First Order ODE's (1B)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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First Order ODE examples (I)

$$\frac{dy}{dx} + y = 0$$

$$y(x) = c e^{-x};$$

$$\frac{dy}{dx} + x y = 0$$

$$y(x) = c e^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2 y = 0$$

$$y(x) = c e^{-\frac{x^3}{3}};$$

y' + y = 0

$$\frac{dy}{dx} + y = 0$$

$$y(x) = c e^{-x};$$

$$\frac{dy}{dx} + x y = 0$$

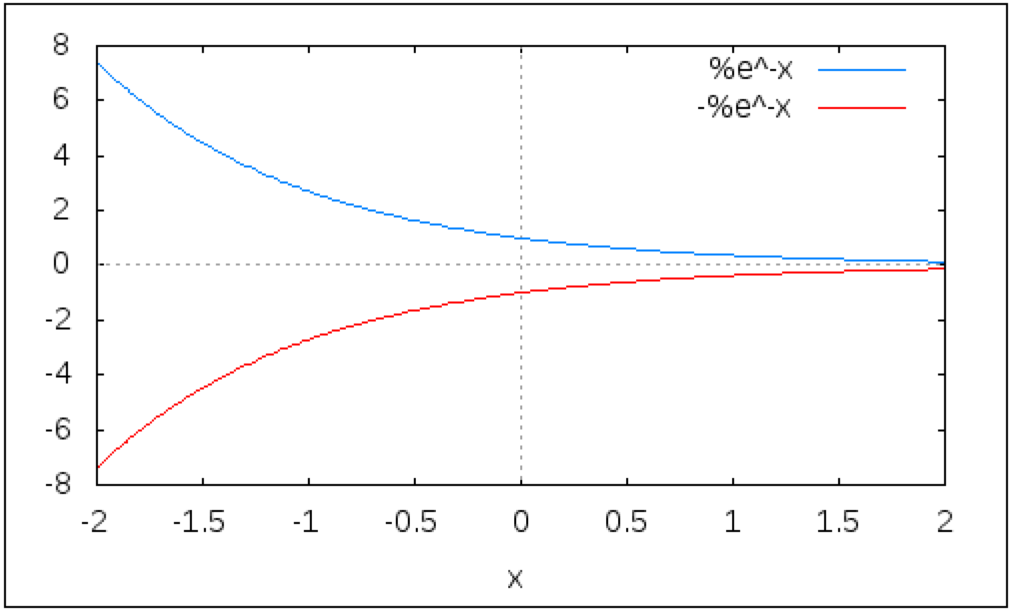
$$y(x) = c e^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2 y = 0$$

$$y(x) = c e^{-\frac{x^3}{3}};$$

```
(%i18) wxplot2d([%e^-x, -%e^-x], [x,-2,2])$
```

(%t18)



$$y = e^{-x};$$

$$y' = -e^{-x};$$

y' + xy = 0

$$\frac{dy}{dx} + y = 0$$

$$y(x) = ce^{-x};$$

$$\frac{dy}{dx} + xy = 0$$

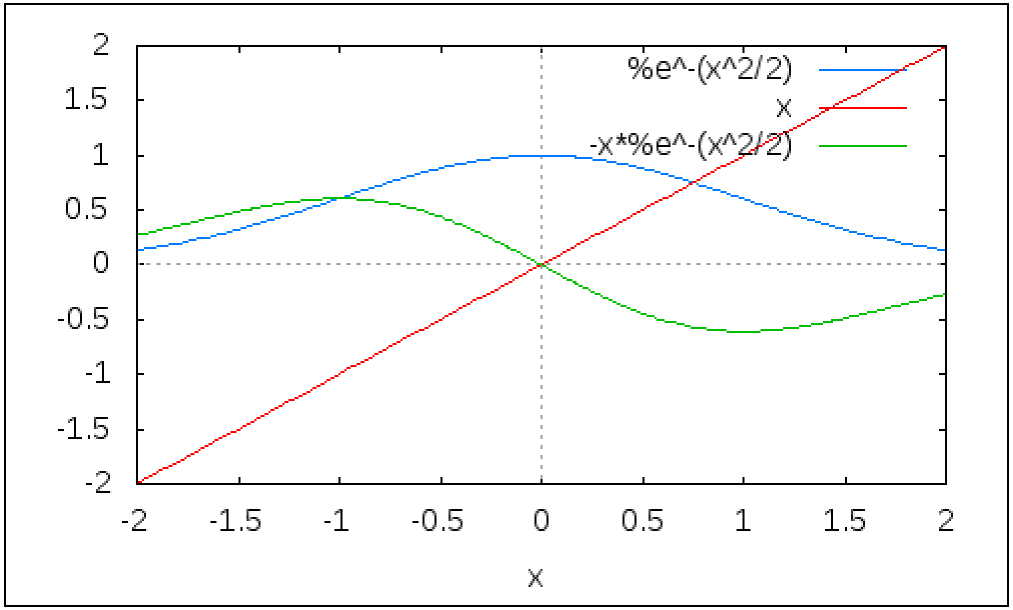
$$y(x) = ce^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2 y = 0$$

$$y(x) = ce^{-\frac{x^3}{3}};$$

```
(%i19) wxplot2d([%e^(-x^2/2),+x,-x*%e^(-x^2/2)], [x,-2,2])$
```

(%t19)



$$y = e^{-\frac{x^2}{2}}$$

$$y'(x) = e^{-\frac{x^2}{2}} \frac{d}{dx}\left(-\frac{x^2}{2}\right)$$

$$y'(x) = -xe^{-\frac{x^2}{2}}$$

$$y'(x) = -\frac{x}{e^{x^2/2}}$$

$y' + x^2y = 0$

$$\frac{dy}{dx} + y = 0$$

$$y(x) = ce^{-x};$$

$$\frac{dy}{dx} + xy = 0$$

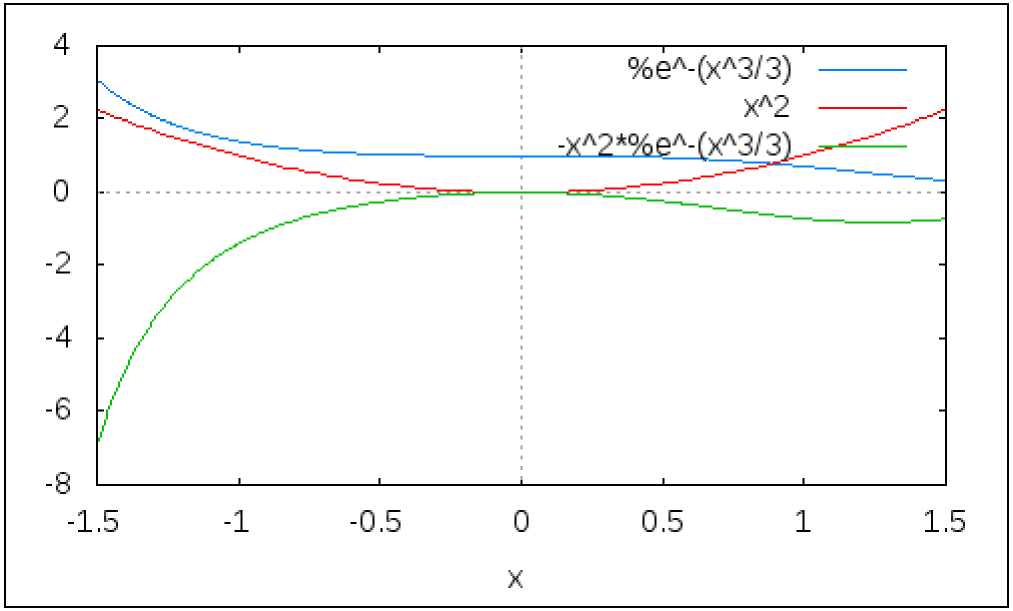
$$y(x) = ce^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2y = 0$$

$$y(x) = ce^{-\frac{x^3}{3}};$$

```
(%i28) wxplot2d([%e^(-x^3/3),+x^2,-x^2*%e^(-x^3/3)], [x,-1.5,1.5])$
```

(%t28)



$$y = e^{-\frac{x^2}{2}}$$

$$y' = e^{-\frac{x^3}{3}}\frac{d}{dx}\left(-\frac{x^3}{3}\right)$$

$$y' = -x^2e^{-\frac{x^3}{3}}$$

$$y' = -\frac{x^2}{e^{x^3/3}}$$

First Order ODE examples (II)

$$\frac{dy}{dx} + y = 1$$

$$e^x \frac{dy}{dx} + e^x y = e^x$$

$$\frac{d}{dx}\left[e^x y\right] = e^x$$

$$e^x y = \int e^x dx + c$$

$$y = 1 + c e^{-x}$$

$$\frac{dy}{dx} + x y = x$$

$$e^{x^2/2} \frac{dy}{dx} + e^{x^2/2} x y = x e^{x^2/2}$$

$$\frac{d}{dx}\left[e^{x^2/2} y\right] = x e^{x^2/2}$$

$$e^{x^2/2} y = \int x e^{x^2/2} dx + c$$

$$e^{x^2/2} y = \int \left\{\frac{d}{dx} e^{x^2/2}\right\} dx + c$$

$$e^{x^2/2} y = e^{x^2/2} + c$$

$$y = 1 + c e^{-x^2/2}$$

$$\frac{dy}{dx} + x^2 y = x^2$$

$$e^{x^3/3} \frac{dy}{dx} + e^{x^3/3} x^2 y = x^2 e^{x^3/3}$$

$$\frac{d}{dx}\left[e^{x^3/3} y\right] = x^2 e^{x^3/3}$$

$$e^{x^3/3} y = \int x^2 e^{x^3/3} dx + c$$

$$e^{x^3/3} y = \int \left\{\frac{d}{dx} e^{x^3/3}\right\} dx + c$$

$$e^{x^3/3} y = e^{x^3/3} + c$$

$$y = 1 + c e^{-x^3/3}$$

y' + y = 1

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + c\,e^{-x}$$

$$\frac{dy}{dx} + x\,y = x$$

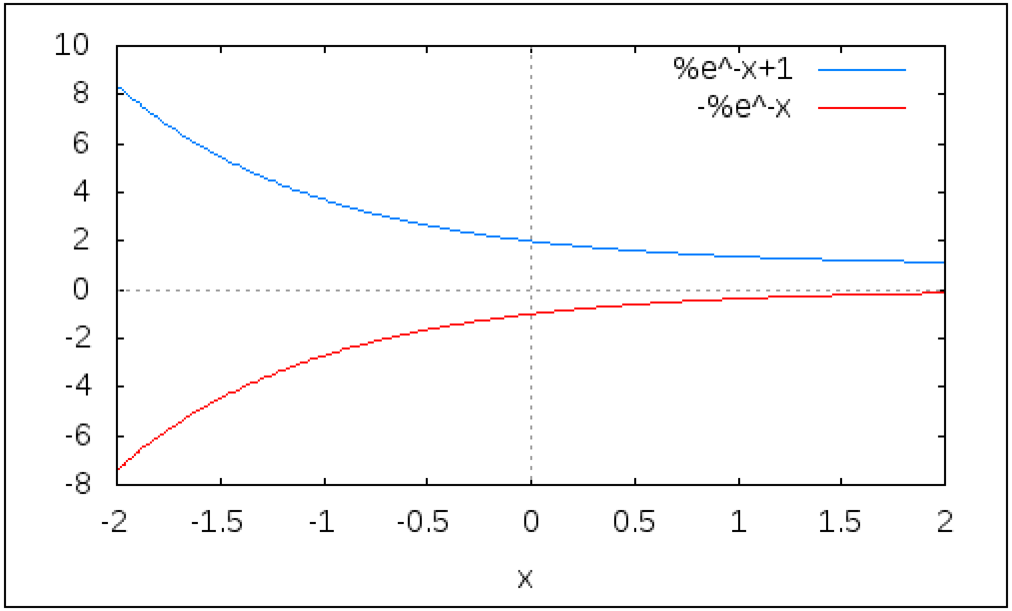
$$y = 1 + c\,e^{-x^2/2}$$

$$\frac{dy}{dx} + x^2 y = x^2$$

$$y = 1 + c\,e^{-x^3/3}$$

```
(%i30) wxplot2d([1+%e^-x, -%e^-x], [x,-2,2])$
```

(%t30)



