Series Solution (H1)
Bessel Functions

20160102

Gamma function

$$\boxed{\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt}$$

$$\boxed{\Gamma(z+1) = \int_0^\infty e^{-t} t^{z} dt}$$

$$\int f g' \, dx = fg - \int f' g \, dx$$

$$\int f' g \, dx = fg - \int f g' \, dx$$

$$\int e^{-t} t^z dt = (-e^{-t})(t^z) - \int (-e^{-t})(t^z)' dt$$

$$= -e^{-t} t^z + z \int e^{-t} t^{(z-1)} dt$$

$$\boxed{\Gamma(z+1)} = \int_0^\infty e^{-t} t^z dt$$

$$\left[-e^{-t} t^z\right]_0^\infty + z \int_0^\infty e^{-t} t^{(z-1)} dt$$

$$= 0 + z \boxed{\Gamma(z)}$$

$$\boxed{\Gamma(z+1)} = z \boxed{\Gamma(z)}$$

$$\boxed{\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1}\, dt}$$

$$\boxed{\Gamma(z+1)}$$

$$= \int_0^{\infty} e^{-t}\, t^{z}\, dt$$

$$= \left[-e^{-t}\, t^{z}\right]_0^{\infty} + z \cdot \int_0^{\infty} e^{-t}\, t^{(z-1)}\, dt$$

$$= \lim_{t\to\infty}\left(-e^{-t}\, t^{z}\right) - \lim_{t\to 0}\left(-e^{-t}\, t^{z}\right) + z \cdot \int_0^{\infty} e^{-t}\, t^{(z-1)}\, dt$$

$$= \quad \Downarrow 0 \qquad \Downarrow 0 \qquad + z \cdot \int_0^{\infty} e^{-t}\, t^{(z-1)}\, dt$$

$$\boxed{\Gamma(z+1)} = z \boxed{\int_0^{\infty} e^{-t}\, t^{(z-1)}\, dt}$$

$$= z \cdot \boxed{\Gamma(z)}$$

$$e^{-t} t^z = \frac{t^z}{e^t}$$

$$= \frac{t^z}{1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^z}{z!} + \frac{t^{z+1}}{(z+1)!}}$$

$$= \frac{1}{\infty} = 0$$

$$\boxed{\Gamma(z+1) = z \cdot \Gamma(z)}$$

$$\Gamma(z) = \underline{(z-1)}\,\Gamma\underline{(z-1)}$$

p342 복습

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad x-1 > -1, \quad \underset{\sim\sim}{x > 0}$$

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt = \int_0^\infty e^{-t} dt$$

$$= \left[-e^{-t}\right]_0^\infty = -e^{-\infty} - (-e^0) = \cancel{\frac{-1}{e^\infty}}^{\,0} + 1$$

$$= 1$$

$$\begin{cases} \Gamma(1) = 1 \\ \boxed{\Gamma(x+1) = x\Gamma(x)} \end{cases}$$

$$\Gamma(2) = \Gamma(1+1) = 1\cdot\Gamma(1) = 1 \qquad = 1!$$

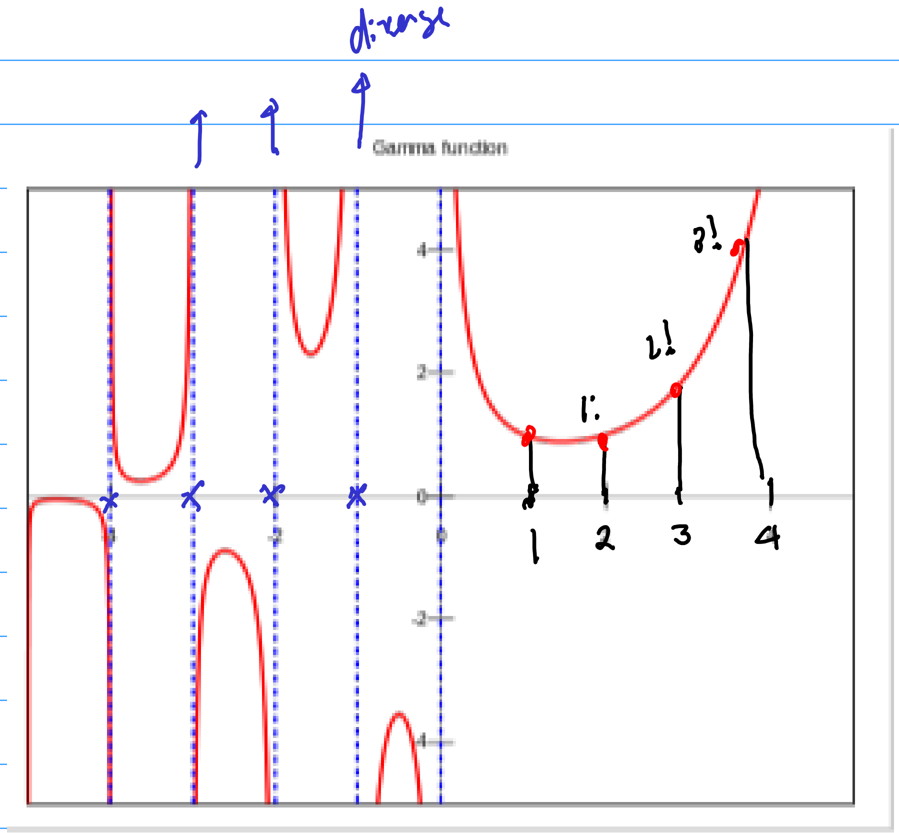
$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2\cdot 1 \qquad = 2!$$

$$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3\cdot 2\cdot 1 \qquad = 3!$$

$$\Gamma(5) \quad \Gamma(4+1) = 4\Gamma(4) = 4\cdot 3\cdot 2\cdot 1 \qquad = 4!$$

x = 양의정수 n

$$\Gamma(n+1) = n!$$



The gamma function along part of the real

$x > 0$

converge

Zill & Wright Sec 5.3 Special Functions

Bessel's Equation

order ν

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

Legendre's Equation

order n

$$(1-x^2) y'' - 2x y' + n(n+1) y = 0$$

Zill & Wright 3.6

Cauchy-Euler Equation

$$x^2 y'' + x y' - \alpha^2 y = 0 \qquad \alpha \geq 0$$

Bessel's Equation

order ν

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$\boxed{y = \sum_{n=0}^{\infty} c_n x^{n+r}}$$

$r = \pm \nu$

* When $r = +\nu$

$C_1 = C_3 = C_5 = \cdots = 0$ odd index

$$C_{2n} = \frac{-C_{2n-2}}{2^2 \cdot n \cdot (n+\nu)}$$ even index

$$C_0 \Leftarrow \frac{1}{2^\nu \Gamma(1+\nu)}$$

$$C_{2n} = \frac{(-1)^n C_0}{2^{2n}\, n!\, (1+\nu)(2+\nu)\cdots(n+\nu)} = \frac{(-1)^n}{2^{2n+\nu}\, n!\, \underline{\Gamma(1+\nu+n)}}$$

$$C_{2n-1} = 0$$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \quad (r=\nu)$$

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, \Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu} \quad (r=-\nu)$$

the assumed series solution

$$x^r$$

$$\boxed{y = \sum_{n=0}^{\infty} c_n x^{n+r}} = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + c_3 x^{r+3} + \cdots$$

$$y' = c_0 r x^{r-1} + c_1 (r+1) x^r + c_2 (r+2) x^{r+1} + \cdots$$

$$y'' = c_0 r(r-1) x^{r-2} + c_1 (r+1)(r) x^{r-1} + c_2 (r+2)(r+1) x^r + \cdots$$

$$y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$x^2 \left(\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} \right) +$$

$$x \left(\sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \right) +$$

$$(x^2 - \nu^2) \left(\sum_{n=0}^{\infty} c_n x^{n+r} \right) = 0$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$x^2\left(\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}\right) +$$

$$x\left(\sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}\right) +$$

$$(x^2-\nu^2)\left(\sum_{n=0}^{\infty} c_n x^{n+r}\right) = 0$$

$$\left(\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r}\right) +$$

$$\left(\sum_{n=0}^{\infty} c_n (n+r) x^{n+r}\right) +$$

$$\left(\sum_{n=0}^{\infty} c_n x^{n+r+2}\right) - \nu^2\left(\sum_{n=0}^{\infty} c_n x^{n+r}\right) = 0$$

Differential Eq.

