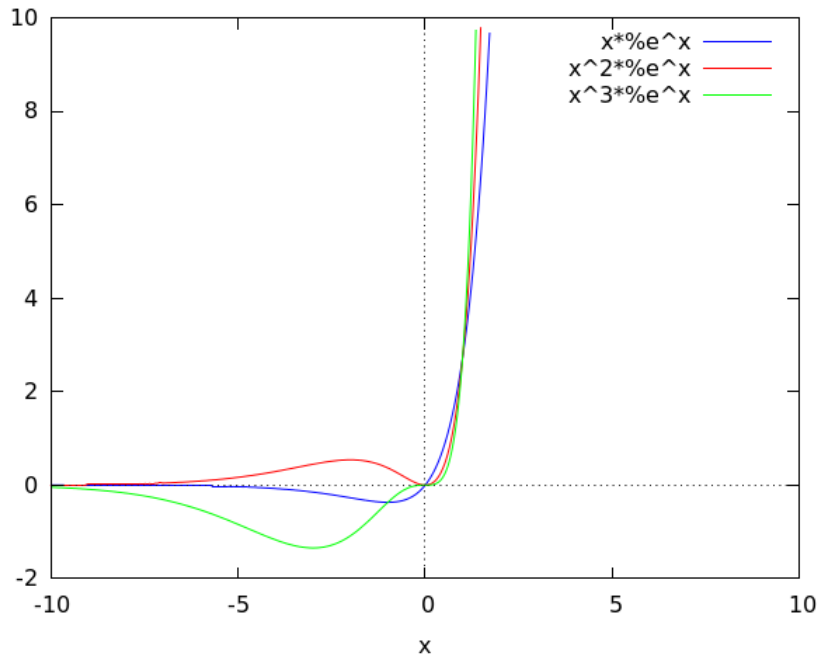
$$\int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c_3 = (x^3 - 3x^2 + 6x - 6)e^x + c_3$$