$$y = 1 + e^{-x}$$

$$y' = -e^{-x}$$

y' + xy = x

$$\frac{dy}{dx} + y = 1$$

$$\frac{dy}{dx} + x\,y = x$$

$$\frac{dy}{dx} + x^2 y = x^2$$

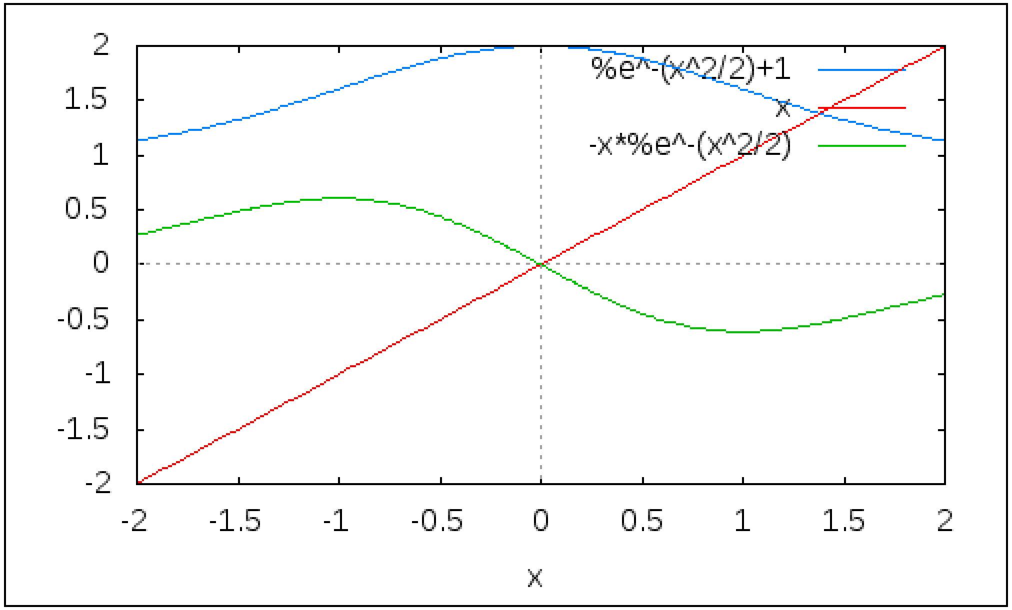
$$y = 1 + c\,e^{-x}$$

$$y = 1 + c\,e^{-x^2/2}$$

$$y = 1 + c\,e^{x^3/3}$$

```
(%i29) wxplot2d([1+%e^(-x^2/2),+x,-x*%e^(-x^2/2)], [x,-2,2])$
```

(%t29)



$$y = 1 + e^{-x^2/2}$$

$$y' = -x\,e^{-x^2/2}$$

y' + x²y = x²

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + c e^{-x}$$

$$\frac{dy}{dx} + x y = x$$

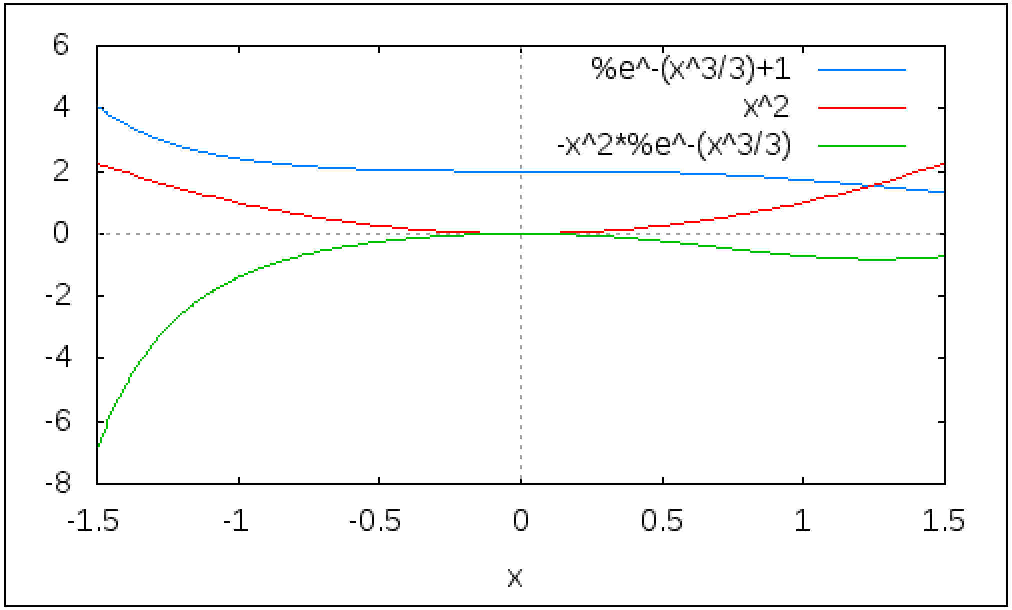
$$y = 1 + c e^{-x^2/2}$$

$$\frac{dy}{dx} + x^2 y = x^2$$

$$y = 1 + c e^{-x^3/3}$$

```
(%i31) wxplot2d([1+%e^(-x^3/3),+x^2,-x^2*%e^(-x^3/3)], [x,-1.5,1.5])$
```

(%t31)



$$y = 1 + e^{-x^3/3}$$

$$y' = -x^2 e^{-x^2/2}$$

First Order ODE examples (III)

$$\frac{dy}{dx} + y = 1$$

$$e^x \frac{dy}{dx} + e^x y = e^x$$

$$\frac{d}{dx}\left[e^x y\right] = e^x$$

$$e^x y = \int e^x dx + c$$

$$y = 1 + c e^{-x}$$

$$\frac{dy}{dx} + y = x$$

$$e^x \frac{dy}{dx} + e^x y = x e^x$$

$$\frac{d}{dx}\left[e^x y\right] = x e^x$$

$$e^x y = \int x e^x dx + c$$

$$e^x y = x e^x - e^x + c$$

$$y = (x-1) + c e^{-x}$$

$$\frac{d}{dx}\left[x e^x\right] = e^x + x e^x$$

$$x e^x = \int e^x dx + \int x e^x dx$$

$$\frac{dy}{dx} + y = x^2$$

$$e^x \frac{dy}{dx} + e^x y = x^2 e^x$$

$$\frac{d}{dx}\left[e^x y\right] = x^2 e^x$$

$$e^x y = \int x^2 e^x dx + c$$

$$e^x y = x^2 e^x - 2(x e^x - e^x) + c$$

$$y = x^2 - 2x + 2 + c e^{-x}$$

$$\frac{d}{dx}\left[x^2 e^x\right] = 2x e^x + x^2 e^x$$

$$x^2 e^x = 2\int x e^x dx + \int x^2 e^x dx$$

y' + y = 1

$$\frac{dy}{dx} + y = 1$$

$$\frac{dy}{dx} + y = x$$

$$\frac{dy}{dx} + y = x^2$$

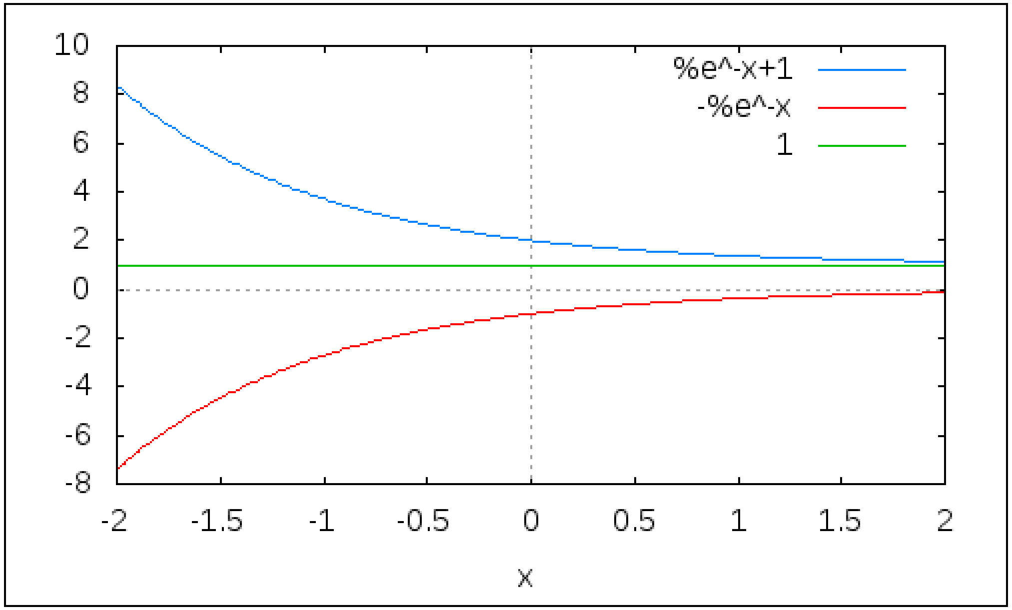
$$y = 1 + c e^{-x}$$

$$y = (x-1) + c e^{-x}$$

$$y = x^2 - 2x + 2 + c e^{-x}$$

```
(%i34) wxplot2d([1+%e^-x, -%e^-x, 1], [x,-2,2])$
```

(%t34)



$$y = 1 + e^{-x}$$

$$y' = -e^{-x}$$

y' + y = x

$$\frac{dy}{dx} + y = 1$$

$$\frac{dy}{dx} + y = x$$

$$\frac{dy}{dx} + y = x^2$$

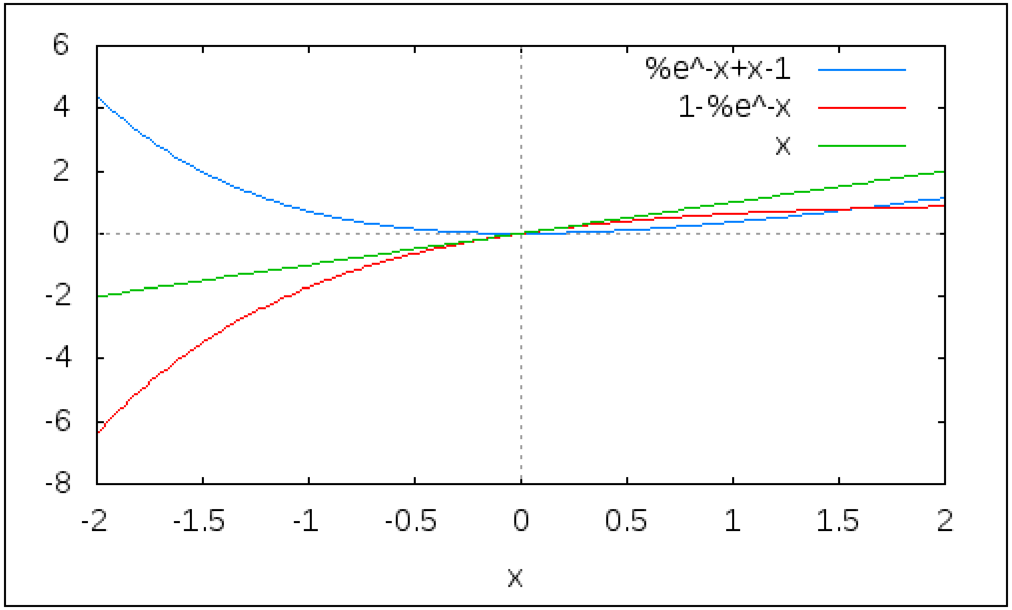
$$y = 1 + c e^{-x}$$

$$y = (x-1) + c e^{-x}$$

$$y = x^2 - 2x + 2 + c e^{-x}$$

```
(%i36) wxplot2d([x-1+%e^(-x),1 -%e^(-x), x], [x,-2,2])$
```