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \qquad y(x) = ?$$

Suppose a solution $\boxed{y = \sum_{n=0}^{\infty} c_n x^{n+r}}$

substitute

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$\boxed{\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0}$$

$$\sum_{n=0}^{\infty} \left\{ \left(\dots \right) \underline{x^{n+r}} \right\} + \underline{\sum_{n=0}^{\infty} c_n x^{n+r+2}} = 0$$

$$\sum_{n=\boxed{0}}^{\infty} \left\{ \left(c_n (n+r)(n+r-1) + c_n (n+r) - \nu^2 c_n \right) x^{n+r} \right\}$$

$n=0, 1, 2, 3 \dots$

$$\sum_{n=0}^{\infty} \left\{ \left(c_n (n+r)(n+r-1) + c_n (n+r) - \nu^2 c_n \right) x^{n+r} \right\}$$

$= \{n=0일 때\} + \{n=1일 때\} + \{n=2일 때\} + \{n=3일 때\}, + \dots$

$= \{n=0일 때\} + \sum_{n=1}^{\infty} \{\dots n \dots\}$

(n=0일 때)

$$\Rightarrow \left(c_0 (0+r)(0+r-1) + c_0 (0+r) - \nu^2 c_0 \right) x^{0+r}$$

$$+ \sum_{n=1}^{\infty} \left(c_n (n+r)(n+r-1) + c_n (n+r) - \nu^2 c_n \right) x^{n+r}$$

$n=1, 2, 3, \dots$

$$\Rightarrow c_0 \left(r(r-1) + r - \nu^2 \right) x^r \quad \leftarrow n=0$$

$$+ x^r \sum_{n=1}^{\infty} \left(c_n (n+r)(n+r-1) + c_n (n+r) - \nu^2 c_n \right) x^n \quad \leftarrow n=1, 2, 3, \dots$$

$$x^r \sum_{n=1}^{\infty} c_n \left((n+r)^2 - \cancel{(n+r)} + \cancel{(n+r)} - \nu^2 \right) x^n$$

$$x^r \sum_{n=1}^{\infty} c_n \left((n+r)^2 - \nu^2 \right) x^n$$

$$+\sum_{n=0}^{\infty} c_n x^{n+r+2}$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$\sum_{n=0}^{\infty} \left\{ \left(\cdots \right) x^{n+r} \right\} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

$n=0$ $\qquad$ $n = 1, 2, 3 \cdots\cdots$

$$c_0 \left(r(r-1) + r - \nu^2 \right) x^r + x^r \sum_{n=1}^{\infty} c_n \left((n+r)^2 - \nu^2 \right) x^n$$

$$r^2 - \cancel{r} + \cancel{r} - \nu^2$$

$$c_0 \left(r^2 - \nu^2 \right) x^r + x^r \sum_{n=1}^{\infty} c_n \left((n+r)^2 - \nu^2 \right) x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$x^r \left[\underbrace{c_0 \left(r^2 - \nu^2 \right)}_{n=0} + \underbrace{\sum_{n=1}^{\infty} c_n \left((n+r)^2 - \nu^2 \right) x^n}_{n=1,2,3,\cdots} + \sum_{n=0}^{\infty} c_n x^{n+2} \right] = 0$$

$$x^r \sum_{n=1}^{\infty} c_n \left((n+r)^2 - \nu^2 \right) x$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$ Bessel's Eq

$$(x^2-\nu^2)\ \boxed{y = \sum_{n=0}^{\infty} c_n x^{n+r}}\ (x^2-\nu^2)$$

$$x\ \boxed{y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}}\ x$$

$$x^2\ \boxed{y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}}\ x^2$$

$$\boxed{\begin{aligned}&\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r} \\ &+ \sum_{n=0}^{\infty} c_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r}\end{aligned}}$$

$$\sum_{n=\boxed{0}}^{\infty} \left(c_n (n+r)(n+r-1) + c_n (n+r) - \nu^2 c_n \right) x^{n+r}$$

$$= c_0 \left(r(r-1) + r - \nu^2 \right) x^r \quad \leftarrow \quad n=0$$

$$+ x^r \sum_{n=1}^{\infty} \left(c_n (n+r)(n+r-1) + c_n (n+r) - \nu^2 c_n \right) x^n \quad \leftarrow n = 1, 2, 3, \cdots$$

$$x^2 y'' + x y' + (x^2-\nu^2) y =$$

$$c_0 (r^2 - \nu^2) x^r + x^r \sum_{n=1}^{\infty} c_n \left((n+r)^2 - \nu^2 \right) x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$= 0$$

$$x^r\left[\boxed{c_0\left(r^2-\nu^2\right)}+\sum_{n=1}^{\infty}c_n\left((n+r)^2-\nu^2\right)x^n+\sum_{n=0}^{\infty}c_n x^{n+2}\right]=0$$

$$\boxed{c_0\left(r^2-\nu^2\right)} = 0$$

x에 대한 항등식

$$x^2y''+xy'+(x^2-\nu^2)y \Rightarrow 0$$

$$c_0\left(\underline{r^2-\nu^2}\right)x^r+x^r\sum_{n=1}^{\infty}c_n\left((n+r)^2-\nu^2\right)x^n+x^r\sum_{n=0}^{\infty}c_n x^{n+2}=0$$

$$r^2=\nu^2$$

① $r=+\nu$

② $r=-\nu$

$$x^2y''+xy'+(x^2-\nu^2)y=0$$

$y(x)=?$

Suppose a solution $\boxed{y=\sum_{n=0}^{\infty}c_n x^{n+r}}$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$c_0 \left(\underline{r^2 - \nu^2}\right) x^r + x^r \sum_{n=1}^{\infty} c_n \left((n+r)^2 - \nu^2\right) x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2}$$

$\boxed{r^2 = \nu^2}$ $\quad r = \pm \nu$ 이면 $\quad 1^{st}$ term $= 0$

$$x^r \sum_{n=1}^{\infty} c_n \left((n+r)^2 - \nu^2\right) x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

case (I) $\quad r = +\nu$

$$x^{\nu} \sum_{n=1}^{\infty} c_n \left((n+\nu)^2 - \nu^2\right) x^n + x^{\nu} \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

case (II) $\quad r = -\nu$

$$x^{-\nu} \sum_{n=1}^{\infty} c_n \left((n-\nu)^2 - \nu^2\right) x^n + x^{-\nu} \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

Case (I) $r = +\nu$

$$x^{\nu}\sum_{n=1}^{\infty} c_n\left((n+\nu)^2 - \nu^2\right)x^n + x^{\nu}\sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

$$x^{\nu}\sum_{n=1}^{\infty} c_n\left(n^2+2\nu n+\cancel{\nu^2-\nu^2}\right)x^n + x^{\nu}\sum_{n=0}^{\infty} c_n x^{n+2}$$

$$x^{\nu}\left[\sum_{n=1}^{\infty} c_n n(n+2\nu)x^n + \sum_{n=0}^{\infty} c_n x^{n+2}\right]$$

$$x^{\nu}\left[(1+2\nu)c_1 x + \sum_{n=2}^{\infty} c_n n(n+2\nu)x^n + \sum_{n=0}^{\infty} c_n x^{n+2}\right]$$

$k = n-2$, $n \leftarrow k+2$ $\qquad$ $k = n$, $n \leftarrow k$

항등식

$$\sum_{k=0}^{\infty}\left[c_{k+2}(k+2)(k+2+2\nu) + c_k\right]x^{k+2}$$

$$x^{\nu}\left[\underbrace{(1+2\nu)c_1}_{=0} x + \sum_{k=0}^{\infty}\underbrace{\left[c_{k+2}(k+2)(k+2+2\nu) + c_k\right]}_{=0}x^{k+2}\right] = 0$$

$$\begin{cases} (1+2\nu)c_1 = 0 \longrightarrow c_1 = 0 \\ c_{k+2}(k+2)(k+2+2\nu) + c_k = 0 \end{cases}$$

$$x^{\nu}\left[(1+2\nu)c_1 x + \sum_{n=2}^{\infty} c_n n(n+2\nu)x^n + \sum_{n=0}^{\infty} c_n x^{n+2}\right]$$

$$\begin{aligned}
n=2 &\Rightarrow c_2\, 2(2+2\nu)x^2 & n=0 &\Rightarrow c_0 x^{0+2} \\
n=3 &\Rightarrow c_3\, 3(3+2\nu)x^3 & n=1 &\Rightarrow c_1 x^{1+2} \\
n=4 &\Rightarrow c_4\, 4(4+2\nu)x^4 & n=2 &\Rightarrow c_2 x^{2+2} \\
&\vdots & &\vdots
\end{aligned}$$

$$\begin{aligned}
k=0 &\quad \left(c_{0+2}(0+2)(0+2+2\nu) + c_0\right)x^{0+2} \\
k=1 &\quad \left(c_{1+2}(1+2)(1+2+2\nu) + c_1\right)x^{1+2} \\
k=2 &\quad \left(c_{2+2}(2+2)(2+2+2\nu) + c_2\right)x^{2+2} \\
&\quad \vdots
\end{aligned}$$