$$\int x e^x dx = (x-1)e^x + c_1$$

$$\int x^2 e^x dx = (x^2 - 2x + 2)e^x + c_2$$

$$\int x^3 e^x dx = (x^3 - 3x^2 + 6x - 6)e^x + c_3$$

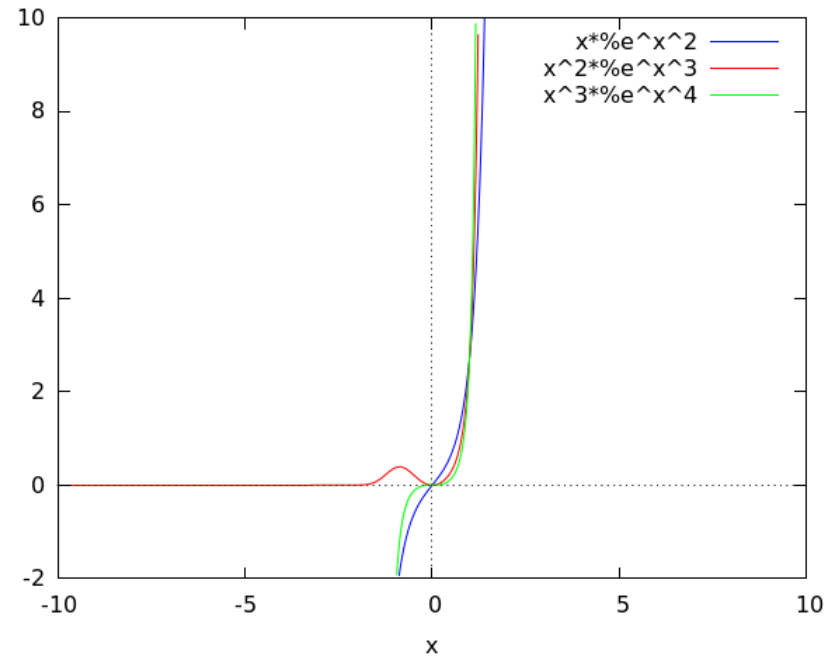
Plots of $x^m e^{x^n}$



$$\int x e^x dx$$

$$\int x^2 e^x dx$$

$$\int x^3 e^x dx$$



$$\int x e^{x^2} dx$$

$$\int x^2 e^{x^3} dx$$

$$\int x^3 e^{x^4} dx$$

First Order ODE examples - Summary

$$\frac{dy}{dx} + y = 0$$

$$y(x) = ce^{-x};$$

$$\frac{dy}{dx} + xy = 0$$

$$y(x) = ce^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2 y = 0$$

$$y(x) = ce^{-\frac{x^3}{3}};$$

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + ce^{-x}$$

$$\frac{dy}{dx} + y = x$$

$$y = (x-1) + ce^{-x}$$

$$\frac{dy}{dx} + y = x^2$$

$$y = x^2 - 2x + 2 + ce^{-x}$$

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + ce^{-x}$$

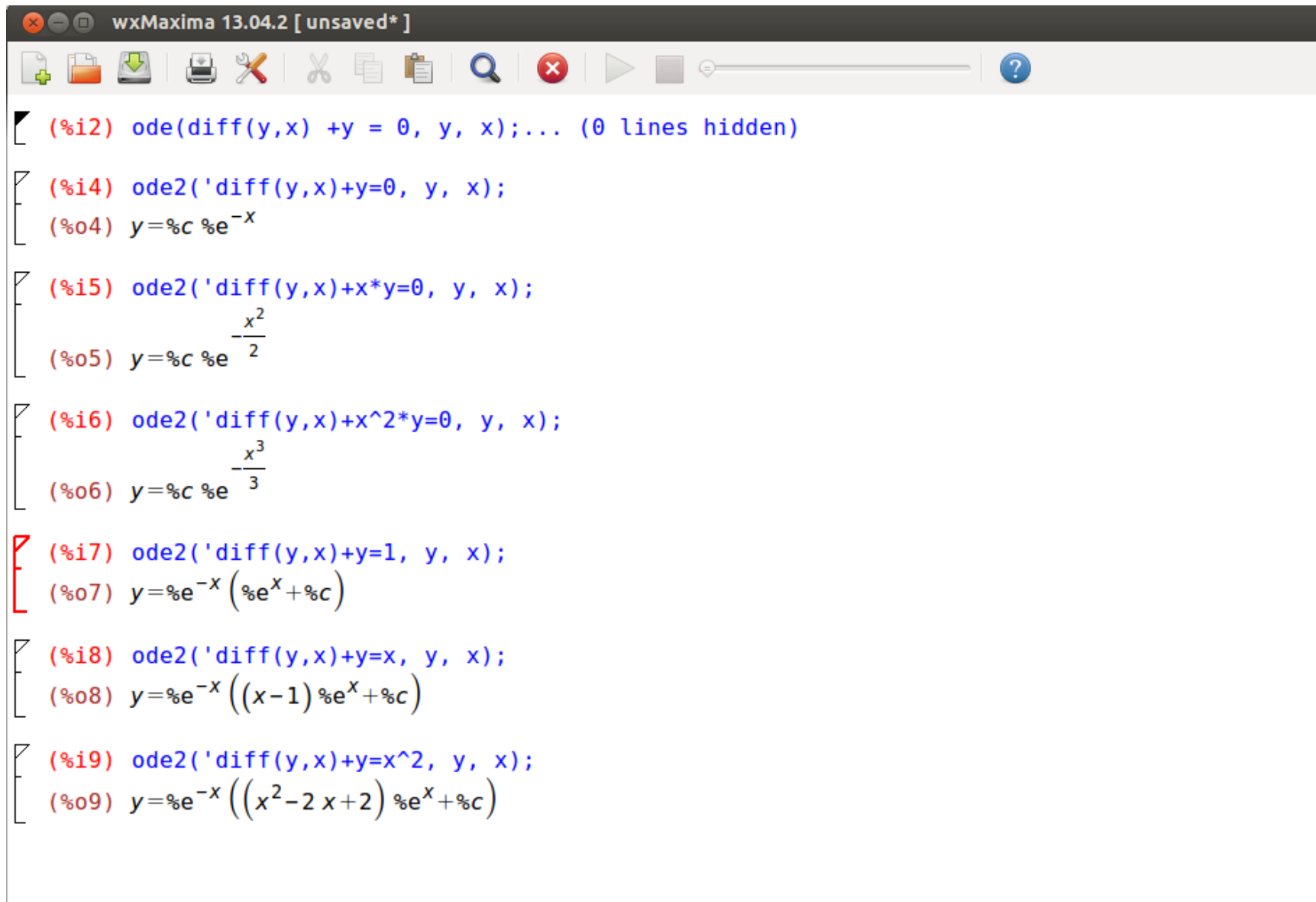
$$\frac{dy}{dx} + xy = x$$

$$y = 1 + ce^{-x^2/2}$$

$$\frac{dy}{dx} + x^2 y = x^2$$

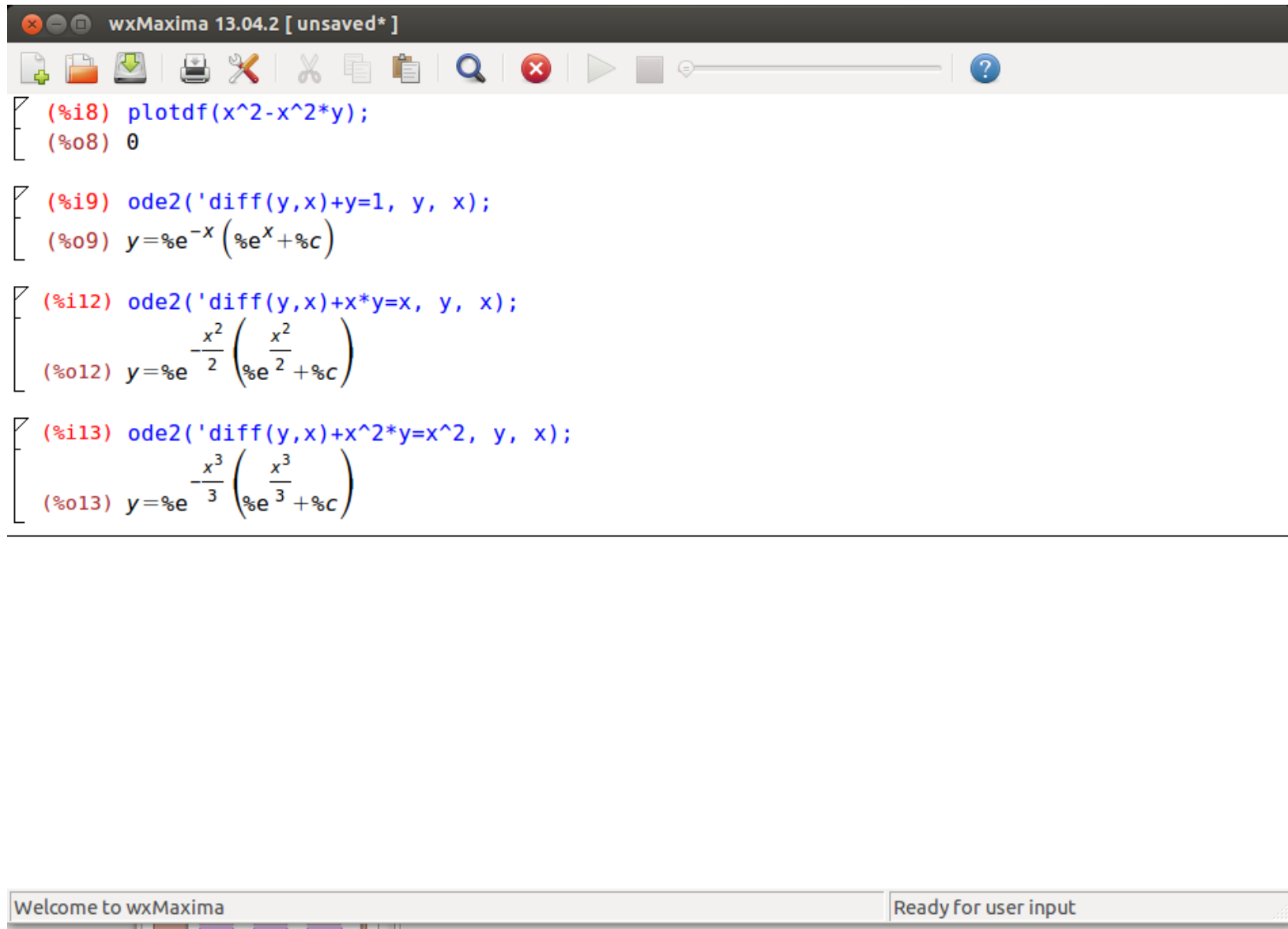
$$y = 1 + ce^{-x^3/3}$$

First Order ODE by wxMaxima (1)



```
wxMaxima 13.04.2 [unsaved*]  
[ (%i2) ode(diff(y,x) + y = 0, y, x);... (0 lines hidden)  
[ (%i4) ode2('diff(y,x)+y=0, y, x);  
[ (%o4) y=%c %e-x  
[ (%i5) ode2('diff(y,x)+x*y=0, y, x);  
[ (%o5) y=%c %e- $\frac{x^2}{2}$   
[ (%i6) ode2('diff(y,x)+x^2*y=0, y, x);  
[ (%o6) y=%c %e- $\frac{x^3}{3}$   
[ (%i7) ode2('diff(y,x)+y=1, y, x);  
[ (%o7) y=%e-x (%ex+%c)  
[ (%i8) ode2('diff(y,x)+y=x, y, x);  
[ (%o8) y=%e-x ((x-1)%ex+%c)  
[ (%i9) ode2('diff(y,x)+y=x^2, y, x);  
[ (%o9) y=%e-x ((x2-2x+2)%ex+%c)
```

First Order ODE by wxMaxima (2)

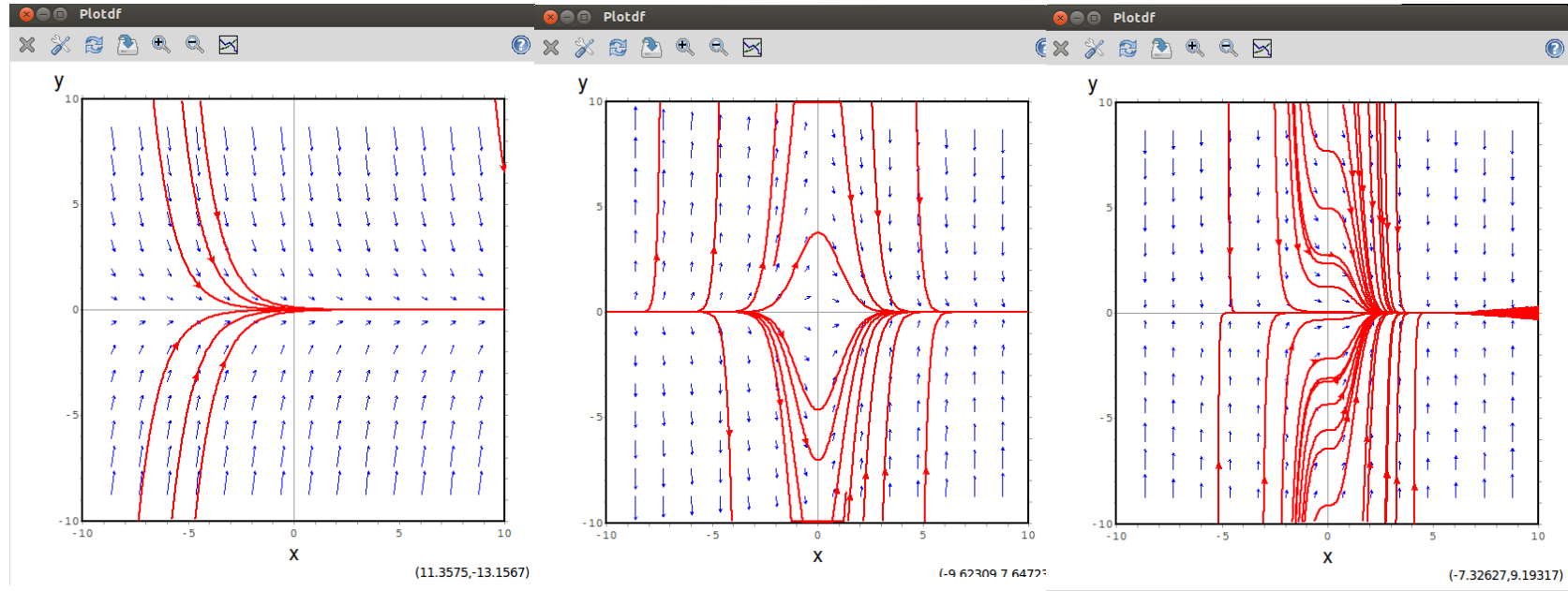


The screenshot shows the wxMaxima 13.04.2 interface with the following content:

```
(%i18) plotdf(x^2-x^2*y);  
(%o18) 0  
  
(%i19) ode2('diff(y,x)+y=1, y, x);  
(%o19) y=%e-x (%ex+%c)  
  
(%i12) ode2('diff(y,x)+x*y=x, y, x);  
(%o12) y=%e-x^2/2 ( %ex^2/2 +%c)  
  
(%i13) ode2('diff(y,x)+x^2*y=x^2, y, x);  
(%o13) y=%e-x^3/3 ( %ex^3/3 +%c)
```

At the bottom of the window, there is a status bar with the text "Welcome to wxMaxima" on the left and "Ready for user input" on the right.

plotdf in wxMaxima (1)



$$\frac{dy}{dx} + y = 0$$

$$y(x) = ce^{-x};$$

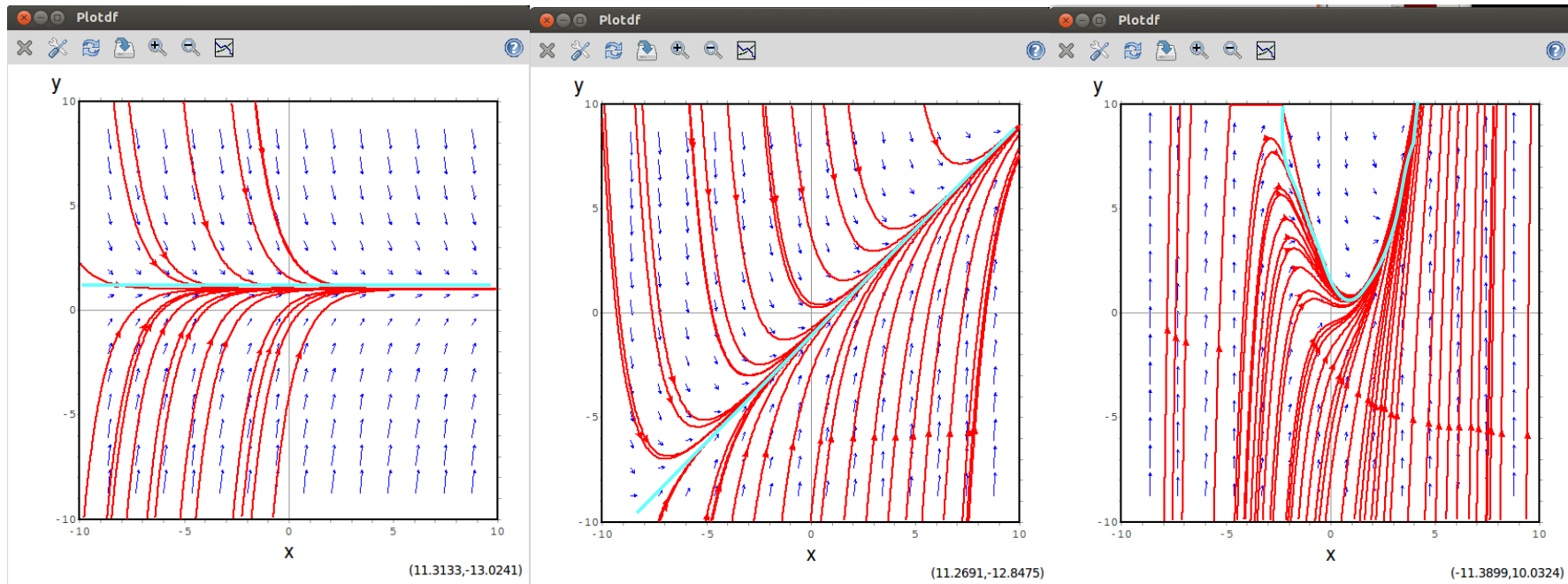
$$\frac{dy}{dx} + xy = 0$$

$$y(x) = ce^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2 y = 0$$

$$y(x) = ce^{-\frac{x^3}{3}};$$

plotdf in wxMaxima (2)