(%t36)



$$y = (x-1) + e^{-x}$$

$$y' = 1 - e^{-x}$$

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$y' + y = x^2$

$$\frac{dy}{dx} + y = 1$$

$$\frac{dy}{dx} + y = x$$

$$\frac{dy}{dx} + y = x^2$$

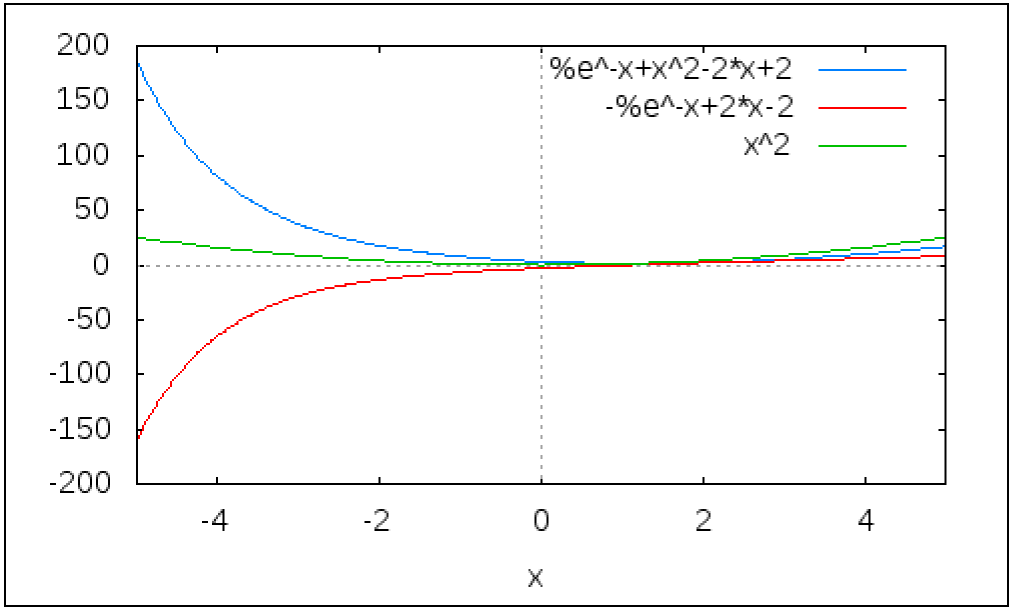
$$y = 1 + c e^{-x}$$

$$y = (x-1) + c e^{-x}$$

$$y = x^2 - 2x + 2 + c e^{-x}$$

```
(%i44) wxplot2d([x^2 -2*x +2+%e^(-x), +2*x -2 -%e^(-x), x^2 ], [x,-5, 5])$
```

(%t44)



$$y = x^2 - 2x + 2 + e^{-x}$$

$$y' = 2x - 2 - e^{-x}$$

Integration by parts

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + c_1 = (x-1)e^x + c_1$$

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx = x^2 e^x - 2x e^x + 2e^x + c_2 = (x^2 - 2x + 2)e^x + c_2$$

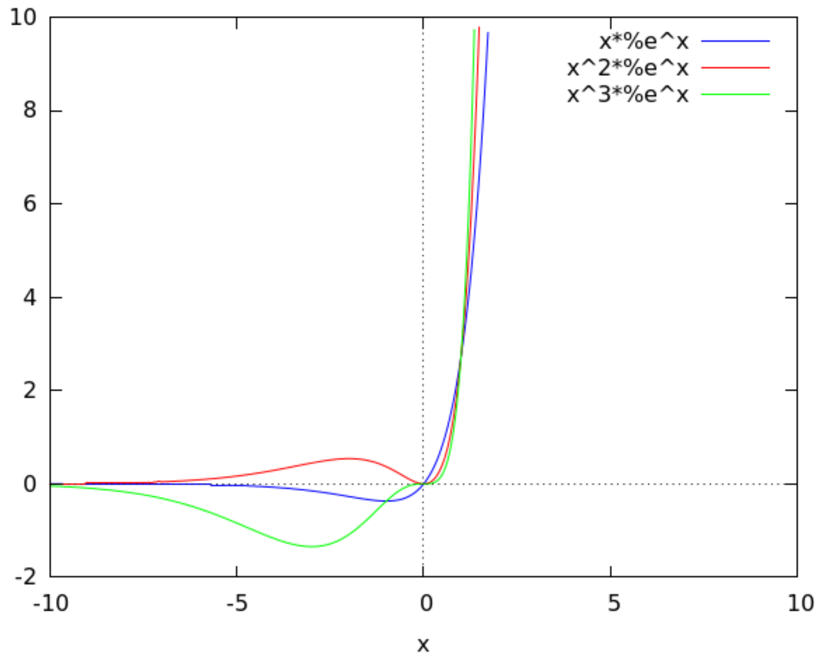
$$\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c_3 = (x^3 - 3x^2 + 6x - 6)e^x + c_3$$

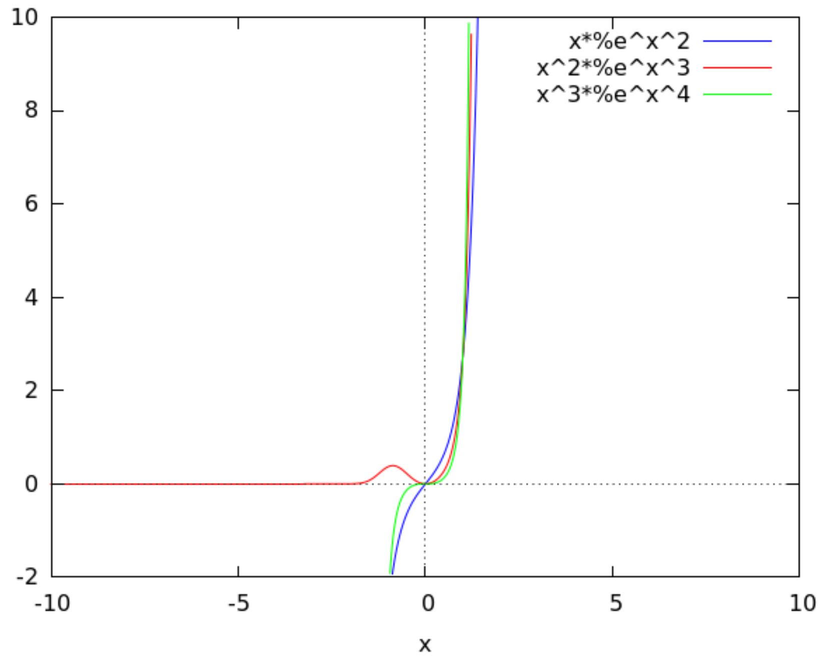
$$\int x e^x\,dx = (x-1)e^x + c_1$$

$$\int x^2 e^x\,dx = (x^2 - 2x + 2)e^x + c_2$$

$$\int x^3 e^x\,dx = (x^3 - 3x^2 + 6x - 6)e^x + c_3$$

Plots of x^m e$^{x^n}$





$$\int x\, e^{x}\, dx$$

$$\int x^2 e^{x}\, dx$$

$$\int x^3 e^{x}\, dx$$

$$\int x\, e^{x^2}\, dx$$

$$\int x^2 e^{x^3}\, dx$$

$$\int x^3 e^{x^4}\, dx$$

First Order ODE examples - Summary

$$\frac{dy}{dx} + y = 0$$

$$y(x) = c e^{-x};$$

$$\frac{dy}{dx} + x y = 0$$

$$y(x) = c e^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2 y = 0$$

$$y(x) = c e^{-\frac{x^3}{3}};$$

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + c e^{-x}$$

$$\frac{dy}{dx} + y = x$$

$$y = (x-1) + c e^{-x}$$

$$\frac{dy}{dx} + y = x^2$$

$$y = x^2 - 2x + 2 + c e^{-x}$$

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + c e^{-x}$$

$$\frac{dy}{dx} + x y = x$$

$$y = 1 + c e^{-x^2/2}$$

$$\frac{dy}{dx} + x^2 y = x^2$$

$$y = 1 + c e^{-x^3/3}$$

First Order ODE by wxMaxima (1)

wxMaxima 13.04.2 [unsaved*]

```
(%i2) ode(diff(y,x) +y = 0, y, x);... (0 lines hidden)
```

```
(%i4) ode2('diff(y,x)+y=0, y, x);
```

(%o4) $y=\%c\,\%e^{-x}$

```
(%i5) ode2('diff(y,x)+x*y=0, y, x);
```

(%o5) $y=\%c\,\%e^{-\frac{x^2}{2}}$

```
(%i6) ode2('diff(y,x)+x^2*y=0, y, x);
```

(%o6) $y=\%c\,\%e^{-\frac{x^3}{3}}$

```
(%i7) ode2('diff(y,x)+y=1, y, x);
```

(%o7) $y=\%e^{-x}\left(\%e^{x}+\%c\right)$

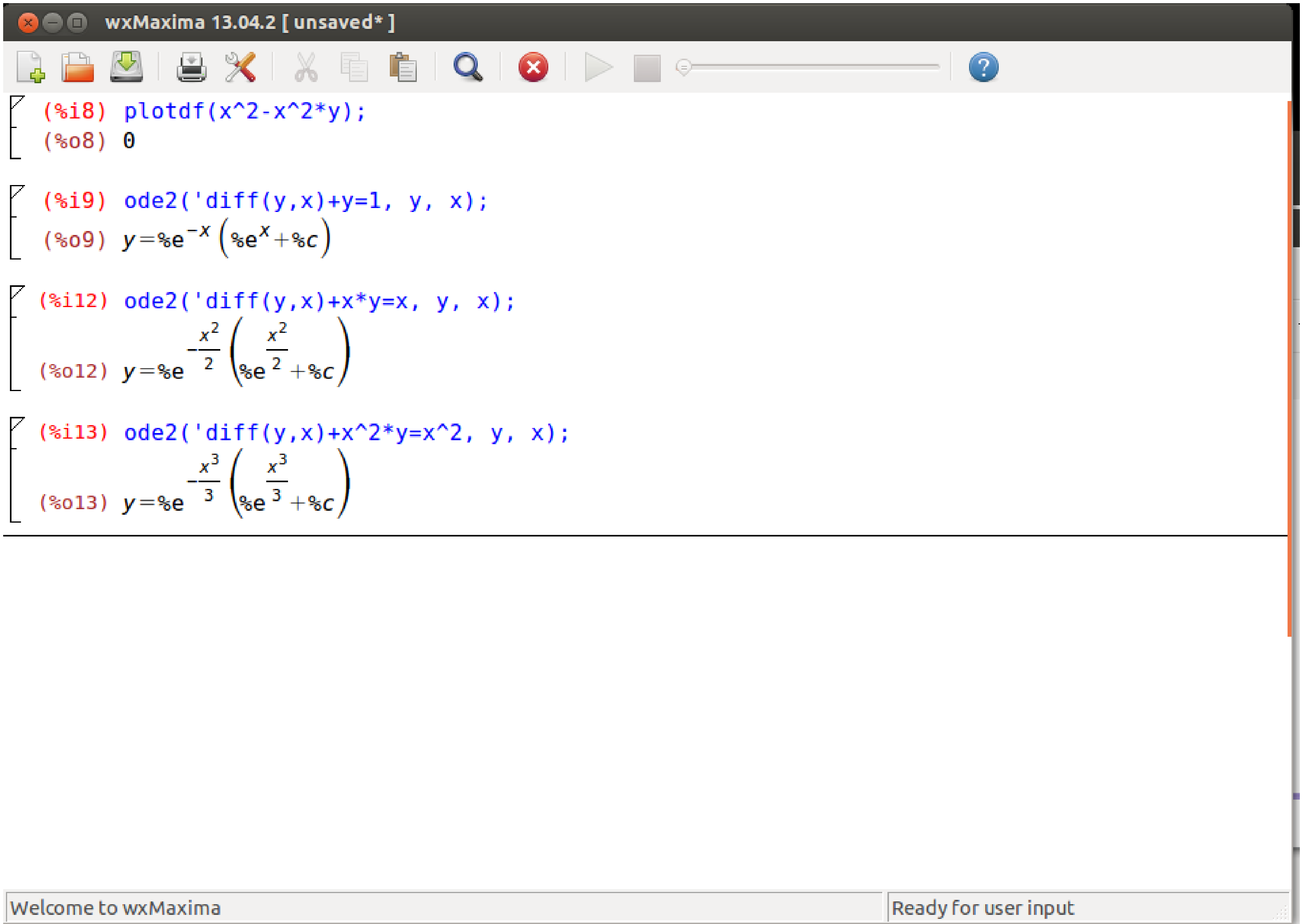
```
(%i8) ode2('diff(y,x)+y=x, y, x);
```