$k = n-2$, $n \leftarrow k+2$

$k = n$, $n \leftarrow k$

$$\sum_{k=0}^{\infty}\left[c_{k+2}(k+2)(k+2+2\nu) + c_k\right]x^{k+2}$$

$$\sum_{n=2}^{\infty} c_n n(n+2\nu)x^n + \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$C_2\, 2(2+2\nu)\, x^2 \qquad\qquad C_2\, x^2$$

$$C_3\, 3(3+2\nu)\, x^3 \qquad\qquad C_3\, x^3$$

$$C_4\, 4(4+2\nu)\, x^4 \qquad\qquad C_4\, x^4$$

$$C_5\, 5(5+2\nu)\, x^5 \qquad\qquad C_5\, x^5$$

$$C_{0+2}\,(0+2)\,(0+2+2\nu)\,x^{0+2} \qquad\qquad C_{0+2}\,x^{0+2}$$

$$C_{1+2}\,(1+2)\,(1+2+2\nu)\,x^{1+2} \qquad\qquad C_{1+2}\,x^{1+2}$$

$$C_{2+2}\,(2+2)\,(2+2+2\nu)\,x^{2+2} \qquad\qquad C_{2+2}\,x^{2+2}$$

$$C_{3+2}\,(3+2)\,(3+2+2\nu)\,x^{3+2} \qquad\qquad C_{3+2}\,x^{3+2}$$

$k = n-2$ $\qquad\qquad$ $k = n$

$n \leftarrow k+2$ $\qquad\qquad$ $n \leftarrow k$

$$\sum_{k=0}^{\infty}\left[C_{k+2}\,(k+2)\,(k+2+2\nu) + C_k\right]x^{k+2}$$

$$x^{\nu}\left[\underline{\underline{(1+2\nu)c_1}}\,x+\sum_{k=0}^{\infty}\left[\underline{\underline{c_{k+2}(k+2)(k+2+2\nu)+c_k}}\right]x^{k+2}\right]=0$$

$$\begin{cases}\underline{\underline{(1+2\nu)c_1}}=0\\[2ex] c_{k+2}=\dfrac{-c_k}{(k+2)(k+2+2\nu)} & k=0,1,2,\ldots\end{cases}$$

$\boxed{c_1=0}\Rightarrow\boxed{c_3=c_5=c_7=\cdots=0}$ odd index

$$c_3=\frac{-\cancel{c_1}^{\,0}}{(1+2)(1+2+2\nu)}=0. \qquad k=1$$

$$c_5=\frac{-\cancel{c_3}^{\,0}}{(3+2)(3+2+2\nu)}=0 \qquad k=3$$

$\boxed{k+2=2n}$ $n=1,2,3,\ldots$ even index

$k=2n-2$

$$c_{k+2}=\frac{-c_k}{(k+2)(k+2+2\nu)}$$

$$c_{2n}=\frac{-c_{2n-2}}{2n(2n+2\nu)}=\frac{-c_{2n-2}}{2\cdot n\cdot 2\cdot(n+\nu)}=\frac{-c_{2n-2}}{2^2\cdot n\cdot(n+\nu)}$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y \Rightarrow 0$$

$$c_0 \left(\underline{r^2 - \nu^2}\right) x^r + x^r \sum_{n=1}^{\infty} c_n \left((n+r)^2 - \nu^2\right) x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2} \Rightarrow 0$$

$$* \ \boxed{r = \nu}$$

$$x^\nu \sum_{n=1}^{\infty} c_n \left(n^2 + 2\nu n + \nu^2 - \nu^2\right) x^n + x^\nu \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$x^\nu \left[\sum_{n=1}^{\infty} c_n n (n + 2\nu) x^n + c_n x^{n+2} \right]$$

$$x^\nu \left[(1+2\nu) c_1 x + \sum_{n=2}^{\infty} c_n n (n + 2\nu) x^n + \sum_{n=0}^{\infty} c_n x^{n+2} \right]$$

$$k = n-2 \qquad\qquad k = n$$

$$\sum_{k=0}^{\infty} \left[c_{k+2} (k+2)(k+2+2\nu) + c_k \right] x^{k+2}$$

$$x^\nu \left[\underline{(1+2\nu)} c_1 x + \sum_{k=0}^{\infty} \left[\underline{c_{k+2}(k+2)(k+2+2\nu) + c_k} \right] x^{k+2} \right] = 0$$

$$\underline{(1+2\nu)} c_1 = 0$$

$$c_{k+2} = \frac{-c_k}{(k+2)(k+2+2\nu)} \qquad k = 0, 1, 2, \dots$$

$$c_1 = 0 \Rightarrow c_3 = c_5 = c_7 = \dots = 0$$

$$k + 2 = 2n \qquad n = 1, 2, 3, \dots$$

$$c_{2n} = \frac{-c_{2n-2}}{2n(2n+2\nu)} = \frac{-c_{2n-2}}{2^2 n (n+\nu)}$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$\boxed{r = +\nu}$

$$C_{2n} = \frac{-C_{2n-2}}{2^2 n (n+\nu)}$$ Bessel Eq Order ν

$$C_2 = \frac{-C_{2\cdot 1-2}}{2^2 \cdot 1 \cdot (1+\nu)} = \frac{-C_0}{2^2 \cdot 1 \cdot (1+\nu)}$$ $n=1$

$$C_4 = \frac{-C_{2\cdot 2-2}}{2^2 \cdot 2 (2+\nu)} = \frac{-C_2}{2^2 \cdot 2 (2+\nu)} = \frac{+C_0}{2^4 \cdot 1 \cdot 2 \cdot (1+\nu)(2+\nu)}$$ $n=2$

$$C_6 = \frac{-C_{2\cdot 3-2}}{2^2 \cdot 3 (3+\nu)} = \frac{-C_4}{2^2 \cdot 3 (3+\nu)} = \frac{-C_0}{2^6 \cdot \underbrace{1 \cdot 2 \cdot 3}_{\text{factorial}} \, (1+\nu)(2+\nu)(3+\nu)}$$ $n=3$

$$\begin{cases} C_0 \Leftarrow \dfrac{1}{2^{\nu} \Gamma(1+\nu)} \\ C_{2n} = \dfrac{(-1)^n C_0}{2^{2n} n! (1+\nu)(2+\nu) \cdots (n+\nu)} \\ C_{2n-1} = 0 \end{cases}$$

$$\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$$

$$\Gamma(2+\nu) = (1+\nu)\Gamma(1+\nu) \qquad = (1+\nu)\Gamma(1+\nu)$$

$$\Gamma(3+\nu) = (2+\nu)\Gamma(2+\nu) \qquad = (2+\nu)(1+\nu)\Gamma(1+\nu)$$

$$\Gamma(4+\nu) = (3+\nu)\Gamma(3+\nu) \qquad = (3+\nu)(2+\nu)(1+\nu)\Gamma(1+\nu)$$

$$\Gamma(n+\nu) = (n-1+\nu)\Gamma(n-1+\nu) \qquad = (n-1+\nu)\cdots(2+\nu)(1+\nu)\Gamma(1+\nu)$$

$$\boxed{\Gamma(n+1+\nu)} = (n+\nu)\Gamma(n+\nu) \qquad = (n+\nu)\cdots(2+\nu)(1+\nu)\Gamma(1+\nu)$$

$$(n+\nu)\cdots(3+\nu)(2+\nu)\underline{(1+\nu)\Gamma(1+\nu)}$$

$$(n+\nu)\cdots(3+\nu)\underline{(2+\nu)\Gamma(2+\nu)}$$

$$(n+\nu)\cdots\underline{(3+\nu)\Gamma(3+\nu)}$$

$$\underline{(n+\nu)\Gamma(n+\nu)}$$

$$\boxed{\Gamma(n+1+\nu)}$$

$$\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$$

$$\Gamma(2+\nu) = (1+\nu)\Gamma(1+\nu) = (1+\nu)\Gamma(1+\nu)$$

$$\Gamma(3+\nu) = (2+\nu)\Gamma(2+\nu) = (2+\nu)(1+\nu)\Gamma(1+\nu)$$

$$\Gamma(4+\nu) = (3+\nu)\Gamma(3+\nu) = (3+\nu)(2+\nu)(1+\nu)\Gamma(1+\nu)$$

$$\Gamma(n+\nu) = (n-1+\nu)\Gamma(n-1+\nu) = (n-1+\nu)\cdots(2+\nu)(1+\nu)\Gamma(1+\nu)$$