$$\frac{dy}{dx} + y = 1$$

$$y = 1 + ce^{-x}$$

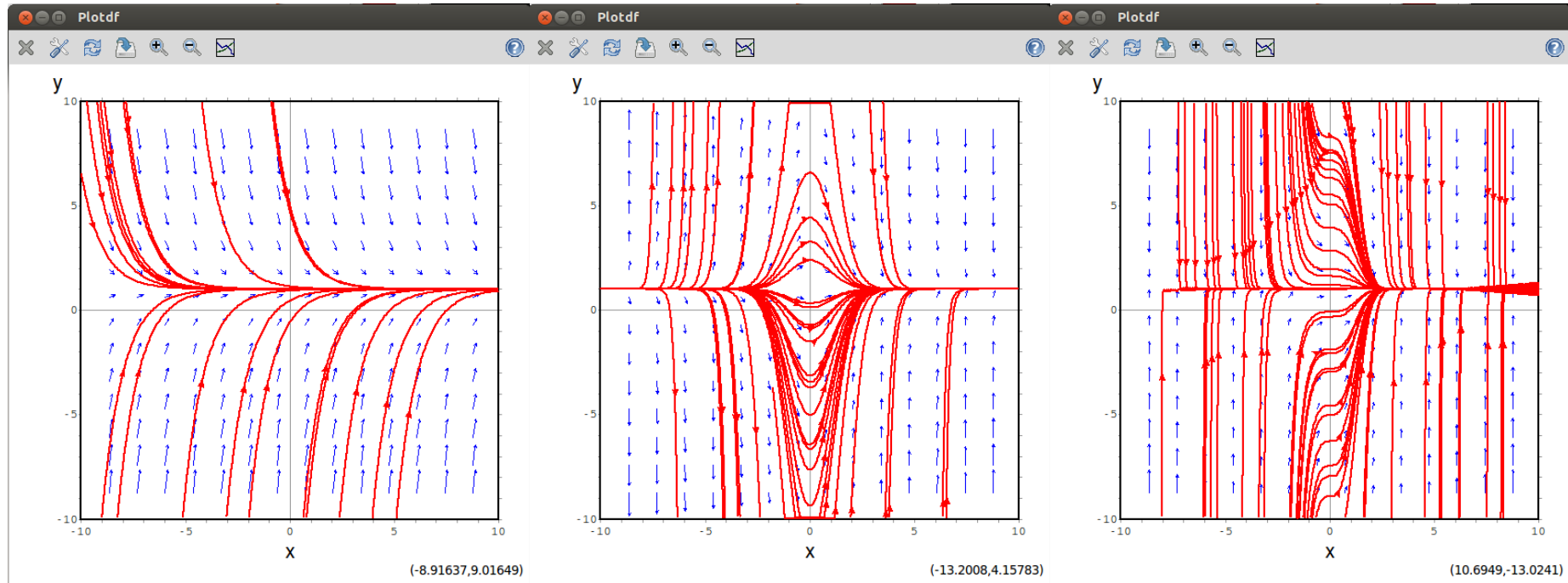
$$\frac{dy}{dx} + y = x$$

$$y = (x-1) + ce^{-x}$$

$$\frac{dy}{dx} + y = x^2$$

$$y = x^2 - 2x + 2 + ce^{-x}$$

plotdf in wxMaxima (3)



$$\frac{dy}{dx} + y = 1$$

$$y = 1 + ce^{-x}$$

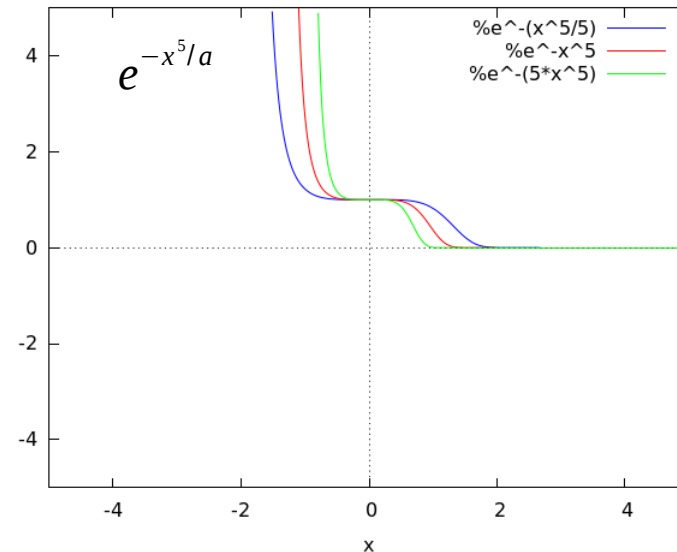
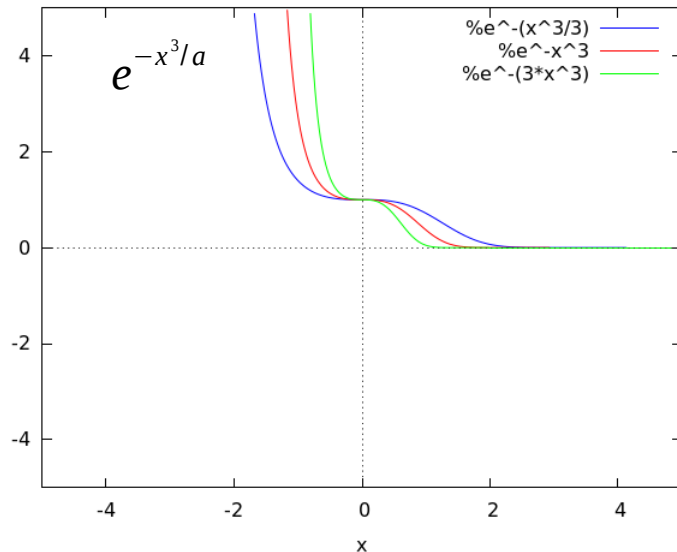
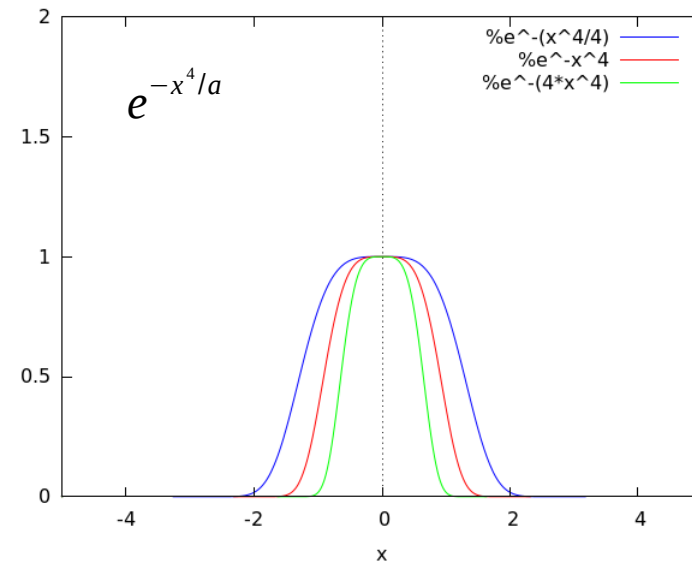
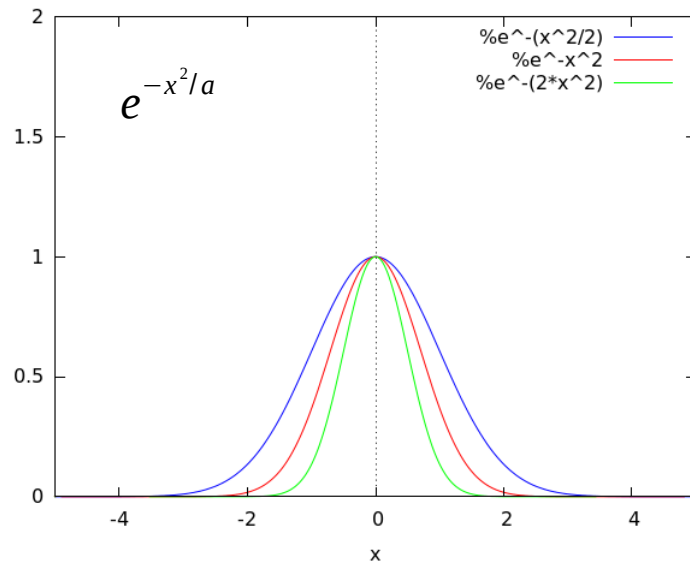
$$\frac{dy}{dx} + xy = x$$