(%o8) $y=\%e^{-x}\left(\left(x-1\right)\%e^{x}+\%c\right)$

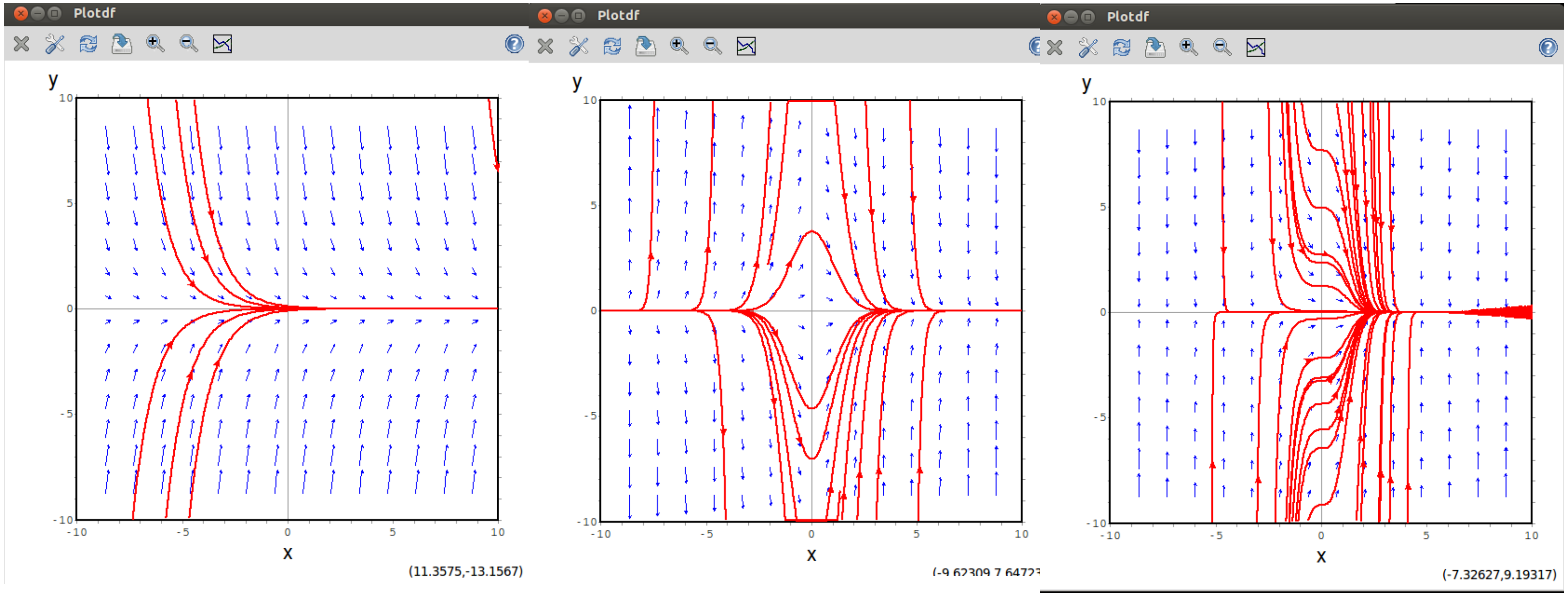
```
(%i9) ode2('diff(y,x)+y=x^2, y, x);
```

(%o9) $y=\%e^{-x}\left(\left(x^2-2\,x+2\right)\%e^{x}+\%c\right)$

First Order ODE by wxMaxima (2)



plotdf in wxMaxima (1)



$$\frac{dy}{dx} + y = 0$$

$$y(x) = c e^{-x};$$

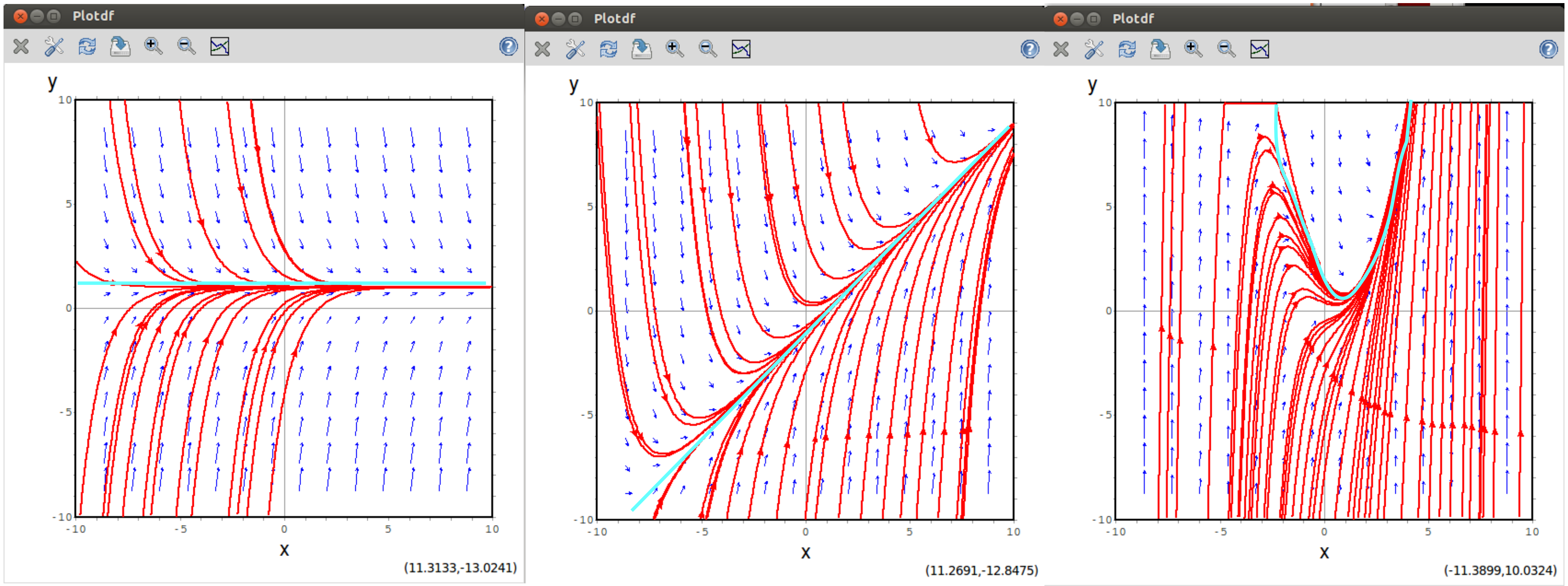
$$\frac{dy}{dx} + x y = 0$$

$$y(x) = c e^{-\frac{x^2}{2}};$$

$$\frac{dy}{dx} + x^2 y = 0$$

$$y(x) = c e^{-\frac{x^3}{3}};$$

plotdf in wxMaxima (2)



$$\frac{dy}{dx} + y = 1$$

$$y = 1 + c e^{-x}$$

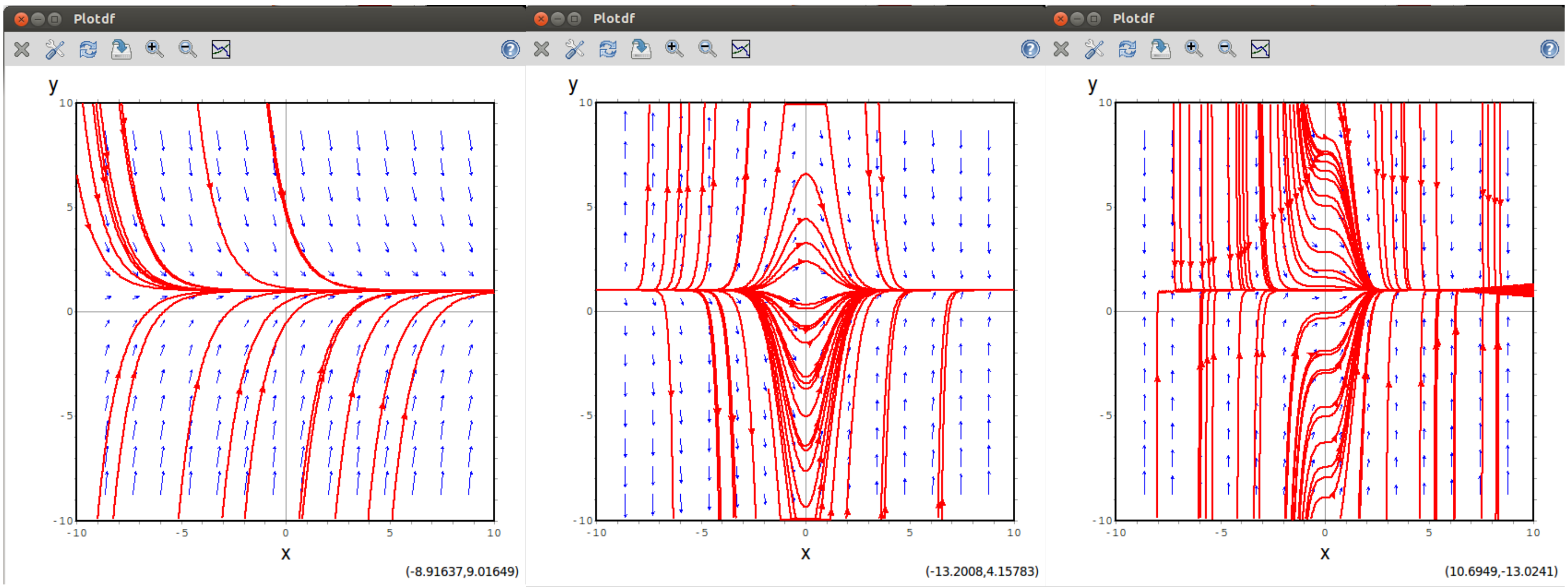
$$\frac{dy}{dx} + y = x$$

$$y = (x-1) + c e^{-x}$$

$$\frac{dy}{dx} + y = x^2$$

$$y = x^2 - 2x + 2 + c e^{-x}$$

plotdf in wxMaxima (3)