$$\boxed{\Gamma(n+1+\nu)} = (n+\nu)\Gamma(n+\nu) = (n+\nu)\cdots(2+\nu)(1+\nu)\Gamma(1+\nu)$$

$$C_0 \Leftarrow \frac{1}{2^{\nu}\Gamma(1+\nu)}$$

$$C_{2n} = \frac{(-1)^n C_0}{2^{2n} n!\,(1+\nu)(2+\nu)\cdots(n+\nu)}$$

$$= \frac{(-1)^n}{2^{2n+\nu} n!\,(1+\nu)(2+\nu)\cdots(n+\nu)\Gamma(1+\nu)}$$

$$= \frac{(-1)^n}{2^{2n+\nu} n!\,\underline{\Gamma(1+\nu+n)}}$$

$$C_0 \Leftarrow \frac{1}{2^{\nu}\Gamma(1+\nu)}$$

$$C_{2n} = \frac{(-1)^n}{2^{2n+\nu} n!\,\underline{\Gamma(1+\nu+n)}}$$

$$C_{2n-1} = 0$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \qquad y(x) = ?$$

Suppose a solution $\boxed{y = \sum_{n=0}^{\infty} c_n x^{n+r}}$

Condition $\boxed{r = \nu}$

$$x^{\nu}\left[(1+2\nu) c_1 x + \sum_{k=0}^{\infty} \left[c_{k+2}(k+2)(k+2+2\nu) + c_k \right] x^{k+2} \right] = 0$$

$$\begin{cases} (1+2\nu) c_1 = 0 \\ c_{k+2} = \dfrac{-c_k}{(k+2)(k+2+2\nu)} \qquad k = 0, 1, 2, \cdots \end{cases}$$

$$\boxed{c_1 = 0} \Rightarrow \boxed{c_3 = c_5 = c_n = \cdots = 0}$$

$$c_0 \Leftarrow \frac{1}{2^{\nu}\Gamma(1+\nu)}$$

$$c_{2n} = \frac{(-1)^n}{2^{2n+\nu} n! \, \Gamma(1+\nu+n)}$$

$$(r=\nu) \quad J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$(r=-\nu) \quad J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, \Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

$$\boxed{y = \sum_{m=0}^{\infty} c_n x^{m+r}}$$

Bessel Functions of the 1st kind

$(r=\nu)$

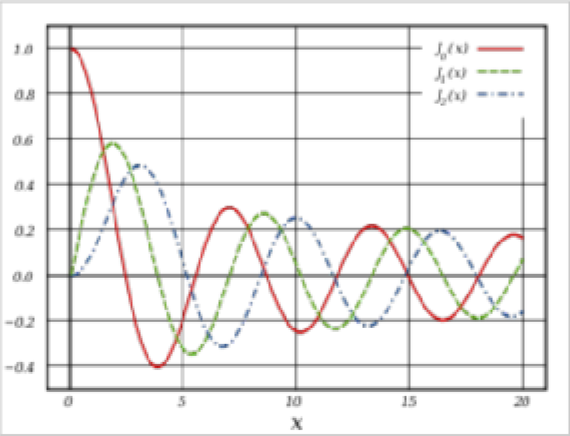
$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$(r=-\nu)$

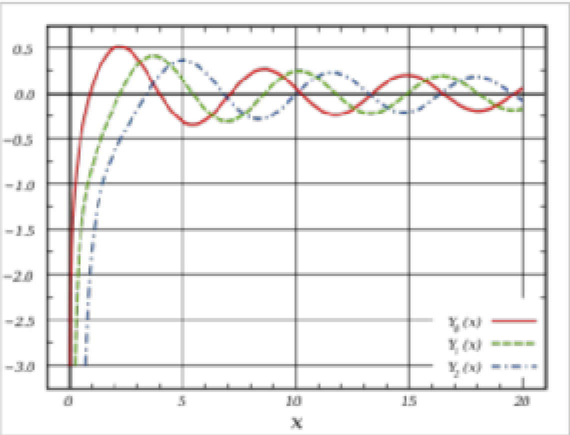
$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

Bessel Functions of the 2nd kind

$$Y_\nu(x) = \frac{\cos(\nu\pi)\,J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$



Plot of Bessel function of the first kind, $J_\alpha(x)$, for integer orders $\alpha = 0, 1, 2$



Plot of Bessel function of the second kind, $Y_\alpha(x)$, for integer orders $\alpha = 0, 1, 2$.

Bessel Functions of the First Kind

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

Series Solution $\quad y = \sum_{n=0}^{\infty} c_n x^{n+r}$

$$\begin{cases} c_0 \Leftarrow \dfrac{1}{2^{\nu}\,\Gamma(1+\nu)} \\ c_{2n} = \dfrac{(-1)^n}{2^{2n+\nu}\, n!\, \Gamma(1+\nu+n)} \\ c_{2n+1} = 0 \end{cases}$$

$$y = \sum_{n=0}^{\infty} c_{2n} x^{2n+r} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu}\, n!\, \Gamma(1+\nu+n)} x^{2n+\nu}$$

$(r=\nu)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} = J_\nu(x)$$

$(r=\nu)$ $$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$(r=-\nu)$ $$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, \Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

Linear Combination of $J_\nu(x)$ & $J_{-\nu}(x)$

$$Y_\nu(x) = \frac{\cos(\nu\pi)\,J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

if ν = integer, then $J_\nu(x)$ & $Y_\nu(x)$ linearly independent

$$y = C_1 J_\nu(x) + C_2 J_{-\nu}(x) \leftarrow \nu \neq \text{integer}$$

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x) \leftarrow \nu = \text{integer \& fraction}$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$\nu = \frac{1}{2} \neq$ integer

$$x^2 y'' + x y' + (x^2 - \tfrac{1}{4}) y = 0$$

$$y = C_1 J_{\frac{1}{2}}(x) + C_2 J_{-\frac{1}{2}}(x)$$

$\nu = 3 =$ integer

$$x^2 y'' + x y' + (x^2 - 9) y = 0$$

$$y = C_1 J_3(x) + C_2 Y_3(x)$$

p. 343

5.3.1 Bessel 함수

일반해

① $x^2 y'' + xy' + (x^2 - \frac{1}{9})y = 0$

③ $4x^2 y'' + 4xy' + (4x^2 - 25)y = 0$

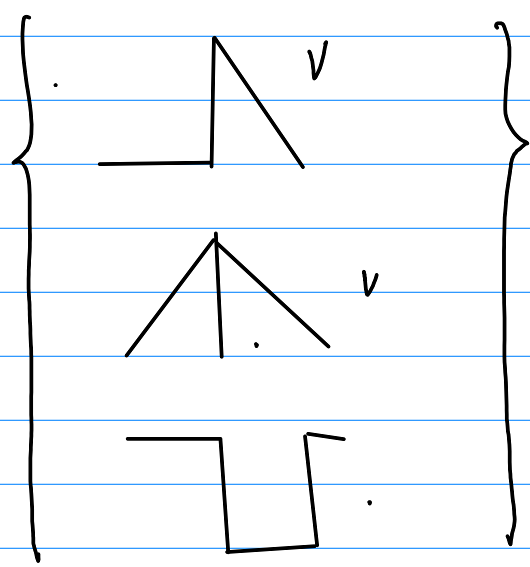
⑤ $xy'' + y' + xy = 0$

p346 5.3.2 Legendre 함수

48 (a) $P_0(x) = ?$

$P_1(x) = ?$

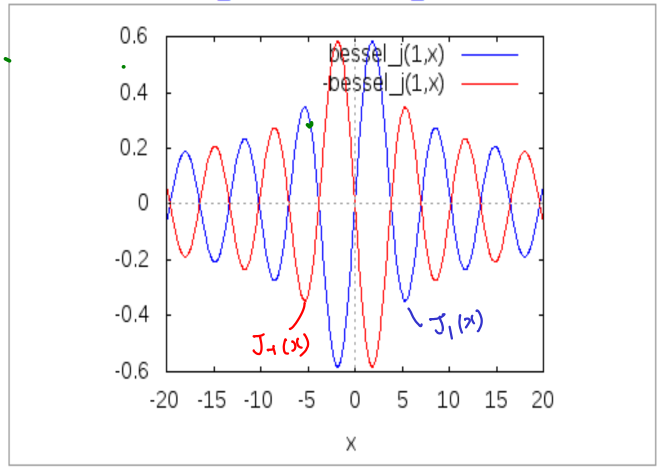
(b) $P_0(x)$와 $P_1(x)$를 particular solution으로 갖는 미분방정식은?