$$y = 1 + ce^{-x^2/2}$$

$$\frac{dy}{dx} + x^2 y = x^2$$

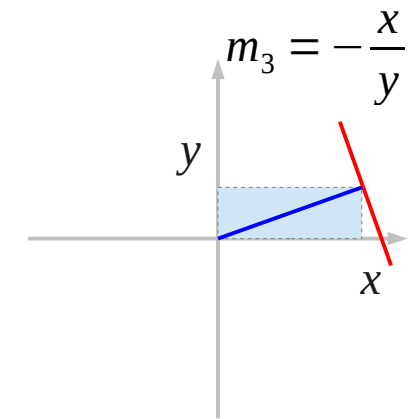
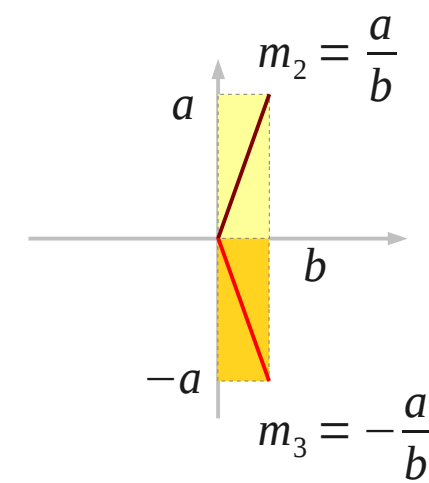
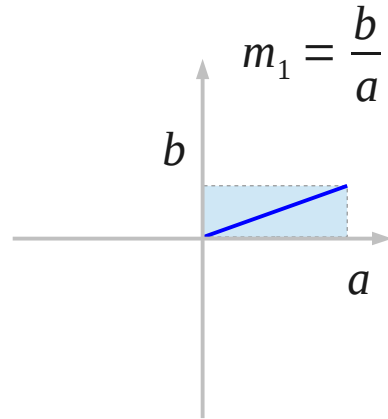
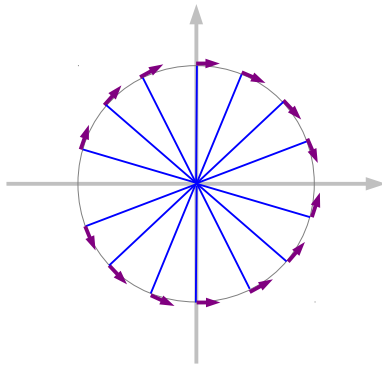
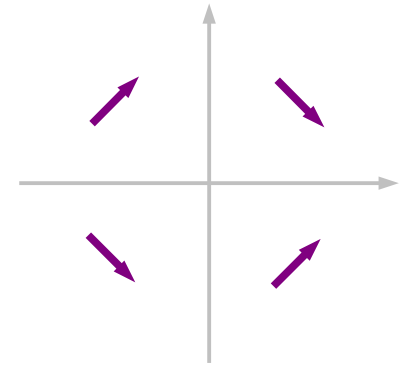
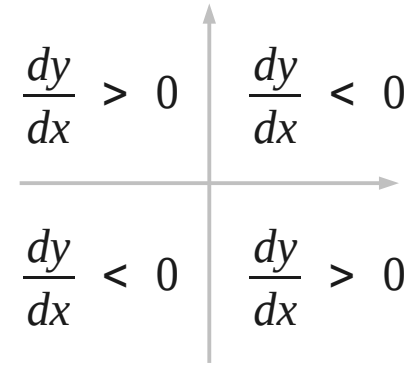
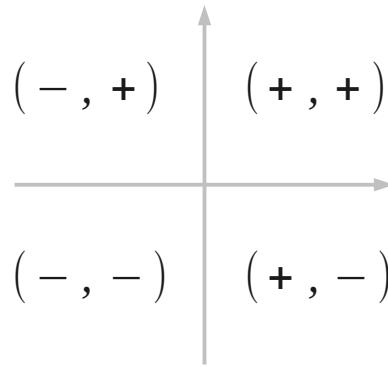
$$y = 1 + ce^{-x^3/3}$$

Plots of $e^{-x^2/a}$, $e^{-x^3/a}$, $e^{-x^4/a}$, $e^{-x^5/a}$ functions



Slope of $(-x/y)$

$$\frac{dy}{dx} = -\frac{x}{y}$$

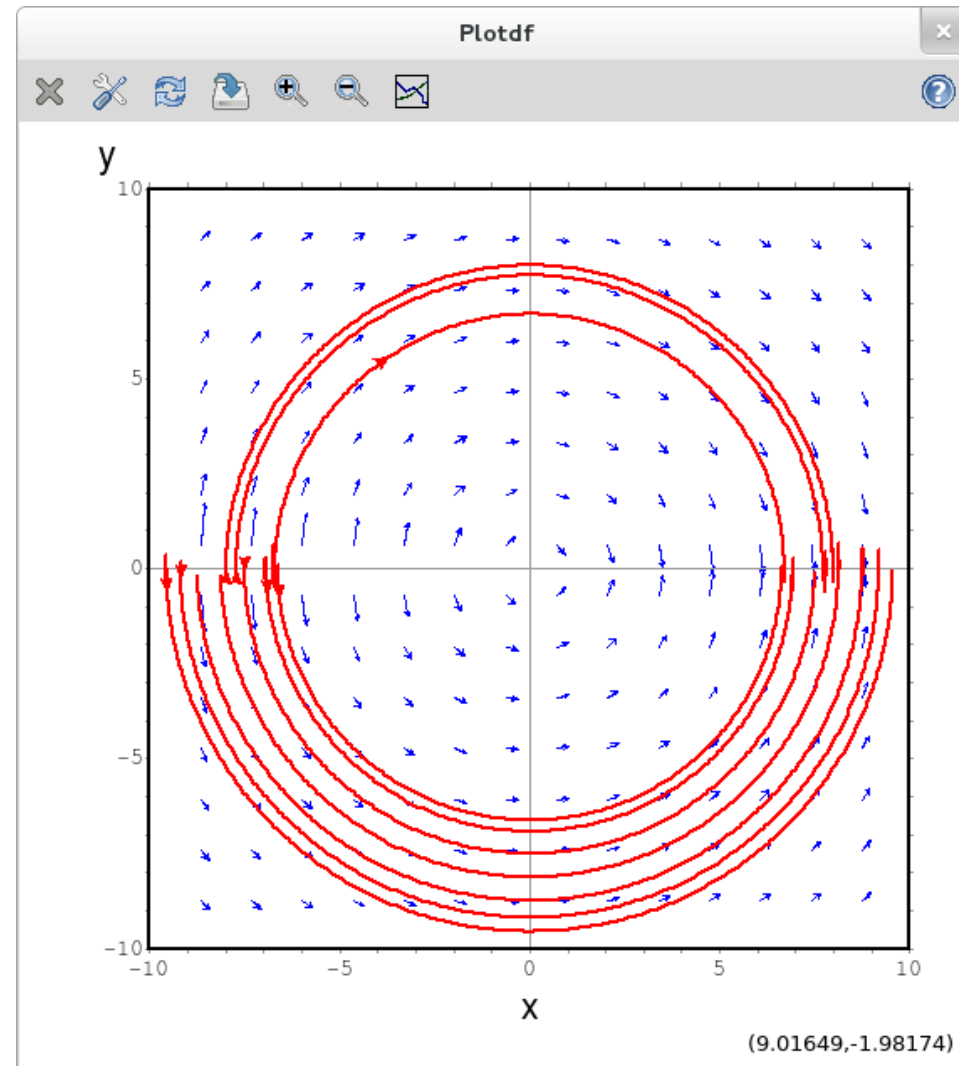
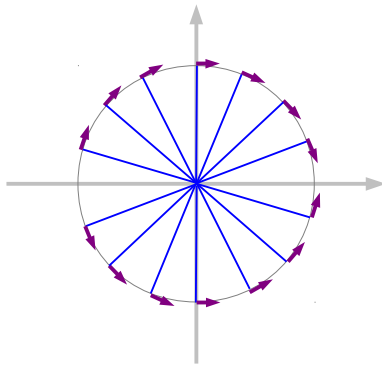