$$\frac{dy}{dx} + y = 1$$

$$y = 1 + c e^{-x}$$

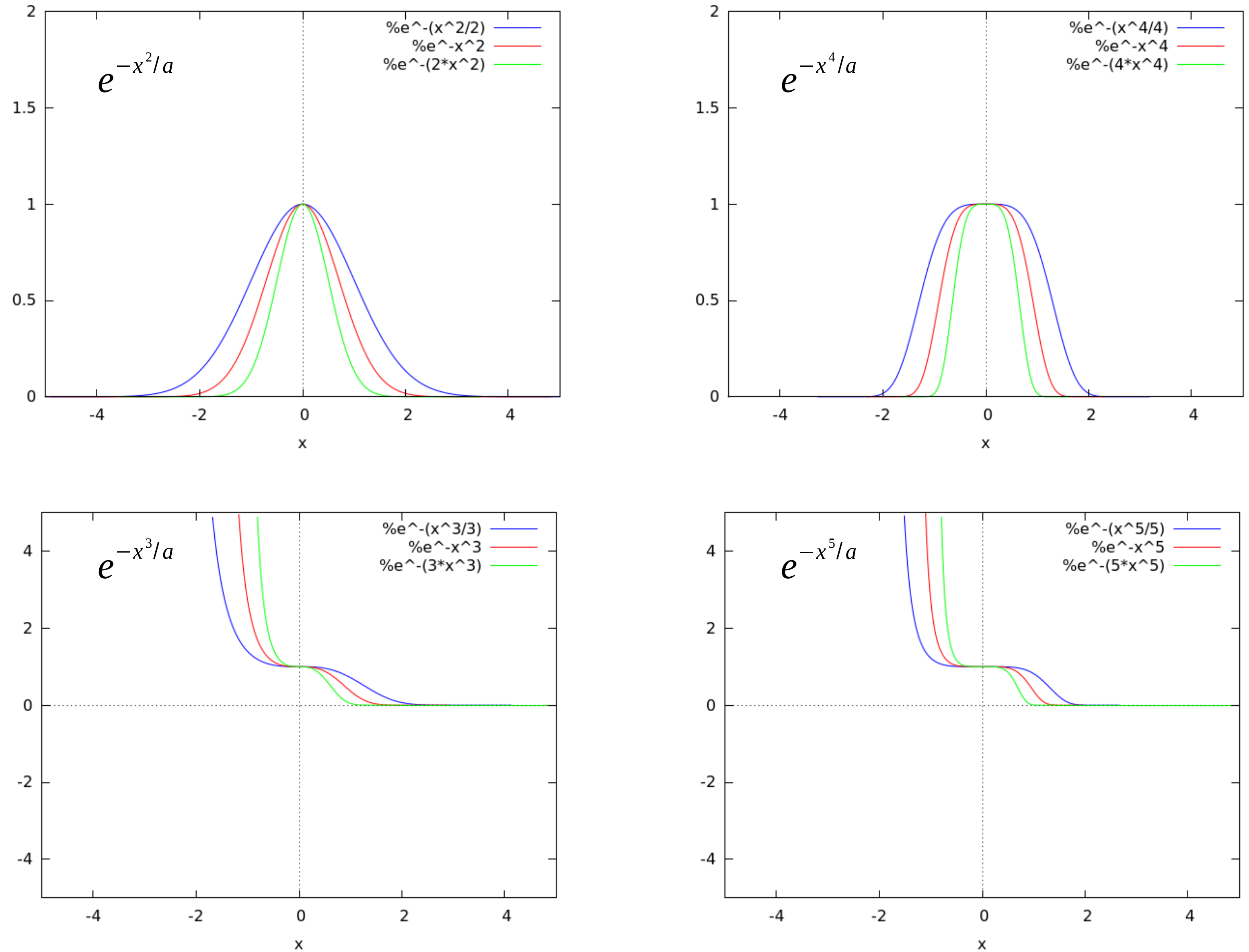
$$\frac{dy}{dx} + x y = x$$

$$y = 1 + c e^{-x^2/2}$$

$$\frac{dy}{dx} + x^2 y = x^2$$

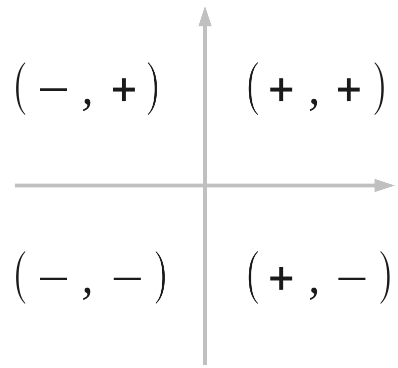
$$y = 1 + c e^{-x^3/3}$$

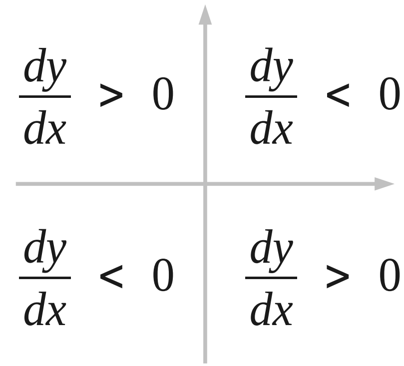
Plots of e^{-x^2}, e^{-x^3}, e^{-x^4}, e^{-x^5} functions

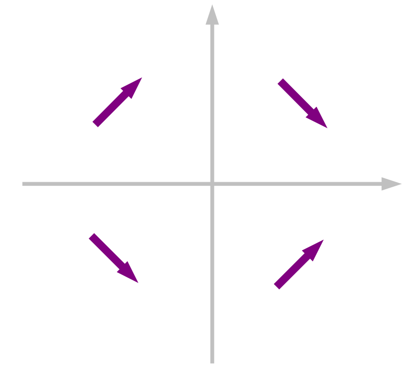


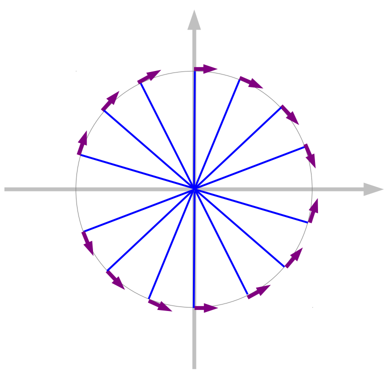
Slope of (– x/y)

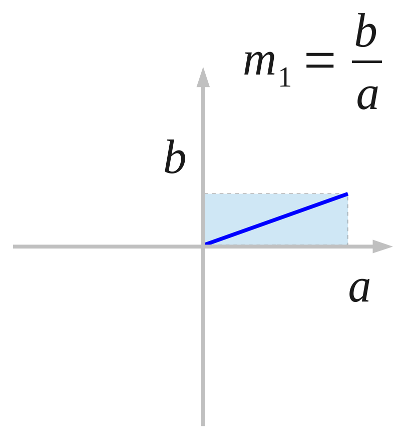
$$\frac{dy}{dx} = -\frac{x}{y}$$

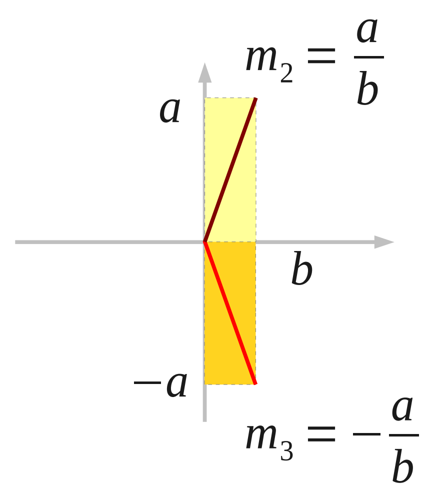


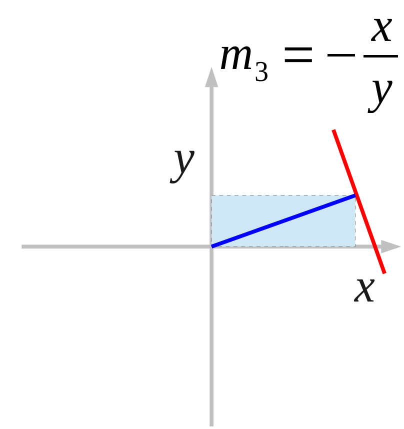










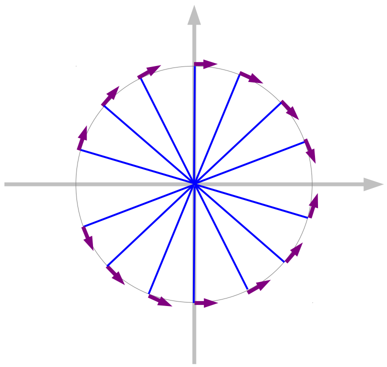


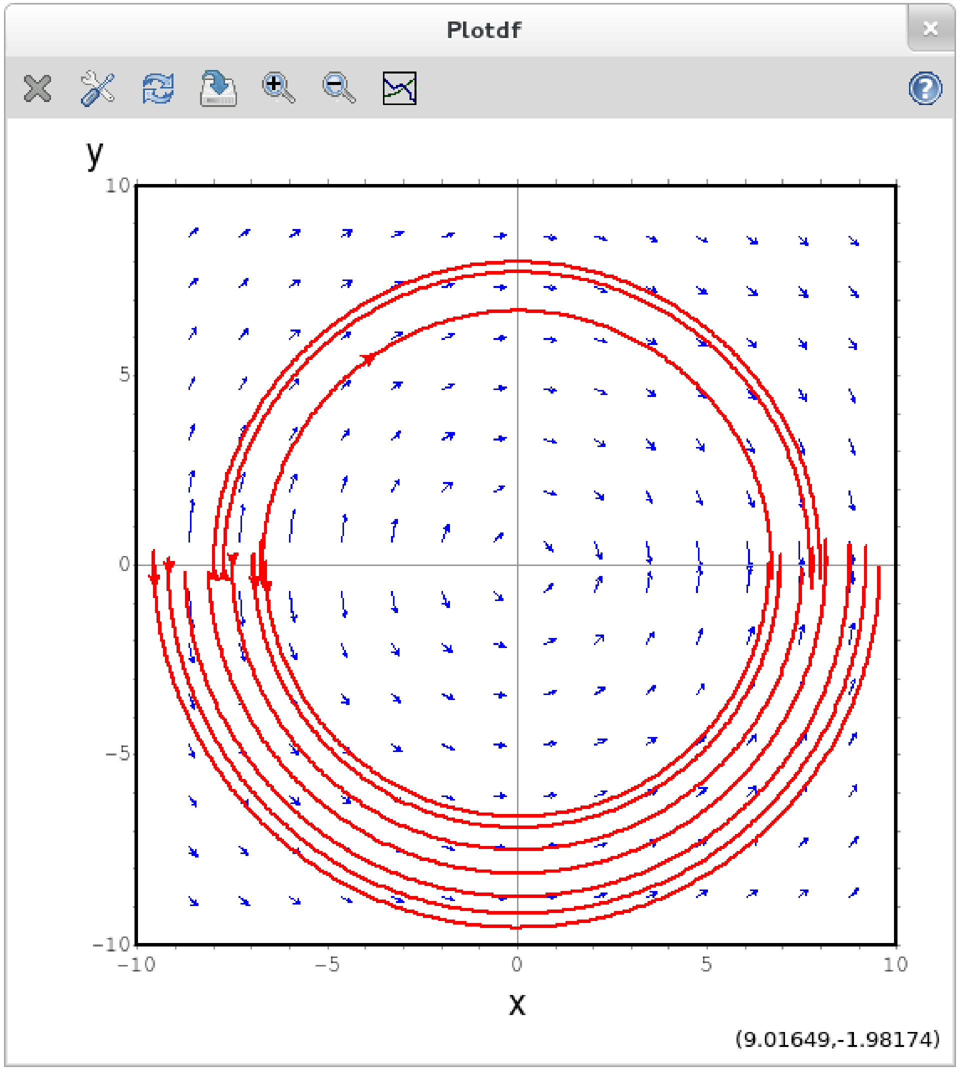
Direction Field of (– x/y)

$$\frac{dy}{dx} = -\frac{x}{y}$$

2-d version of F(x,y)

$$F(x, y) = -\frac{x}{y}$$

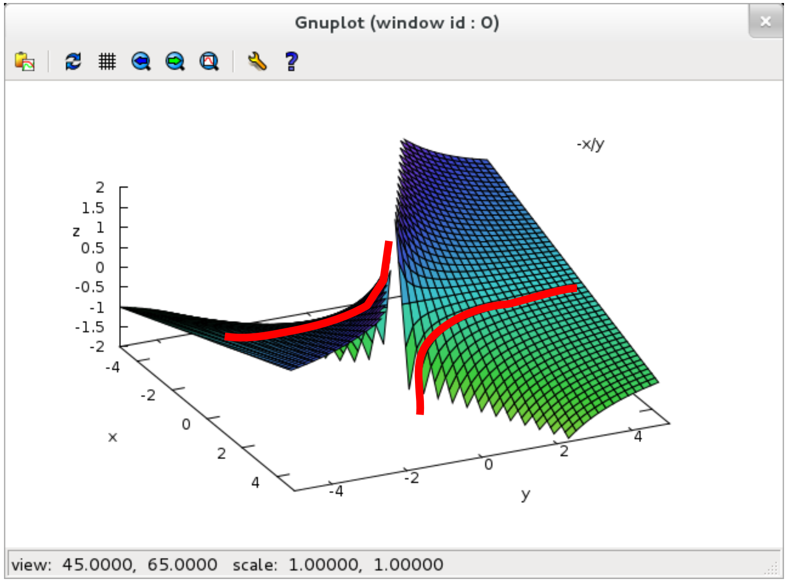


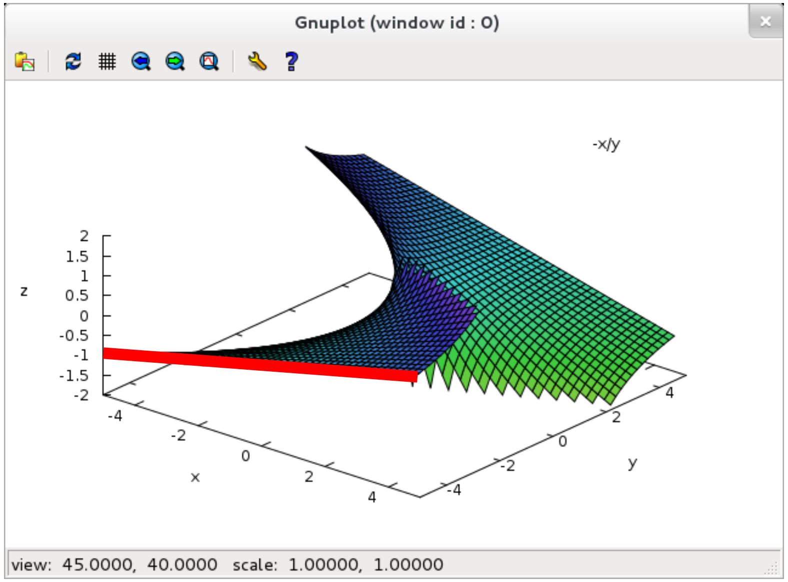


$$\frac{dy}{dx} = -\frac{x}{y}$$

$$F(x, y) = -\frac{x}{y}$$

3-d plot of F(x,y)





$$-\frac{1}{y}$$

$$x$$

Separable Equation Method

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx}dx = -\frac{x}{y}dx$$

$$dy = -\frac{x}{y}dx$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 = -x^2 + C$$

$$y = \pm\sqrt{C - x^2}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} - \frac{x}{y} = 0$$