integer $\nu = 1, 2, 3, \cdots$ J_ν , $J_{-\nu}$: ~~linearly independent~~

--> wxplot2d([bessel_j(1,x), bessel_j(-1, x)], [x,-20,20])$

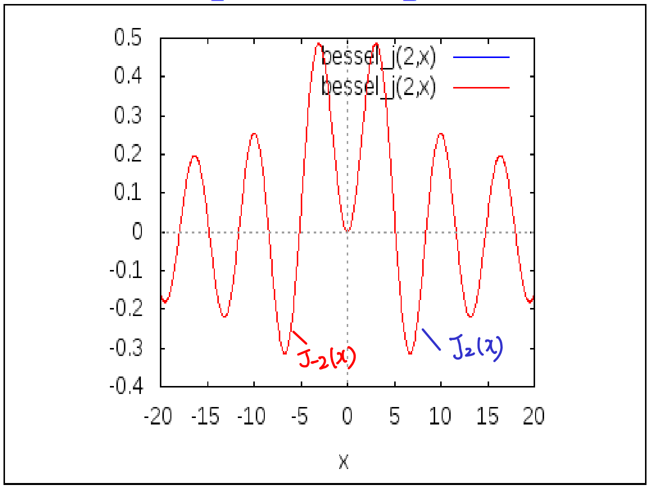
(%t15)



Odd function $J_1(x)$, $J_{-1}(x)$

(%i16) wxplot2d([bessel_j(2,x), bessel_j(-2, x)], [x,-20,20])$

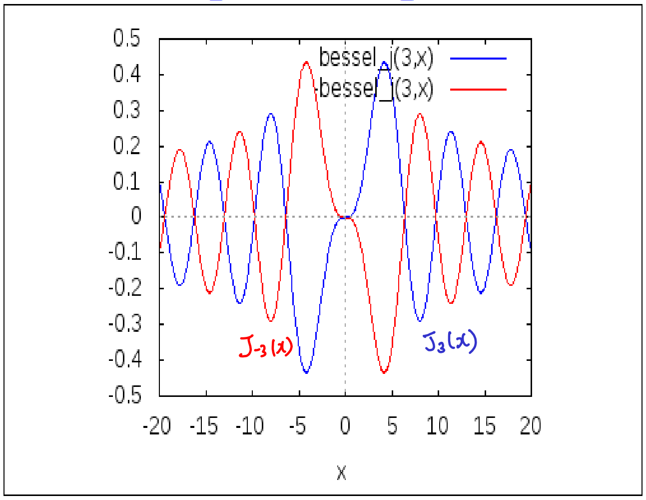
(%t16)



even function $J_2(x)$, $J_{-2}(x)$

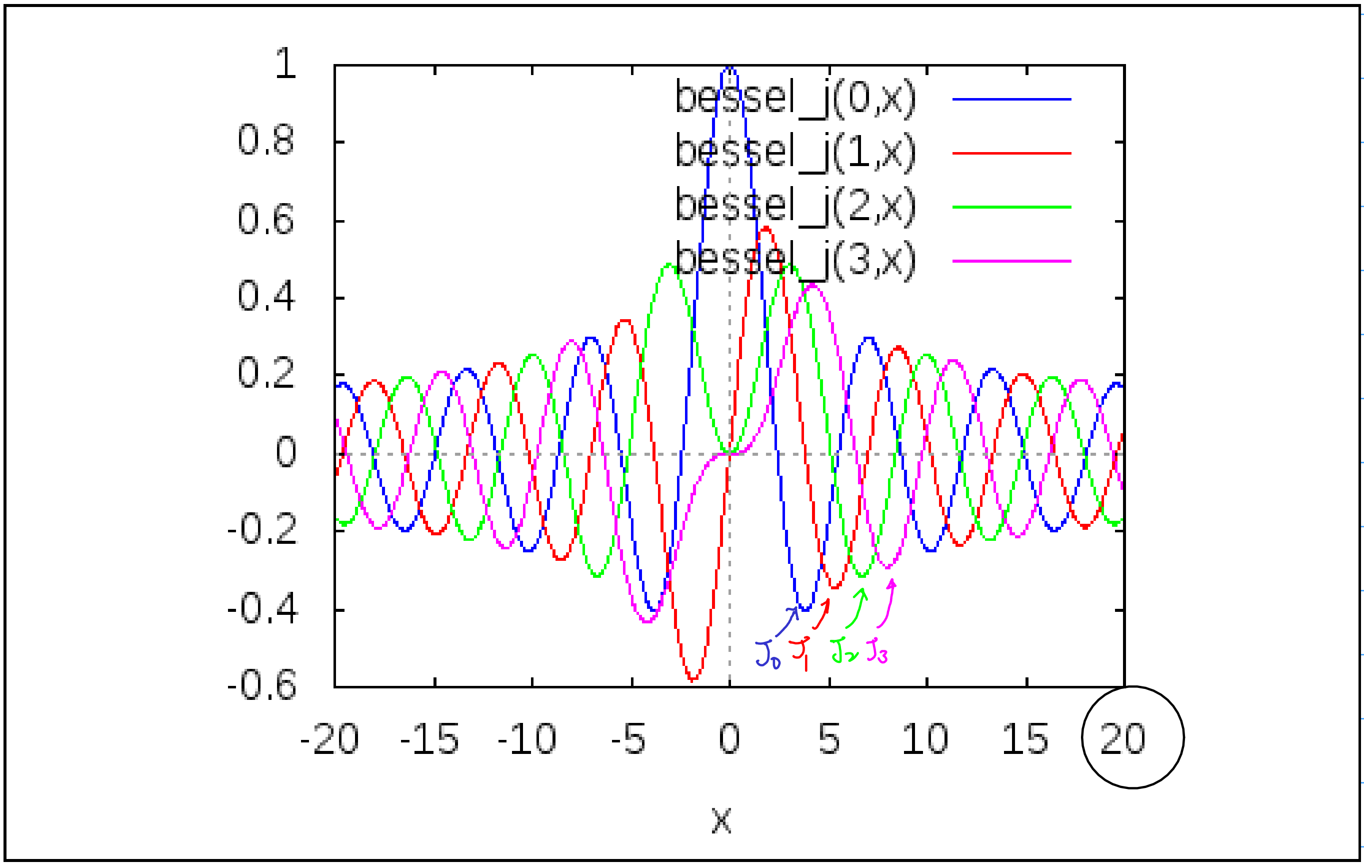
(%i17) wxplot2d([bessel_j(3,x), bessel_j(-3, x)], [x,-20,20])$

(%t17)