Direction Field of $(-x/y)$

$$\frac{dy}{dx} = -\frac{x}{y}$$

2-d version of $F(x,y)$

$$F(x, y) = -\frac{x}{y}$$

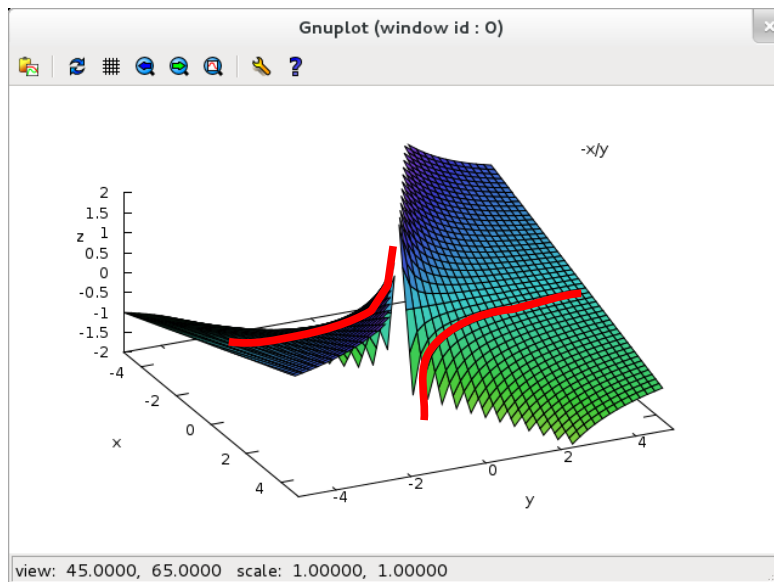


3-d Plot of $(-x/y)$

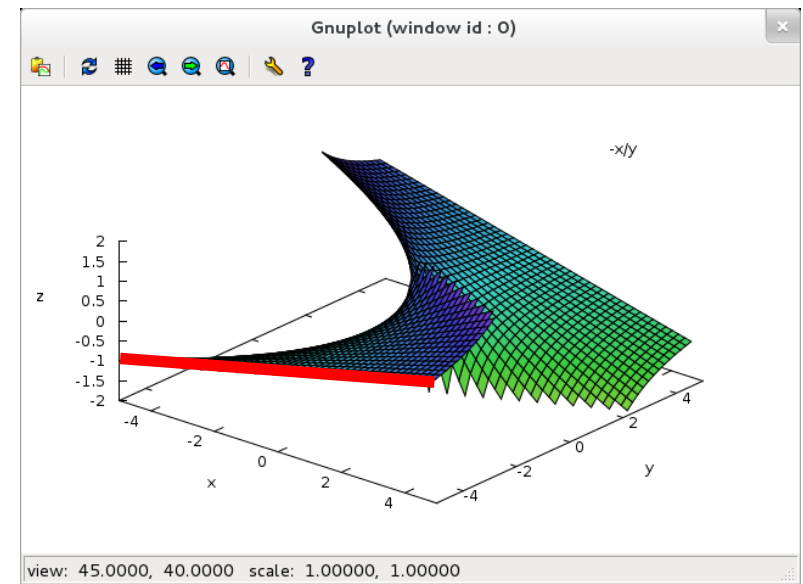
$$\frac{dy}{dx} = -\frac{x}{y}$$

$$F(x, y) = -\frac{x}{y}$$

3-d plot of $F(x,y)$



$$-\frac{1}{y}$$



x

Separable Equation Method

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} dx = -\frac{x}{y} dx$$

$$y dy = -x dx$$

$$y dy = -x dx$$

$$\int y dy = -\int x dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 = -x^2 + C$$

$$y = \pm\sqrt{C - x^2}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} - \frac{x}{y} = 0$$

Not Linear Equation

Exact Equation Method (1)

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$dy = -\frac{x}{y} dx$$

$$y dy = -x dx$$

$$x dx + y dy = 0$$

$$\frac{\partial f}{\partial x} = x \quad \frac{\partial f}{\partial y} = y$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0 \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial f}{\partial x} = x$$

$$\int \frac{\partial f}{\partial x} dx = \int x dx$$

$$f(x, y) = \frac{x^2}{2} + g(y)$$

$$\frac{\partial f}{\partial y} = g'(y)$$

$$\frac{\partial f}{\partial y} = y$$

$$g'(y) = y$$

$$g(y) = \frac{y^2}{2} + c_1$$

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1$$

Exact Equation Method (2)

A differential form

$$P(x, y)dx + Q(x, y)dy$$

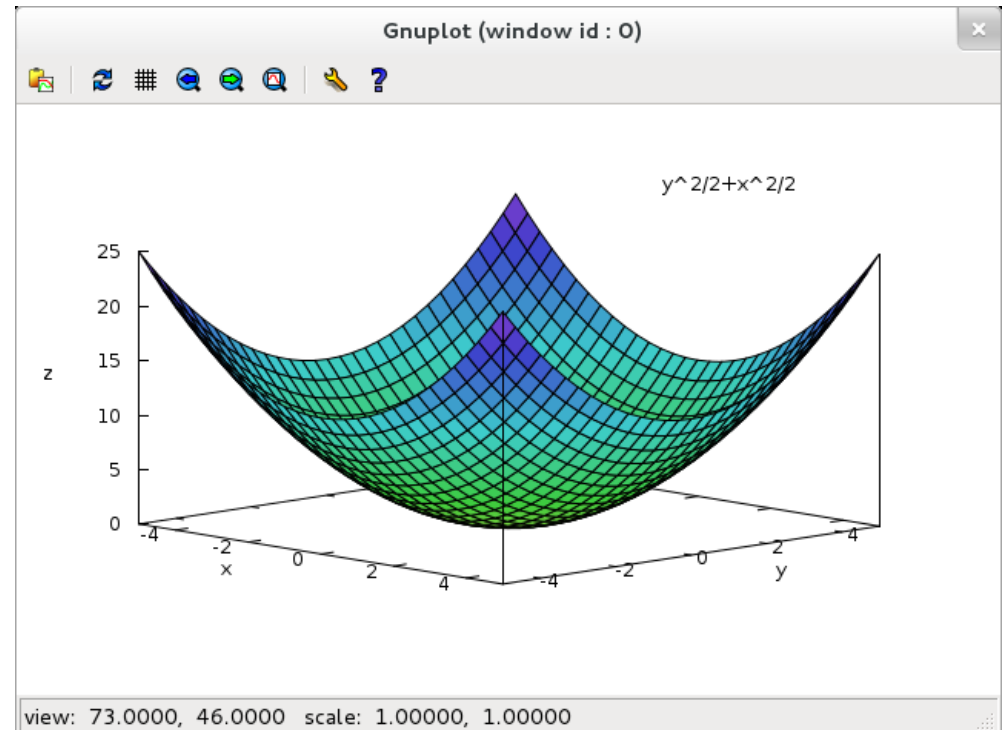
this differential form is **exact** in a region **R** if there is a function **f(x,y)** such that

$$\begin{aligned} P(x, y)dx + Q(x, y)dy \\ = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df \end{aligned}$$

differential form

$$x dx + y dy \quad \rightarrow$$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df$$



exact differential form

Since there exists such

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1$$

Exact Equation Method (3)

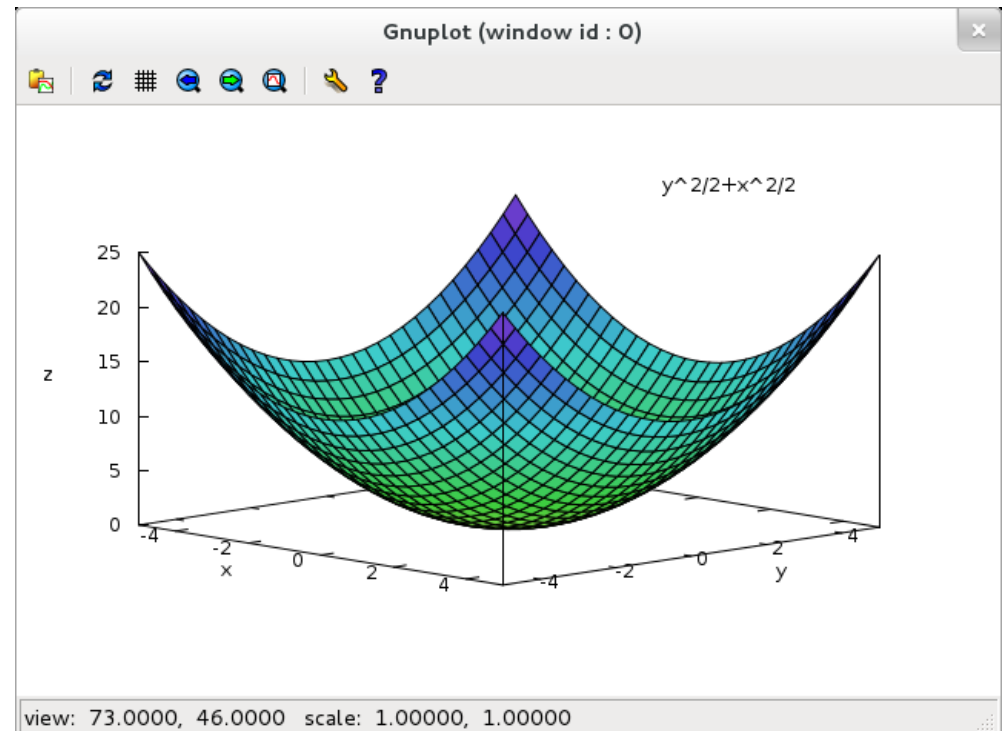
A first order differential equation

$$P(x, y)dx + Q(x, y)dy = 0$$

exact equation

in a region R if there is a function $f(x, y)$ such that

$$\begin{aligned} P(x, y)dx + Q(x, y)dy \\ = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df = 0 \end{aligned}$$



differential equation

$$x dx + y dy = 0 \quad \rightarrow$$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df = 0$$

exact differential equation

The implicit solution is

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1 = c$$

Exact Equation Method (4)