Not Linear Equation

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Exact Equation Method (1)

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$dy = -\frac{x}{y}dx$$

$$y\,dy = -x\,dx$$

$$x\,dx + y\,dy = 0$$

$$\frac{\partial f}{\partial x} = x \qquad \frac{\partial f}{\partial y} = y$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial f}{\partial x} = x$$

$$\int \frac{\partial f}{\partial x} dx = \int x\,dx$$

$$f(x,y) = \frac{x^2}{2} + g(y)$$

$$\frac{\partial f}{\partial y} = g'(y)$$

$$\frac{\partial f}{\partial y} = y$$

$$g'(y) = y$$

$$g(y) = \frac{y^2}{2} + c_1$$

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1$$

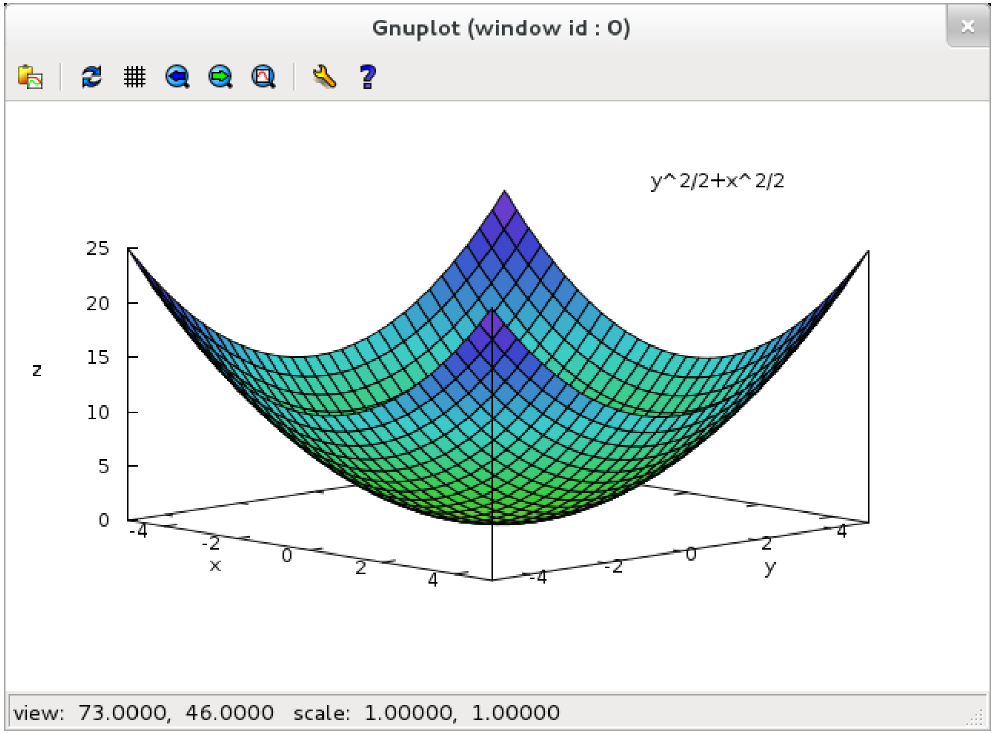
A differential form

$$P(x,y)dx + Q(x,y)dy$$

this differential form is **exact**
in a region R if there is a function
f(x,y) such that

$$P(x,y)dx + Q(x,y)dy$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$



differential form

$$x\,dx + y\,dy$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dy = df$$

exact differential form

Since there exists such

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1$$

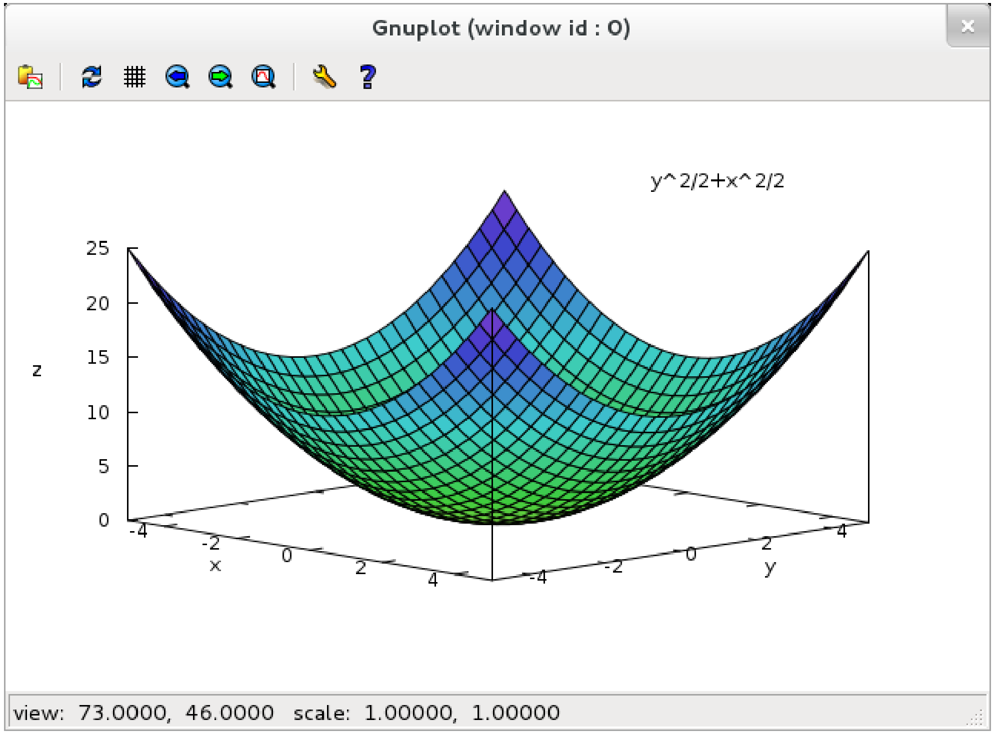
Exact Equation Method (3)

A first order **differential equation**

$$P(x,y)dx + Q(x,y)dy = 0$$

exact equation
in a region R if there is a function
f(x,y) such that

$$P(x,y)dx + Q(x,y)dy$$
$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df = 0$$



differential equation

$$x\,dx + y\,dy = 0$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dy = df = 0$$

exact differential equation

The implicit solution is

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1 = c$$

Exact Equation Method (4)

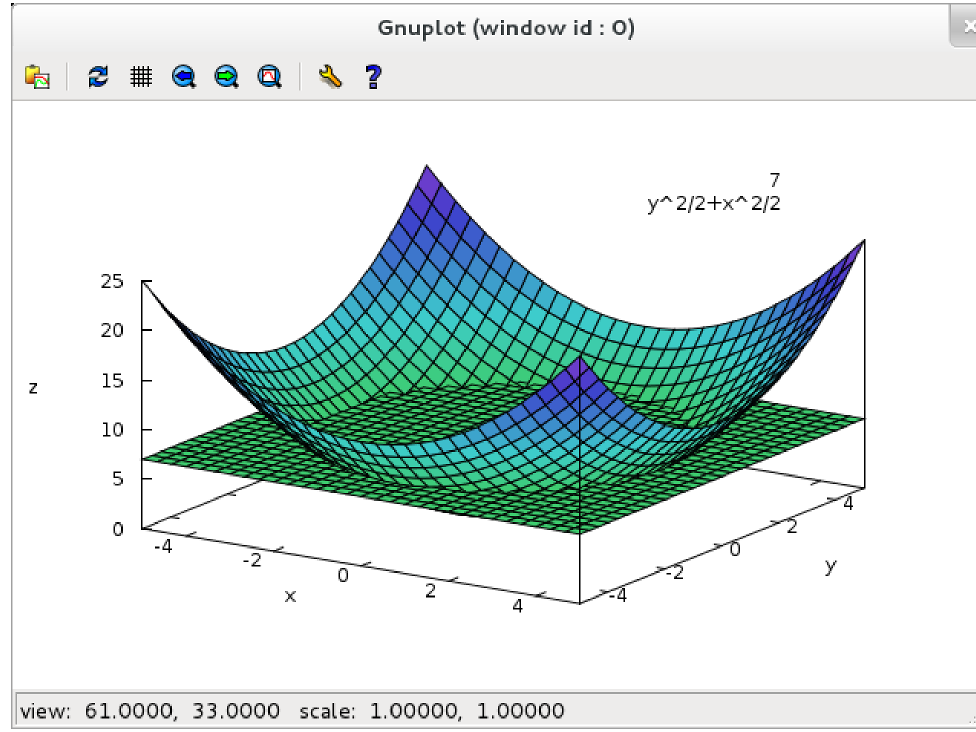
differential equation

$$x\,dx + y\,dy = 0$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dy = df = 0$$

The implicit solution is

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1 = c$$

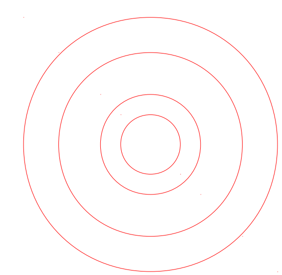


$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} = 7$$

The solution is all the points (x, y) that satisfies f(x,y) = 7

different c
In 3-d space

different c
In 2-d space



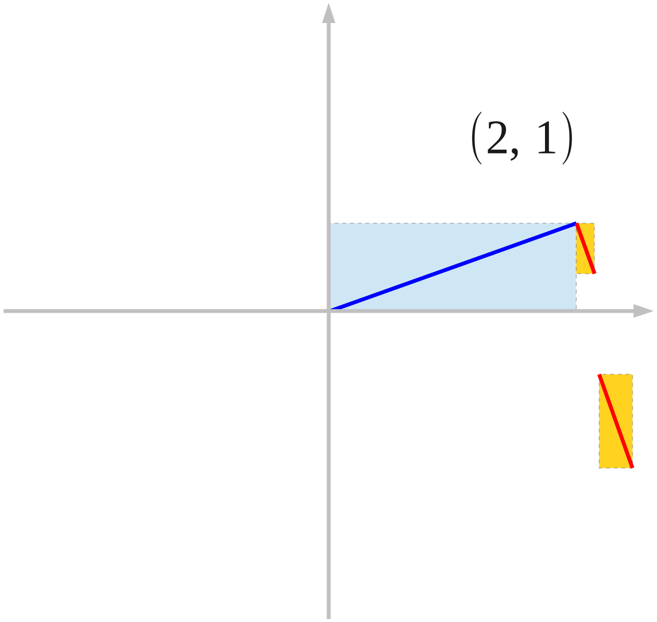
Exact Equation Method (5)

The implicit solution is

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} = C$$

$$\frac{\partial f}{\partial x} = x \qquad \frac{\partial f}{\partial y} = y$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial x} dy = 0$$