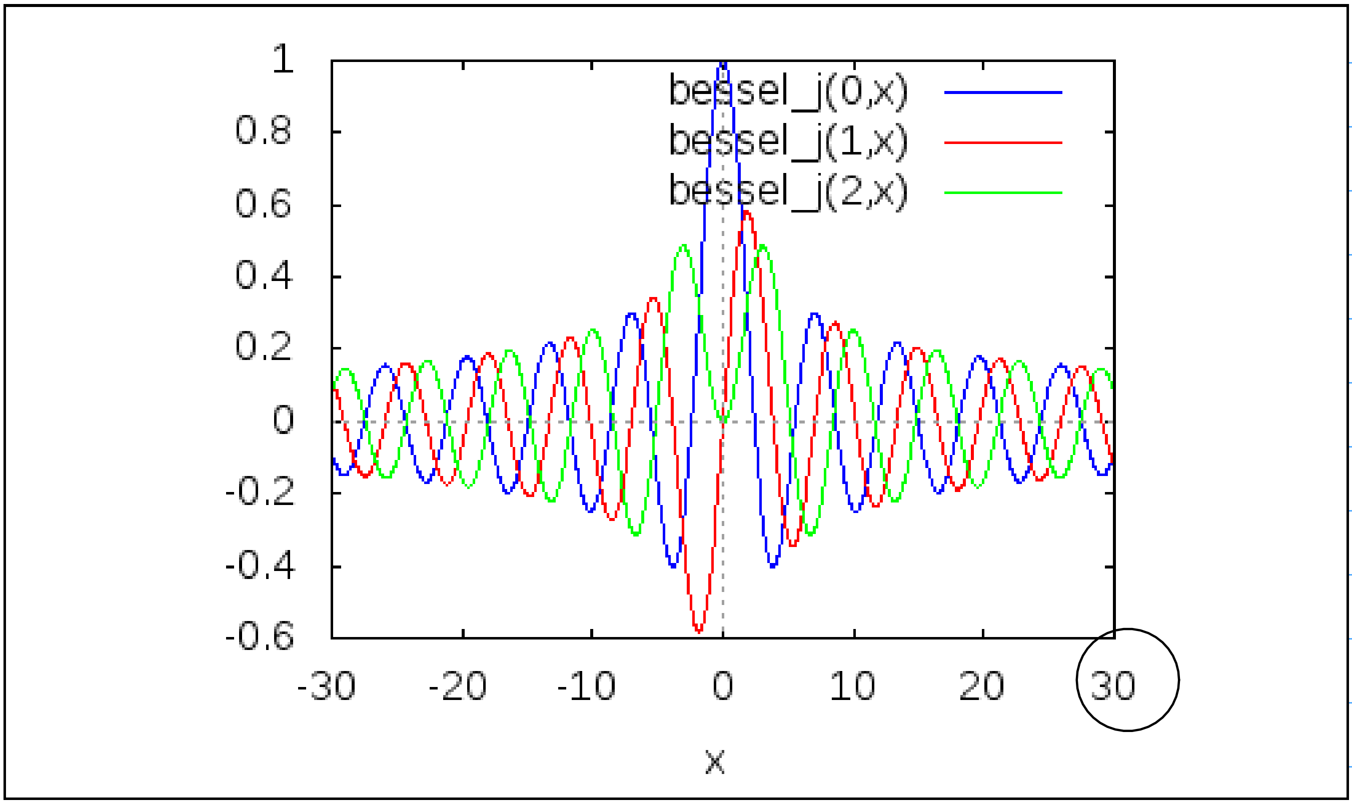


Odd function $J_3(x)$, $J_{-3}(x)$

(%i19) `wxplot2d([bessel_j(0,x), bessel_j(1, x), bessel_j(2, x), bessel_j(3,x)], [x,-20,20])$`

(%t19) 

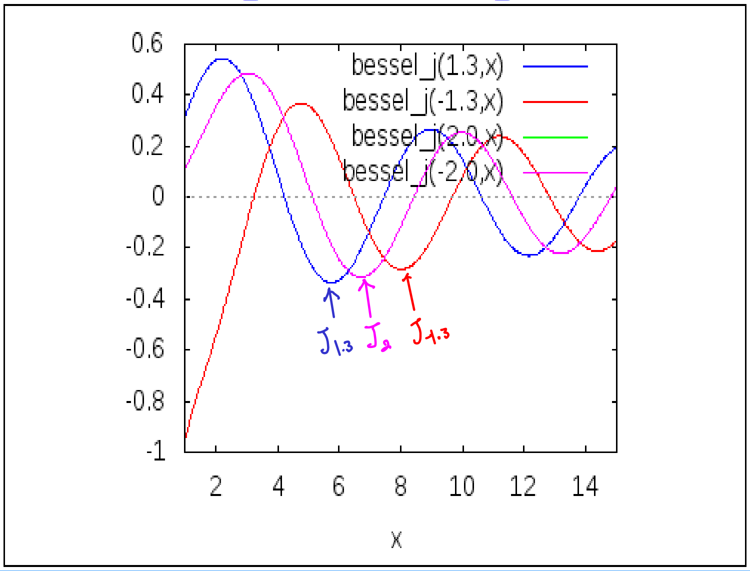
(%i21) `wxplot2d([bessel_j(0,x), bessel_j(1, x), bessel_j(2, x)], [x,-30,30])$`

(%t21) 

Fractional Order $\nu = 1.3, 1.5, 1.8, \cdots$ linearly independent $J_\nu(x)$ $J_{-\nu}(x)$

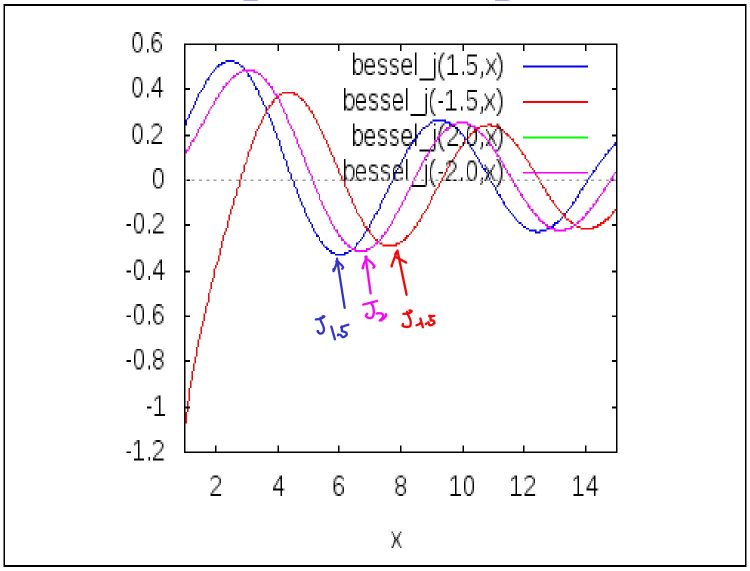
```
(%i36) wxplot2d([bessel_j(1.3,x), bessel_j(-1.3, x),
                 bessel_j(2.0,x), bessel_j(-2.0, x)], [x,1,15])$
```

(%t36)



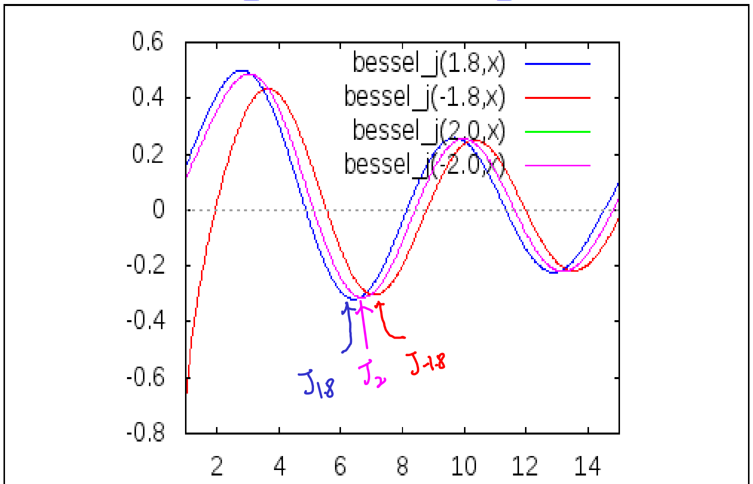

```
(%i37) wxplot2d([bessel_j(1.5,x), bessel_j(-1.5, x),
                 bessel_j(2.0,x), bessel_j(-2.0, x)], [x,1,15])$
```

(%t37)




```
(%i38) wxplot2d([bessel_j(1.8,x), bessel_j(-1.8, x),
                 bessel_j(2.0,x), bessel_j(-2.0, x)], [x,1,15])$
```

(%t38)



Bessel Functions of the 2nd kind

$$Y_\nu(x) = \frac{\cos(\nu\pi)\, J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

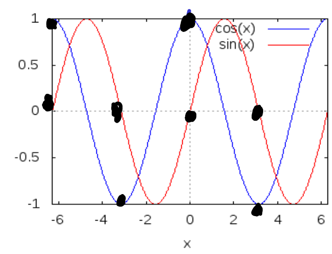
integer order $\boxed{\nu = n}$

www.solitaryroad.com/c678.html

$$Y_\nu(x) = \lim_{\nu \to n} \frac{\cos(n\pi)\, J_n(x) - J_{-n}(x)}{\sin(n\pi)}$$