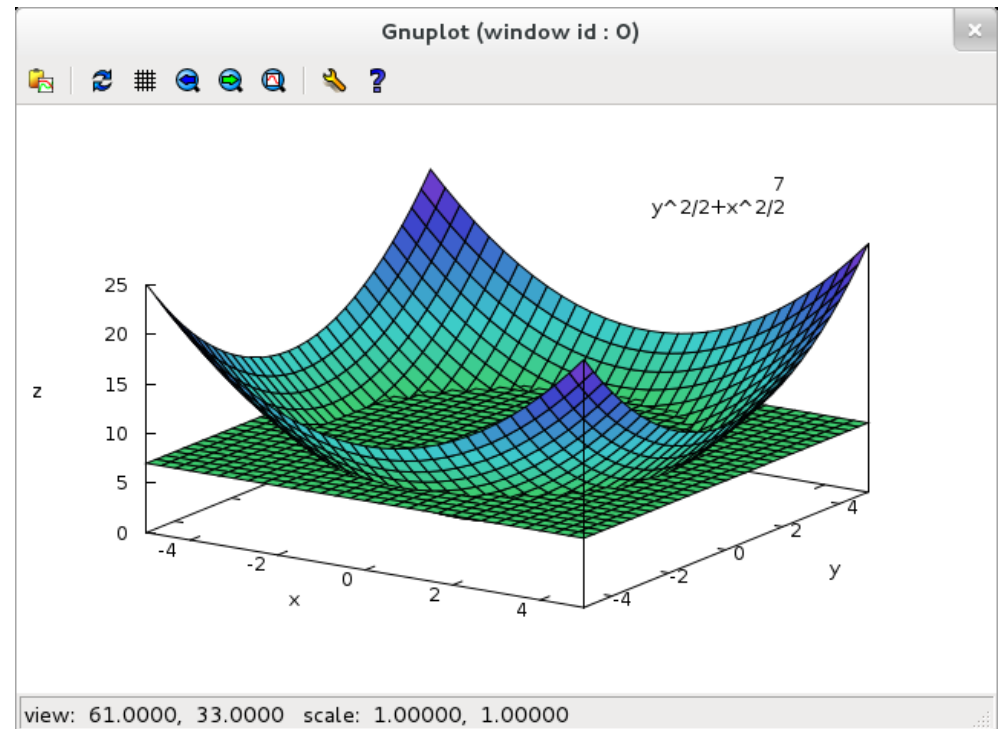
differential equation

$$x dx + y dy = 0 \quad \rightarrow$$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df = 0$$

The implicit solution is

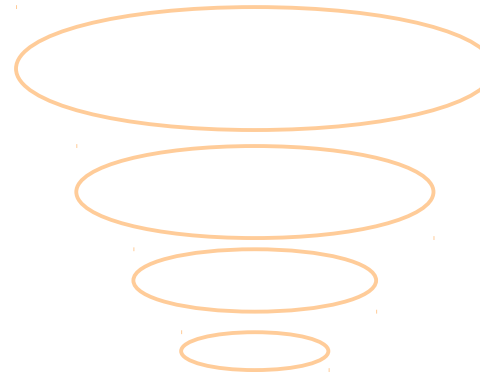
$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1 = c$$



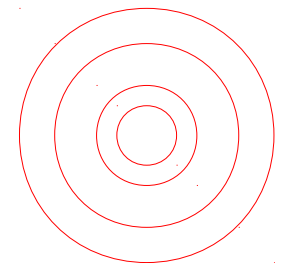
$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} = 7$$

The solution is all the points (x, y) that satisfies $f(x, y) = 7$

different c
In 3-d space



different c
In 2-d space



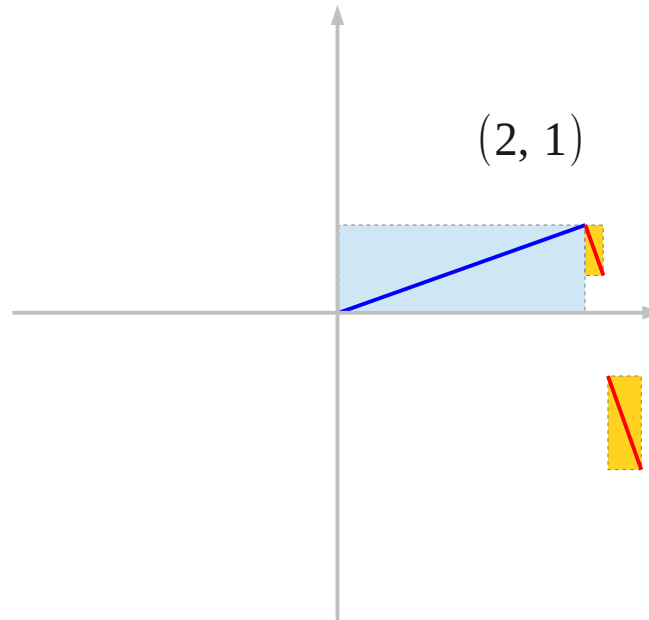
Exact Equation Method (5)

The implicit solution is

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} = C$$

$$\frac{\partial f}{\partial x} = x \quad \frac{\partial f}{\partial y} = y$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

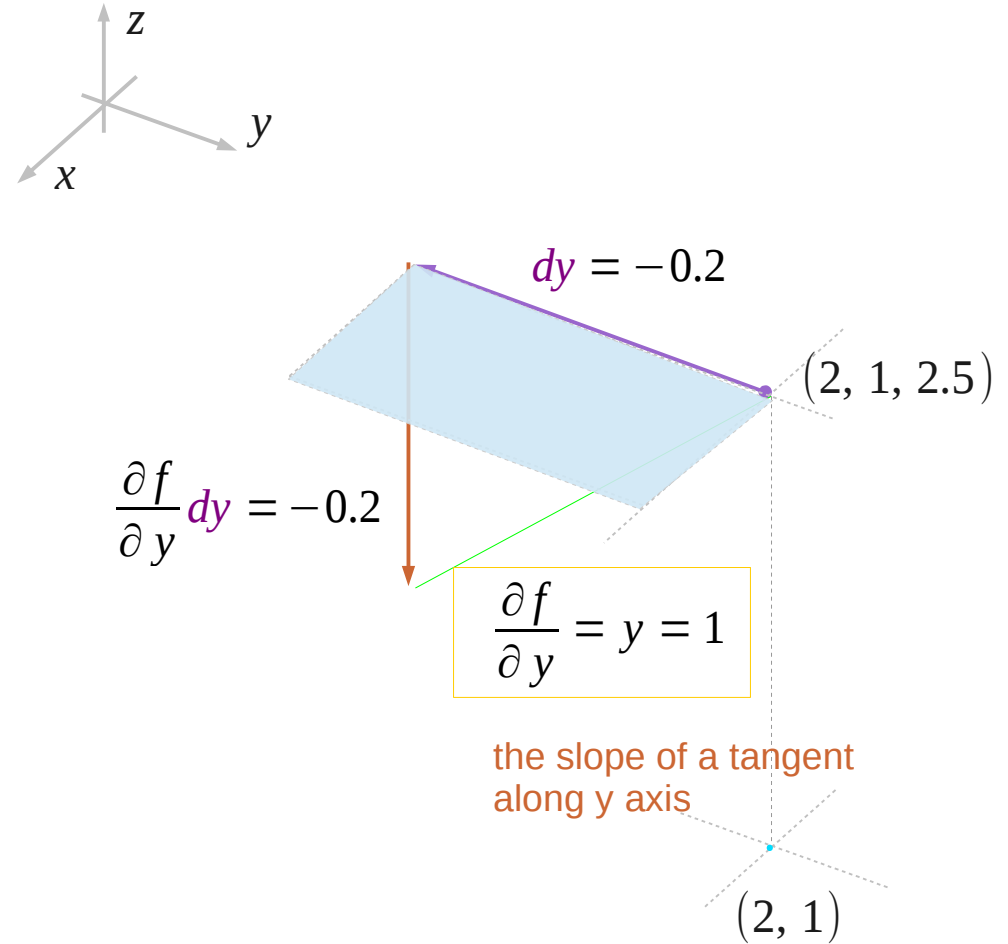
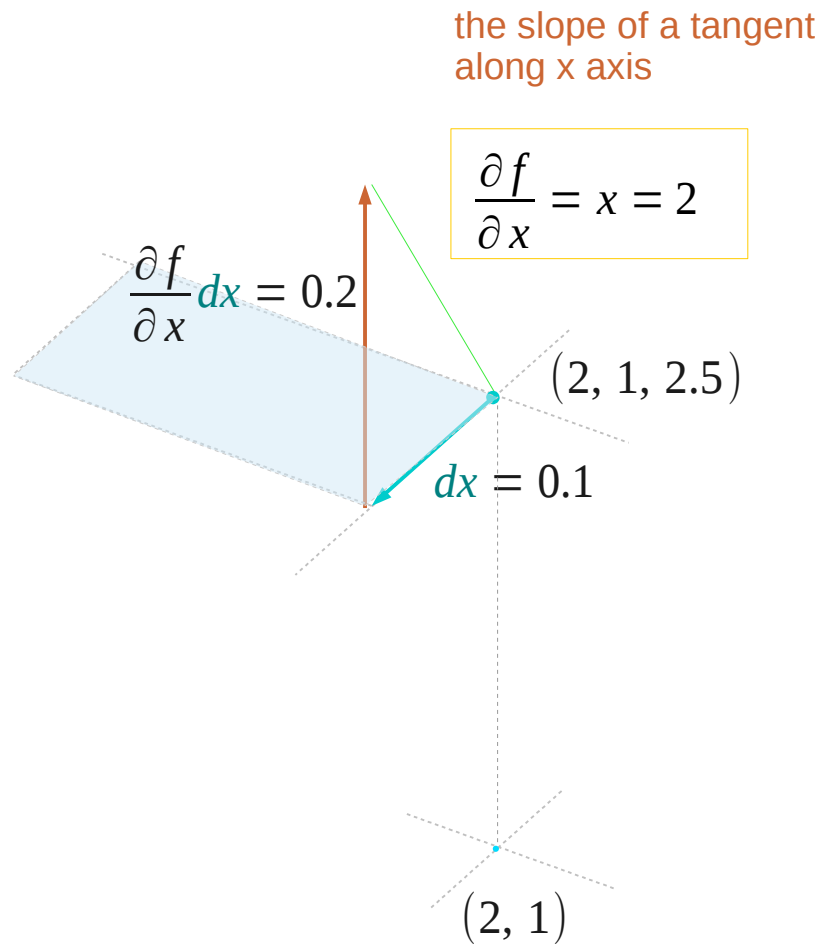