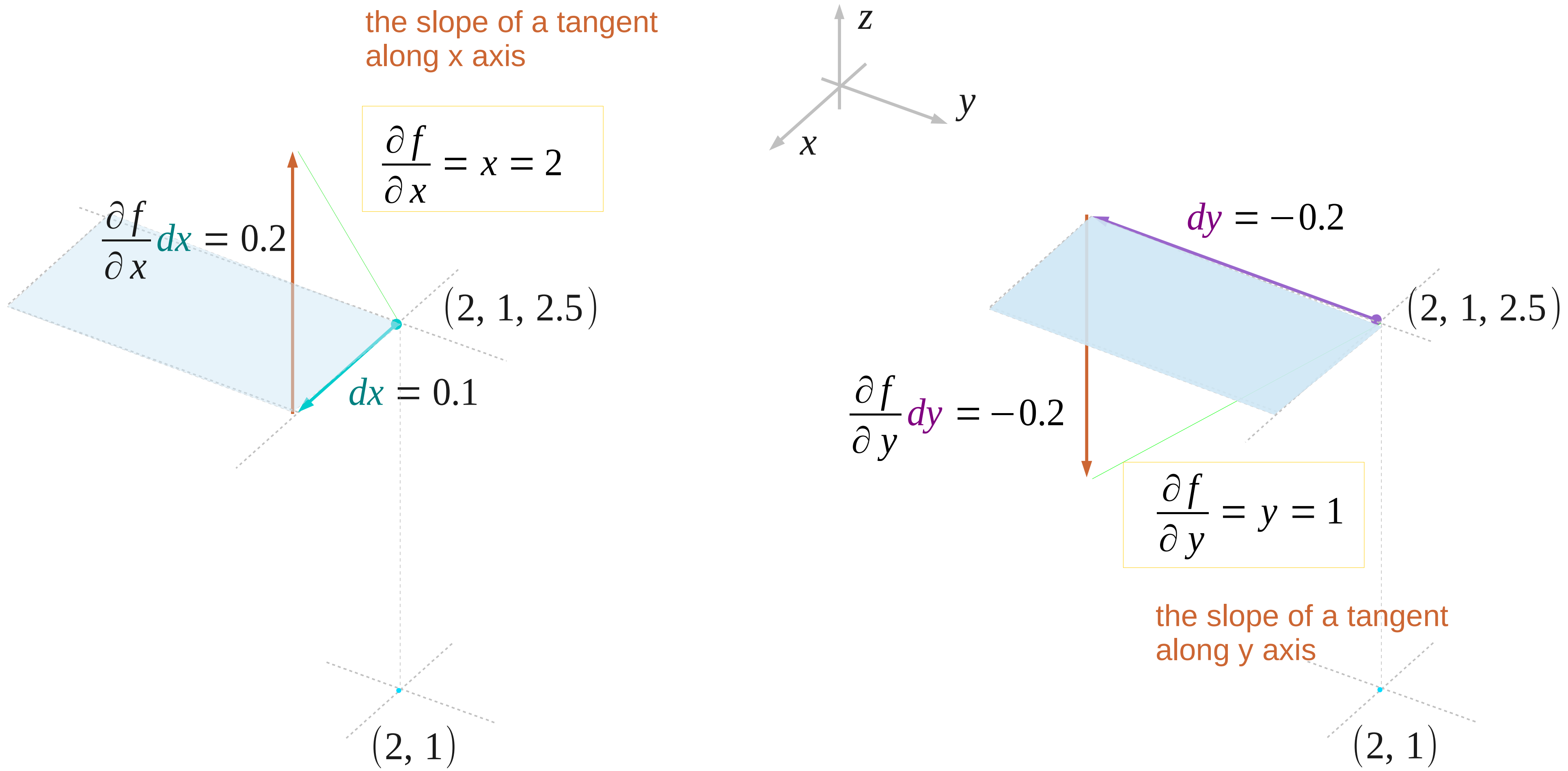


$$df = 2 \cdot dx + 1 \cdot dy = 0$$

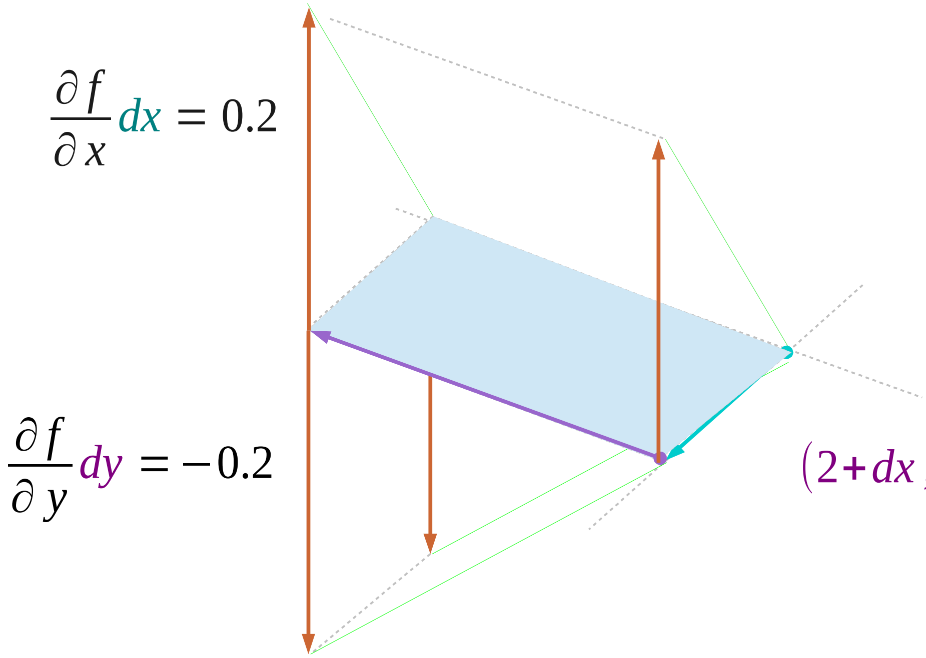
dx	dy
$+0.1$	-0.2
-0.1	$+0.2$
$+0.01$	-0.02
-0.01	$+0.02$

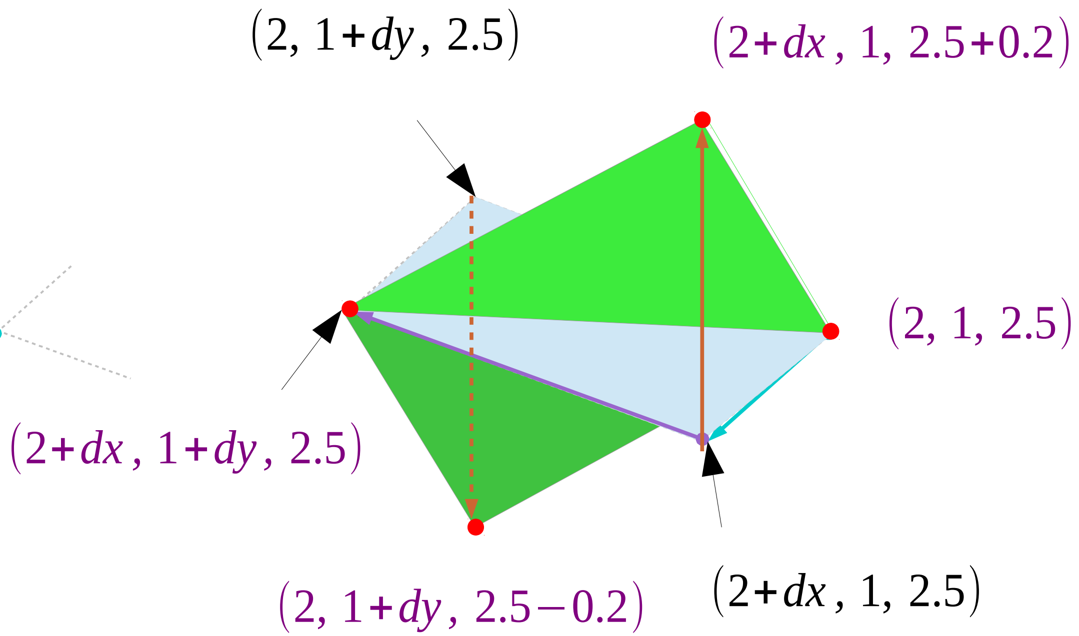
The solution is all the points (x, y)
that satisfies f(x,y) = 7

Exact Equation Method (6)



Exact Equation Method (7)





Exact Equation Method (8)

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$$

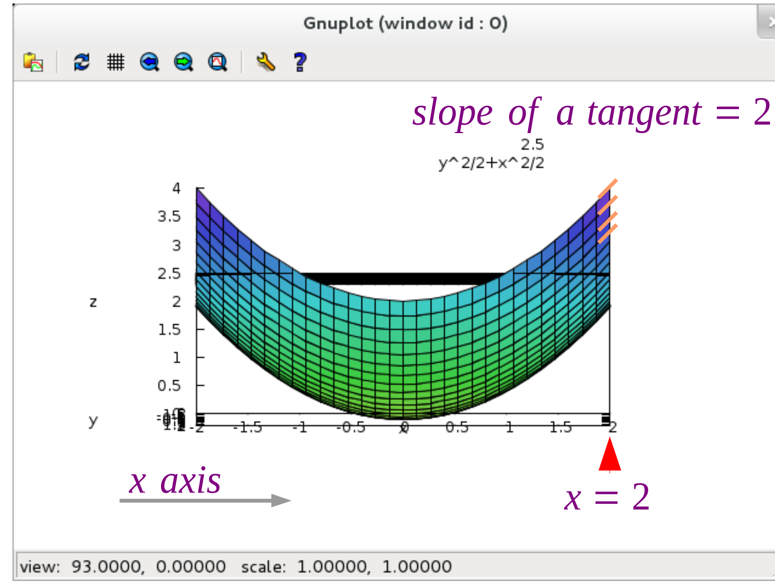
$$f(x, 1) = \frac{x^2}{2} + \frac{1}{2}$$

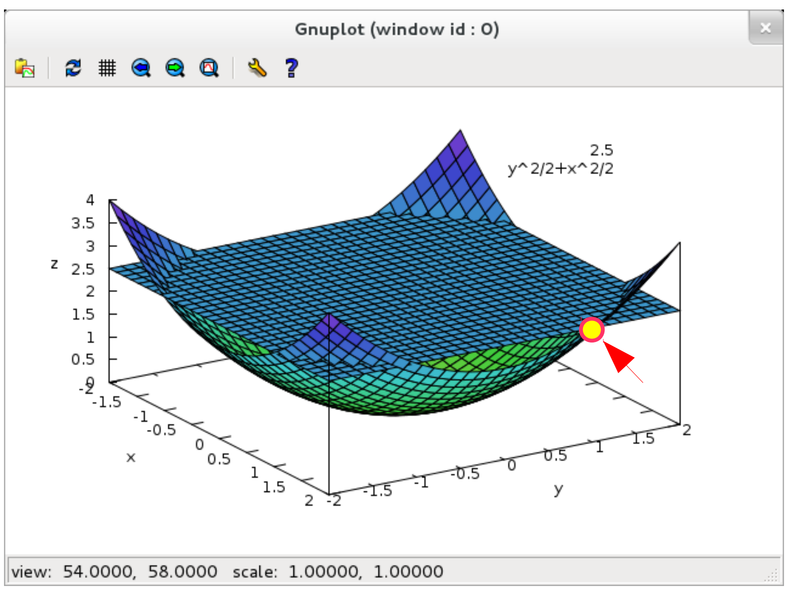
$$f(2, y) = 2 + \frac{y^2}{2}$$

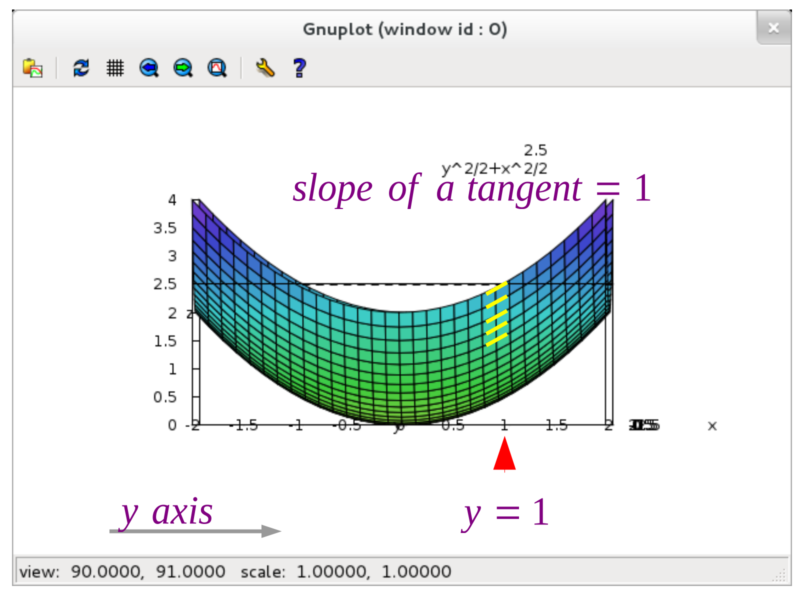
$$f(2, 1) = 2.5$$

$$\frac{\partial f}{\partial x}(2, 1) = 2$$

$$\frac{\partial f}{\partial y}(2, 1) = 1$$







Integrating Total Differentials

$$z = f(x, y)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\int df = \int \left\{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right\} + c$$