$$\cos(n\pi) = \pm 1$$

$$\sin(n\pi) = 0$$

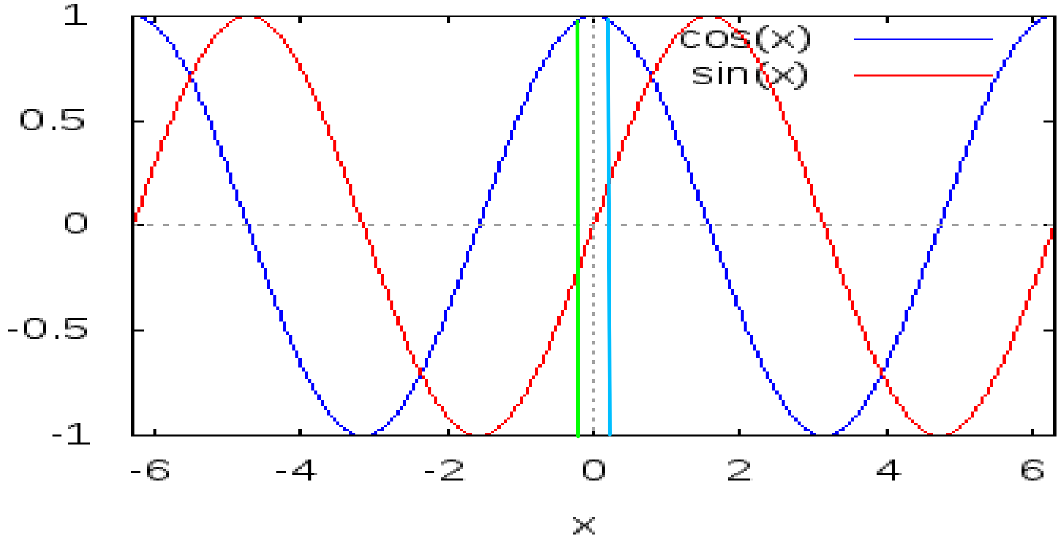


odd n $\quad \cos(n\pi) = -1 \qquad J_n(x) = -J_{-n}(x)$

$$\cos(n\pi)\, J_n(x) - J_{-n}(x) = 0$$

even n $\quad \cos(n\pi) = +1 \qquad J_n(x) = J_{-n}(x)$

$$\cos(n\pi)\, J_n(x) - J_{-n}(x) = 0$$



$n \neq \nu \qquad \cos(n\pi) J_\nu(x) - J_{-\nu}(x) \neq 0$

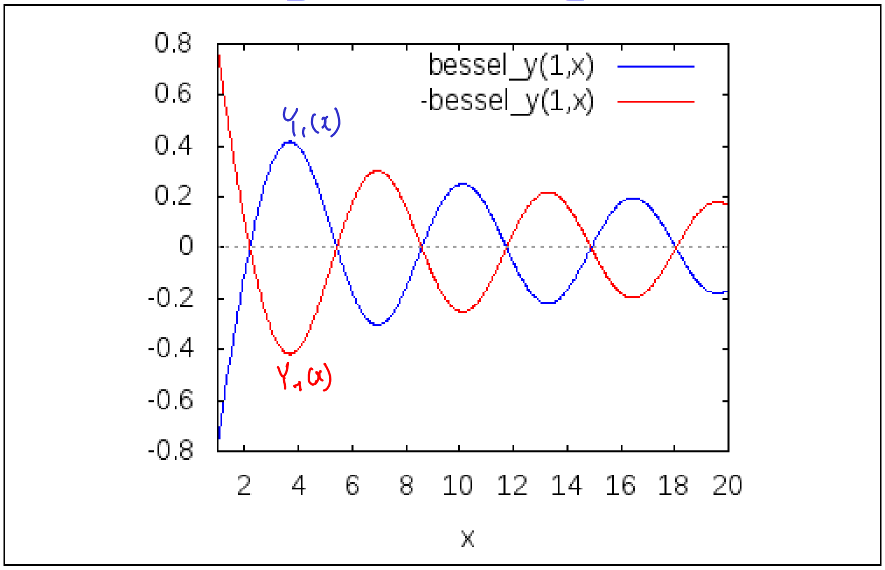
slight difference

$$\frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$ → amplify the slight difference

integer $\nu = 1, 2, 3, \cdots$ Y_{ν}, $Y_{-\nu}$: ~~linearly independent~~

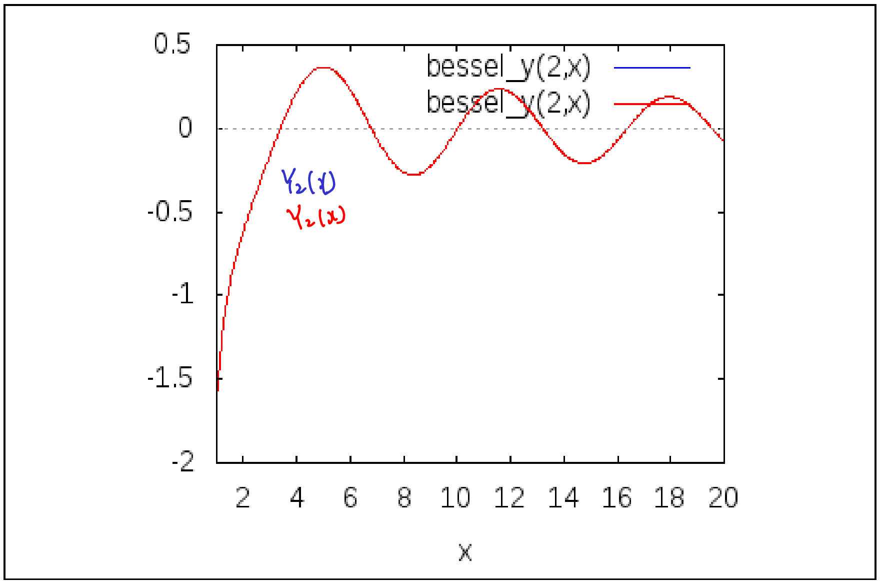
(%i46) wxplot2d([bessel_y(+1,x), bessel_y(-1,x)], [x,1,20])$

(%t46)



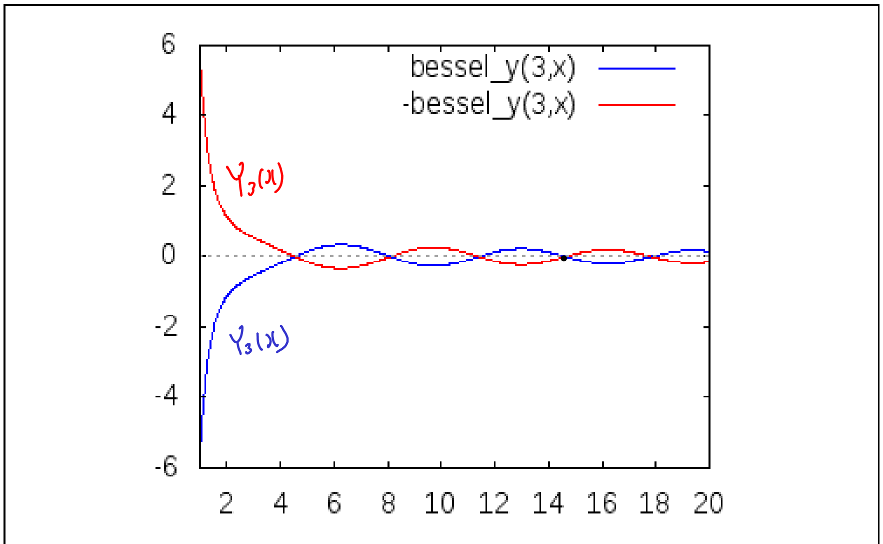
(%i47) wxplot2d([bessel_y(+2,x), bessel_y(-2,x)], [x,1,20])$

(%t47)



(%i48) wxplot2d([bessel_y(+3,x), bessel_y(-3,x)], [x,1,20])$

(%t48)

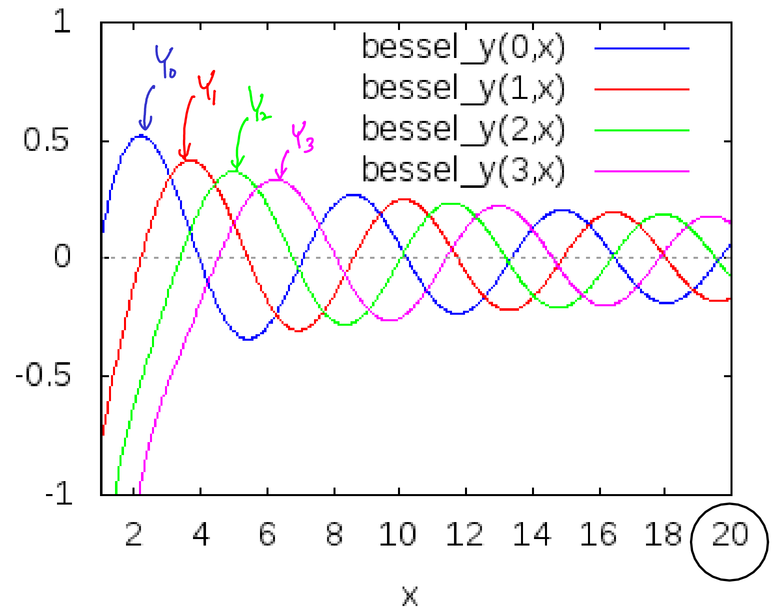


```
(%i52) wxplot2d([bessel_y(0,x), bessel_y(1,x),
        bessel_y(2,x), bessel_y(3,x)],
        [x,1,20], [y,-1, +1])$
```

plot2d: some values were clipped.

plot2d: some values were clipped.