$$df = 2 \cdot dx + 1 \cdot dy = 0$$

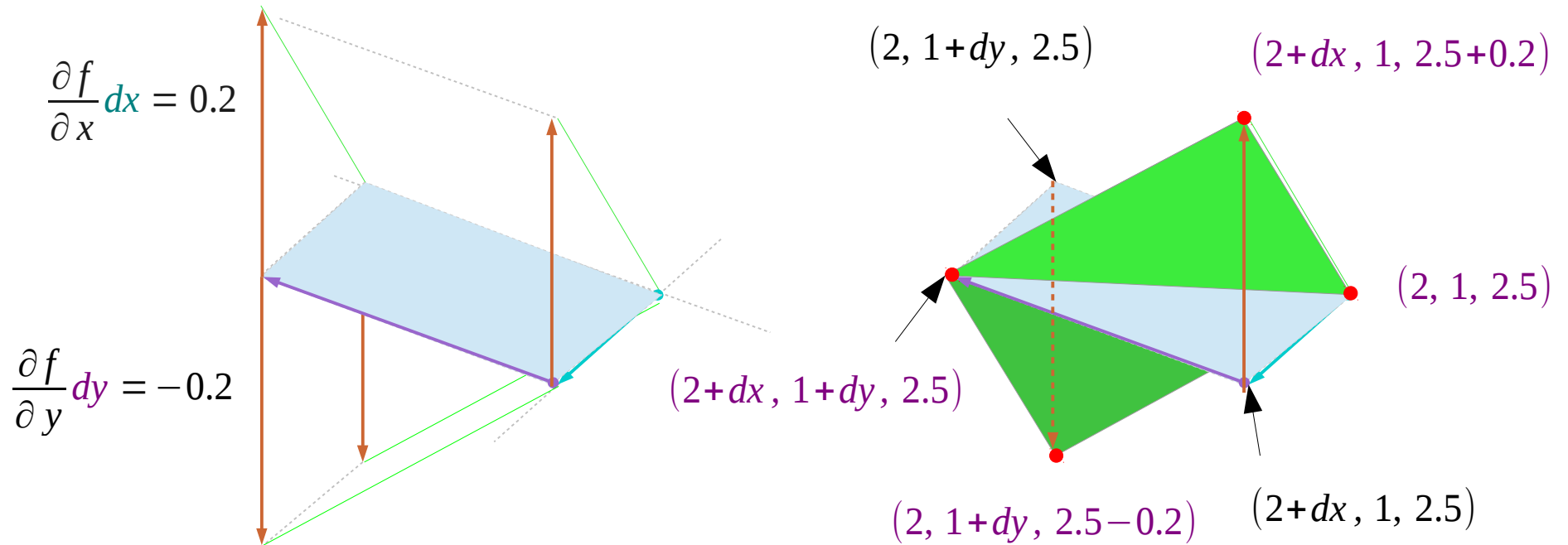
dx	dy
+0.1	-0.2
-0.1	+0.2
+0.01	-0.02
-0.01	+0.02

The solution is all the points (x, y) that satisfies $f(x, y) = 7$

Exact Equation Method (6)



Exact Equation Method (7)



Exact Equation Method (8)

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$$

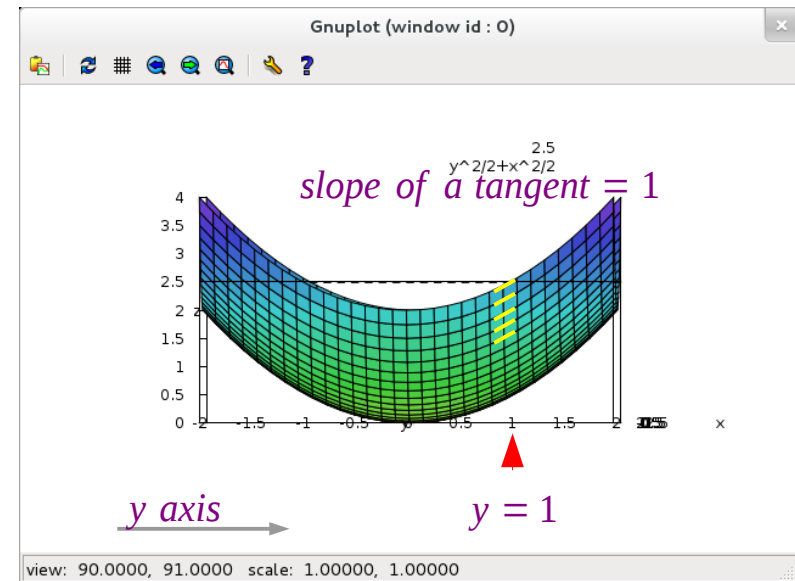
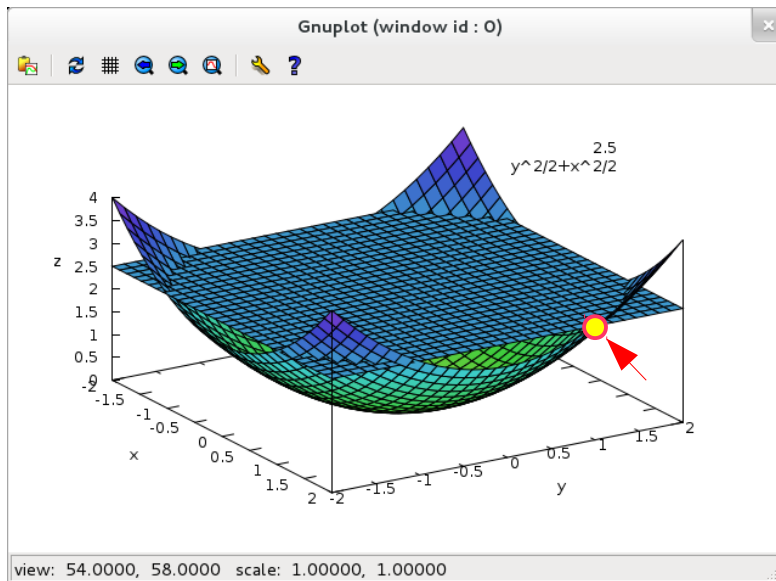
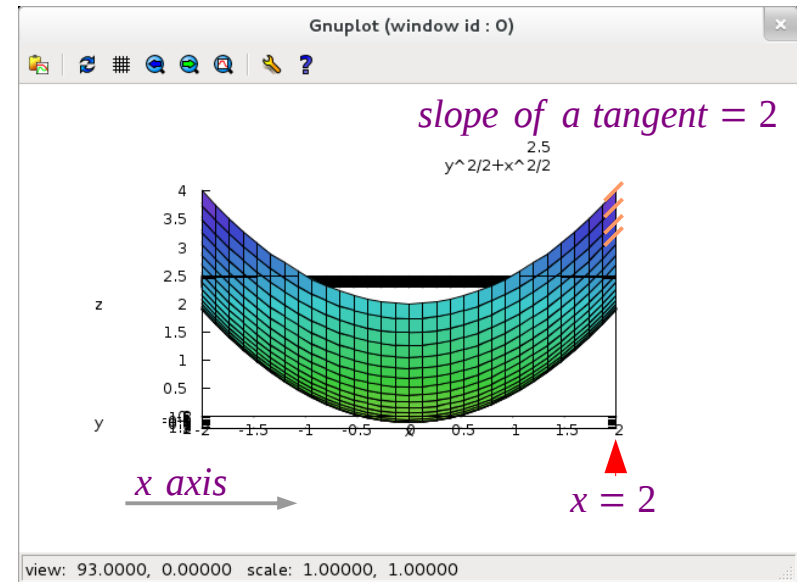
$$f(2, 1) = 2.5$$

$$f(x, 1) = \frac{x^2}{2} + \frac{1}{2}$$

$$\frac{\partial f}{\partial x}(2, 1) = 2$$

$$f(2, y) = 2 + \frac{y^2}{2}$$

$$\frac{\partial f}{\partial y}(2, 1) = 1$$



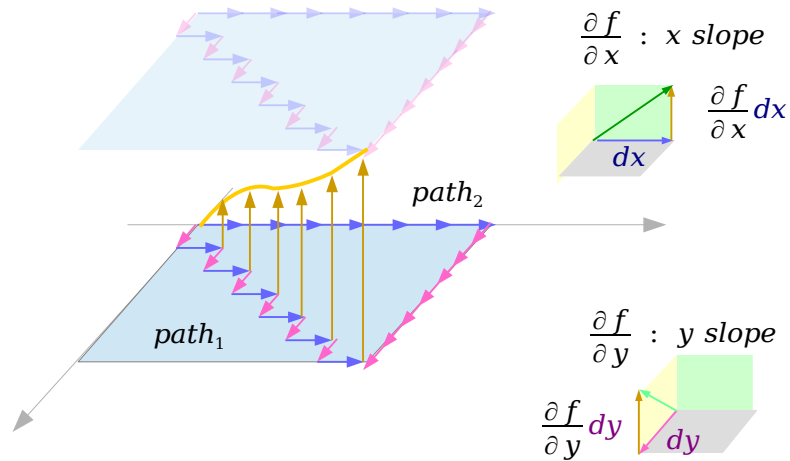
Integrating Total Differentials

$$z = f(x, y)$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\int df = \int \left[\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right] + c$$

$$f = \int \left[\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right] + c$$