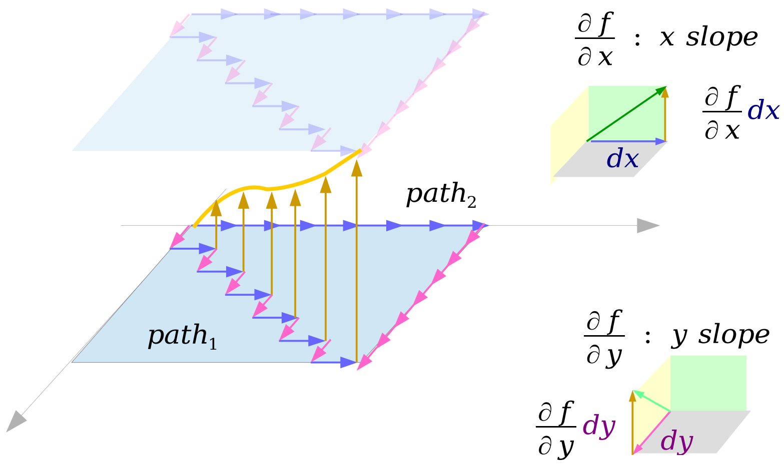
$$f = \int \left\{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right\} + c$$

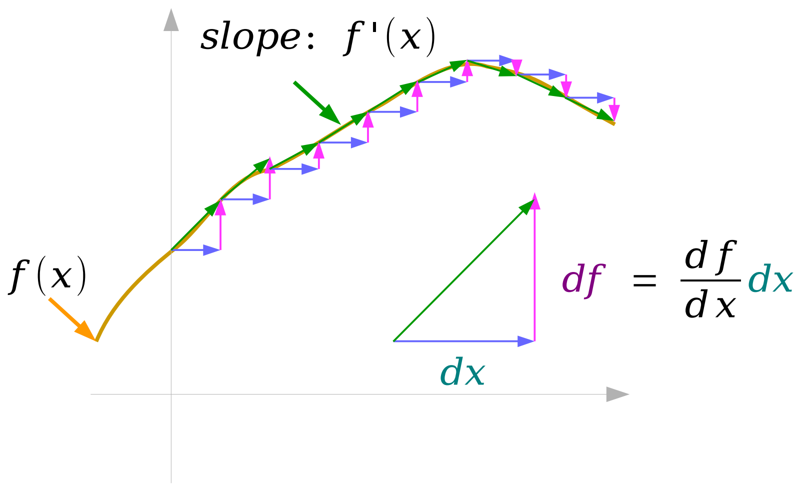
$$y = f(x)$$

$$df = \frac{df}{dx}dx$$

$$\int df = \int \frac{df}{dx}dx + c$$

$$f = \int \frac{df}{dx}dx + c$$





$df = 0 \Rightarrow f(x, y) = c$

$df = 0 \Rightarrow f(x) = c$

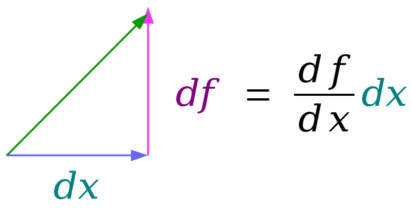
Integrating Differentials

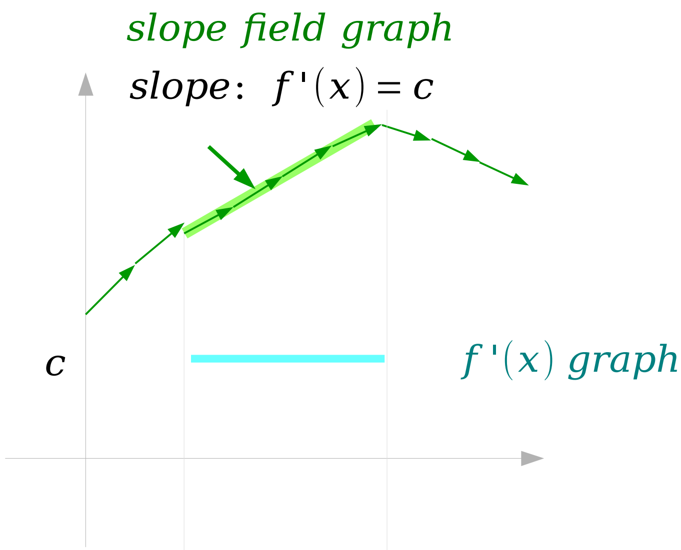
$$y = f(x)$$

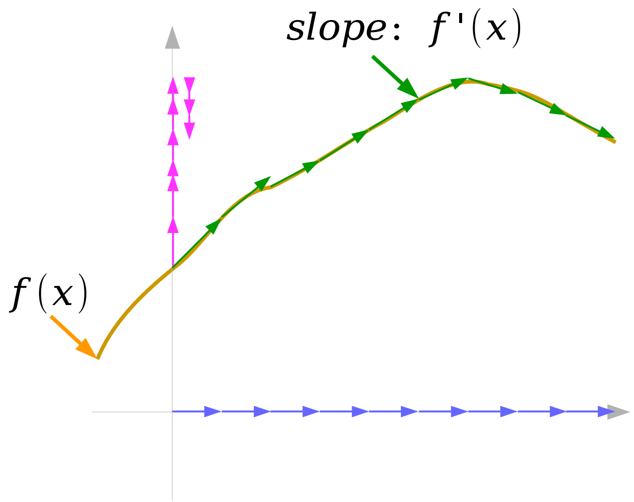
$$df = \frac{df}{dx}dx$$

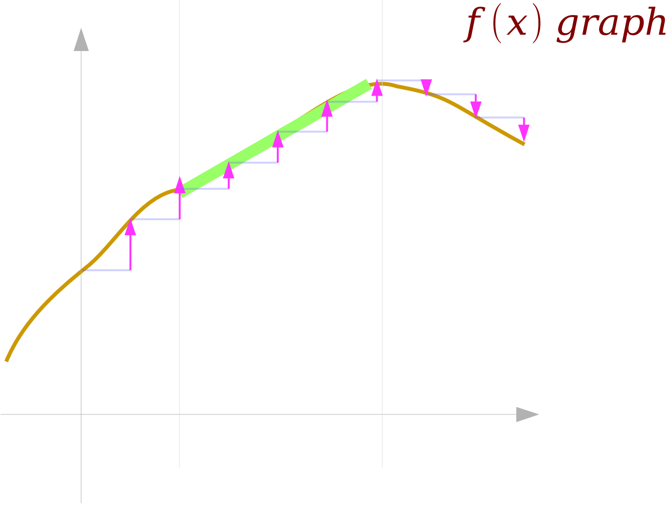
$$\int df = \int \frac{df}{dx}dx$$

$$f = \int \frac{df}{dx}dx + c$$



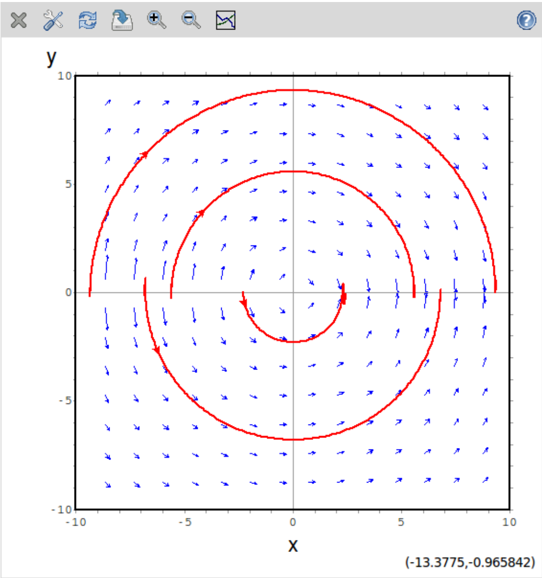


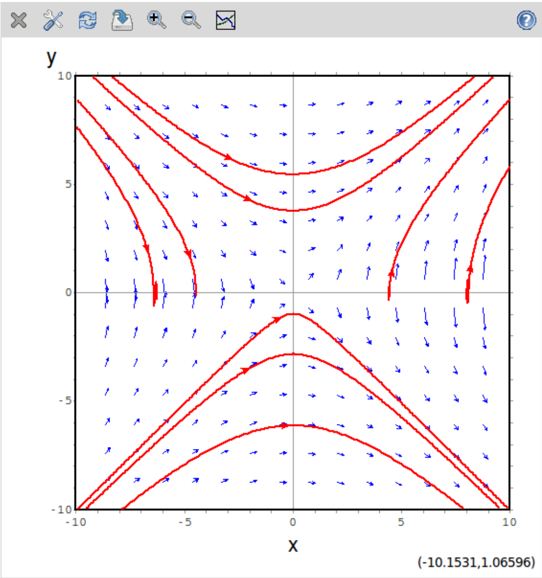




Some other direction field examples

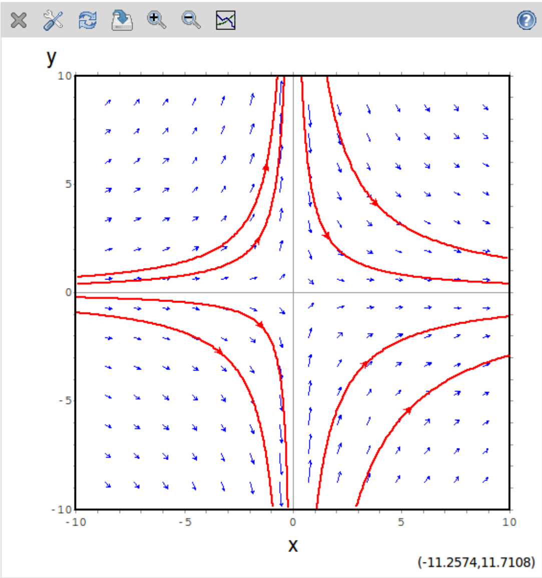
$$\frac{dy}{dx} = -\frac{x}{y}$$

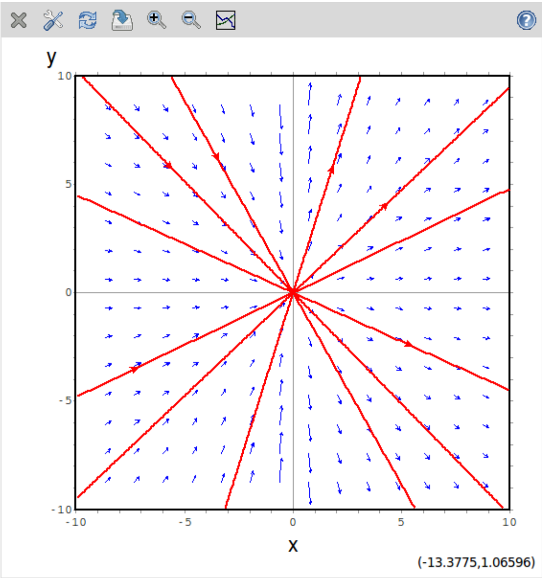




$$\frac{dy}{dx} = +\frac{x}{y}$$

$$\frac{dy}{dx} = -\frac{y}{x}$$





$$\frac{dy}{dx} = +\frac{y}{x}$$

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, "Mathematical Methods in the Physical Sciences"
[3] E. Kreyszig, "Advanced Engineering Mathematics"
[4] D. G. Zill, W. S. Wright, "Advanced Engineering Mathematics"