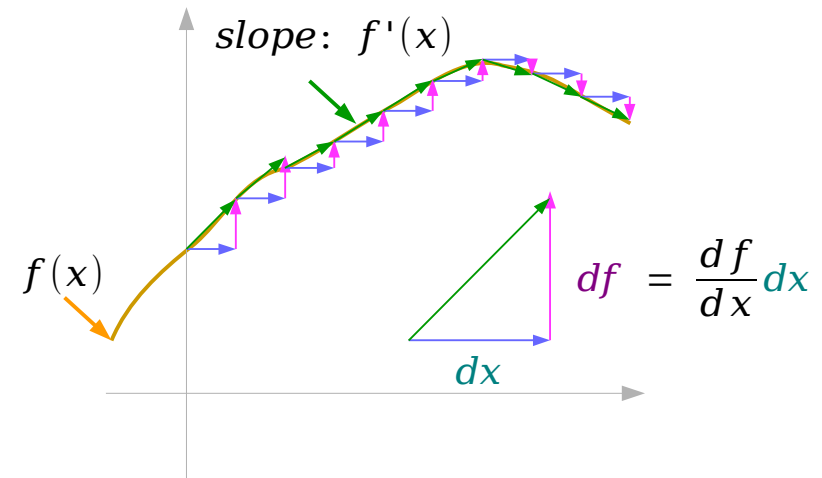
$$df = 0 \quad \Rightarrow \quad f(x, y) = c$$

$$y = f(x)$$

$$df = \frac{df}{dx} dx$$

$$\int df = \int \frac{df}{dx} dx + c$$

$$f = \int \frac{df}{dx} dx + c$$



$$df = 0 \quad \Rightarrow \quad f(x) = c$$

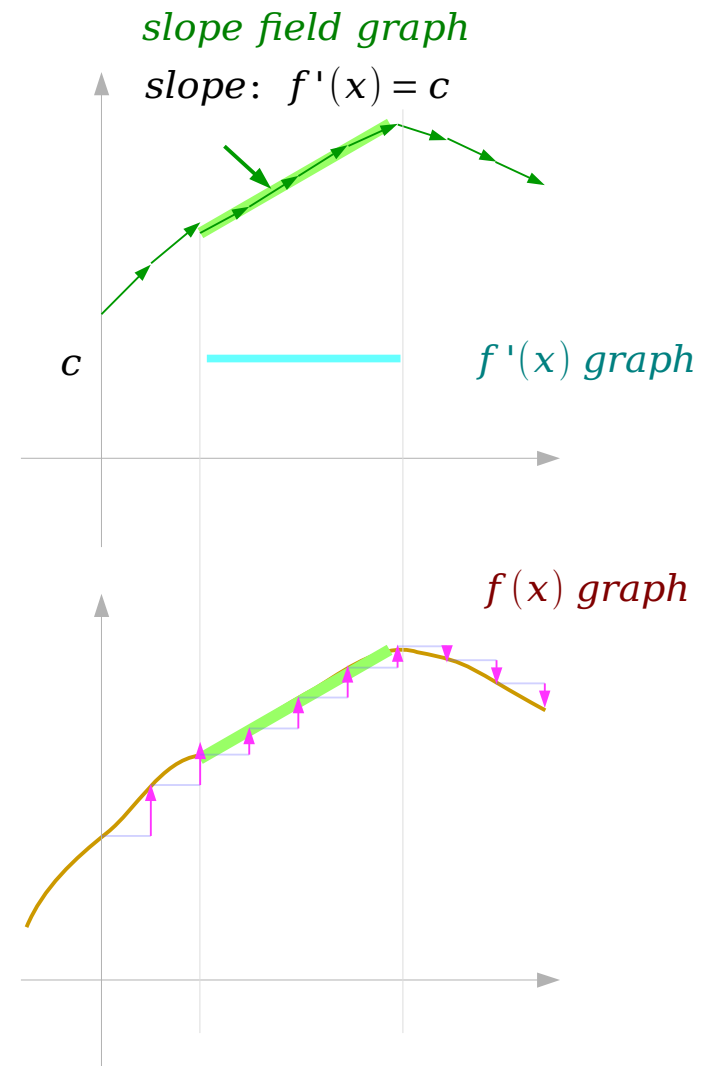
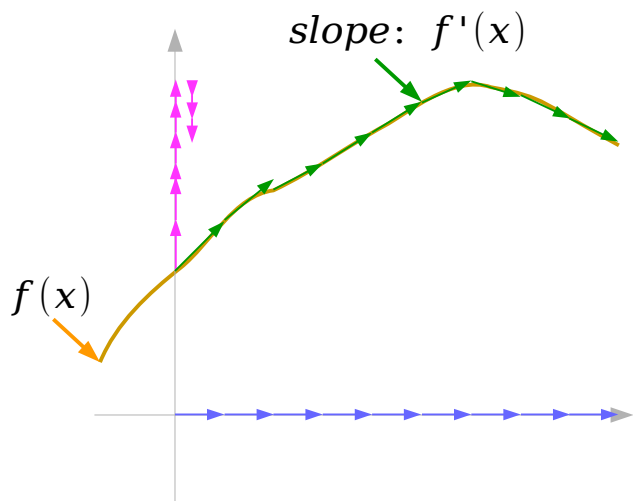
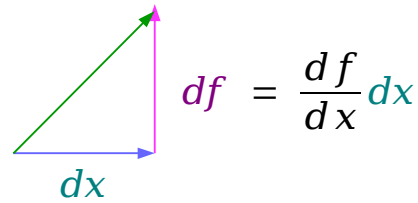
Integrating Differentials

$$y = f(x)$$

$$df = \frac{df}{dx} dx$$

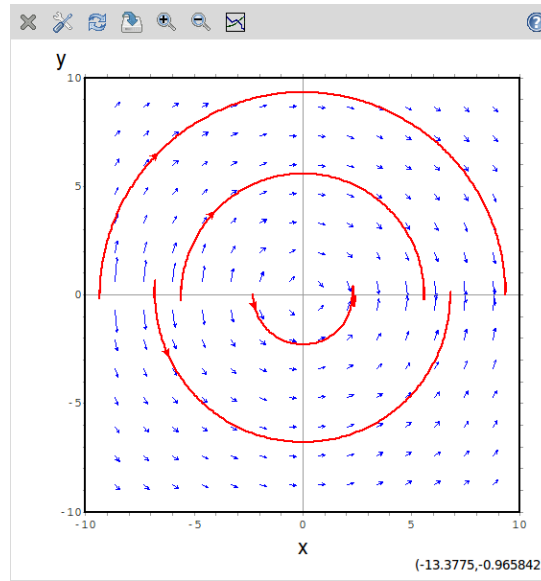
$$\int df = \int \frac{df}{dx} dx$$

$$f = \int \frac{df}{dx} dx + c$$

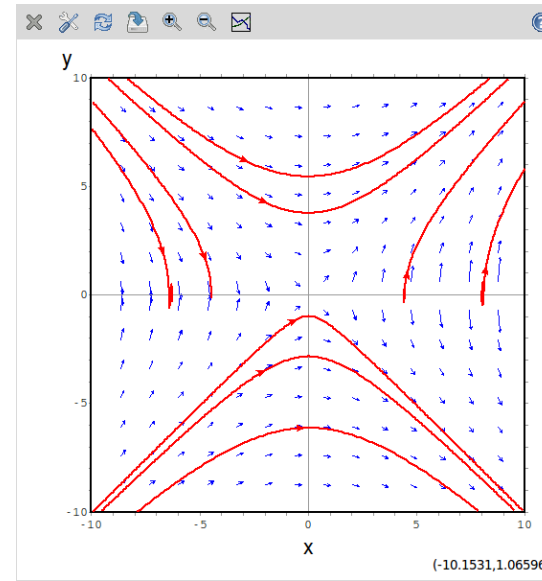


Some other direction field examples

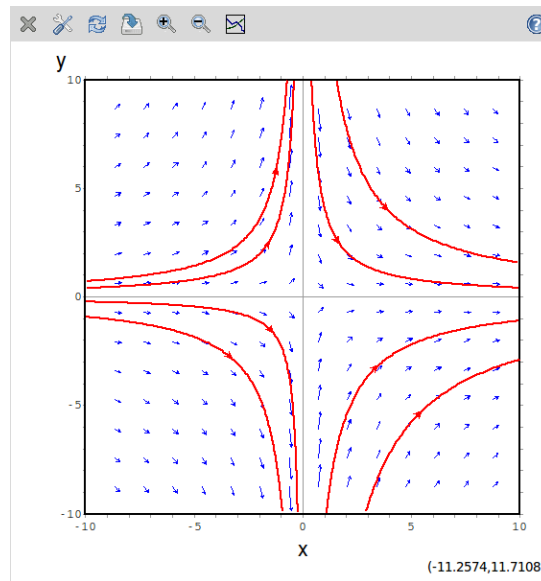
$$\frac{dy}{dx} = -\frac{x}{y}$$



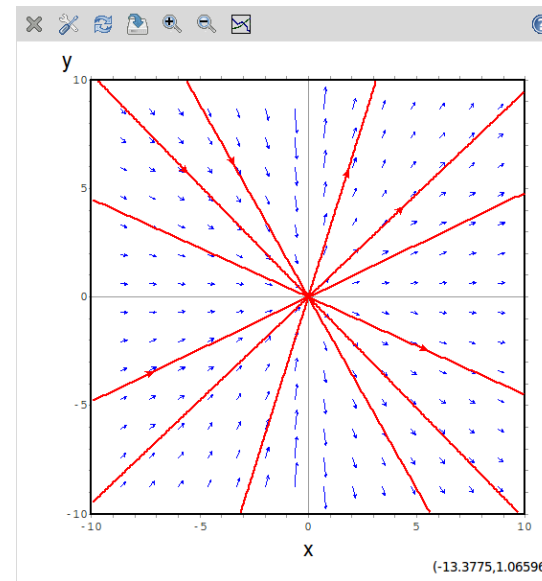
$$\frac{dy}{dx} = +\frac{x}{y}$$



$$\frac{dy}{dx} = -\frac{y}{x}$$



$$\frac{dy}{dx} = +\frac{y}{x}$$



References

- [1] <http://en.wikipedia.org/>
- [2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [3] E. Kreyszig, “Advanced Engineering Mathematics”
- [4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”