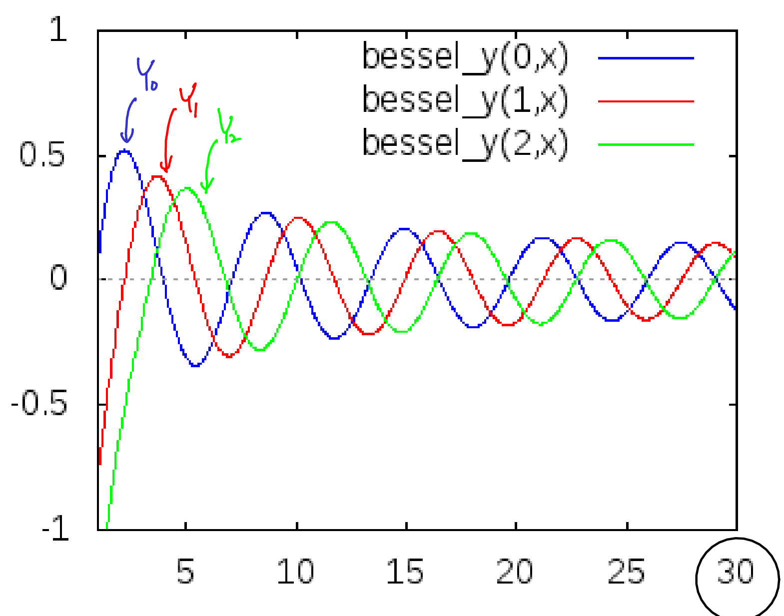
(%t52)




```
(%i53) wxplot2d([bessel_y(0,x), bessel_y(1,x),
        bessel_y(2,x)],
        [x,1,30], [y,-1, +1])$
```

plot2d: some values were clipped.

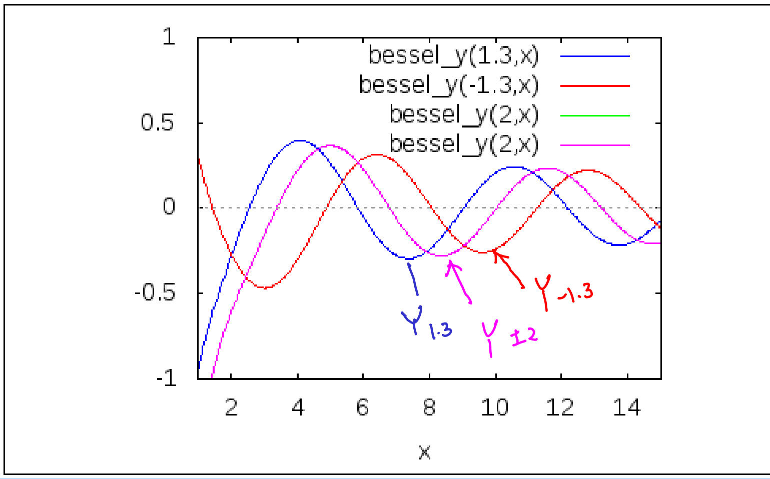
(%t53)



Fractional Order $\nu = 1.3, 1.5, 1.8, \cdots$ linearly independent $Y_\nu(x), Y_{-\nu}(x)$

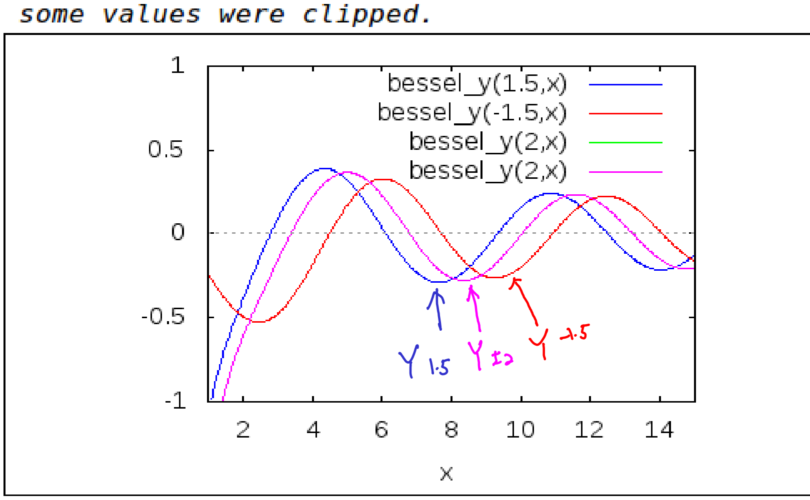
```
(%i57) wxplot2d([bessel_y(1.3,x), bessel_y(-1.3,x),
       bessel_y(2,x), bessel_y(-2, x)],
       [x,1,15], [y,-1, +1])$
plot2d: some values were clipped.
 plot2d: some values were clipped.
```

(%t57)



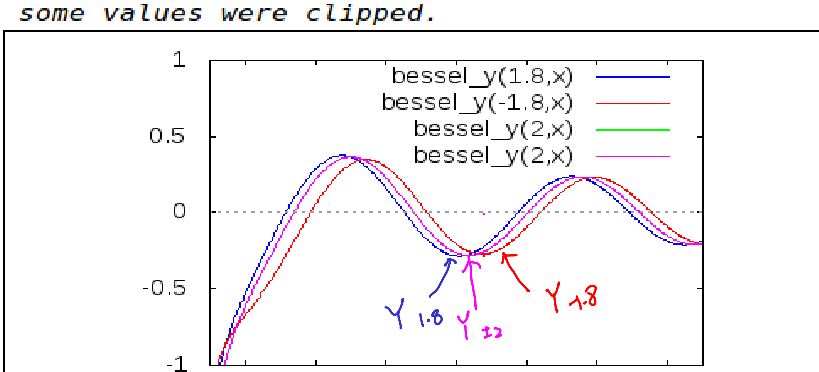

```
(%i56) wxplot2d([bessel_y(1.5,x), bessel_y(-1.5,x),
       bessel_y(2,x), bessel_y(-2, x)],
       [x,1,15], [y,-1, +1])$
plot2d: some values were clipped.
 plot2d: some values were clipped.
 plot2d: some values were clipped.
```

(%t56)




```
(%i58) wxplot2d([bessel_y(1.8,x), bessel_y(-1.8,x),
       bessel_y(2,x), bessel_y(-2, x)],
       [x,1,15], [y,-1, +1])$
plot2d: some values were clipped.
 plot2d: some values were clipped.
 plot2d: some values were clipped.
 plot2d: some values were clipped.
```

(%t58)

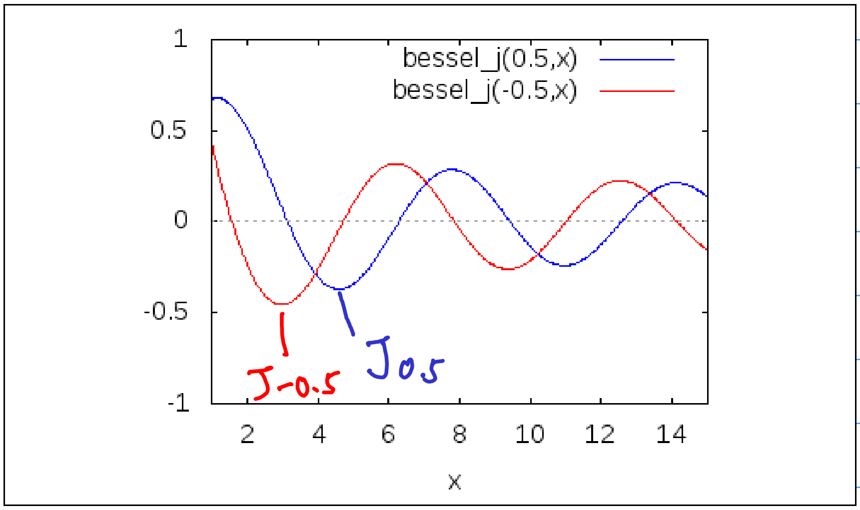


$\nu = \frac{1}{2}$ fractional $\quad x^2 y'' + x\, y' + (x^2 - \frac{1}{4})\, y = 0$

$$y = c_1 J_{\frac{1}{2}}(x) + c_2 J_{-\frac{1}{2}}(x)$$

```
(%i63) wxplot2d([bessel_j(0.5,x), bessel_j(-0.5,x)],
       [x,1,15], [y,-1, +1])$
```

(%t63)

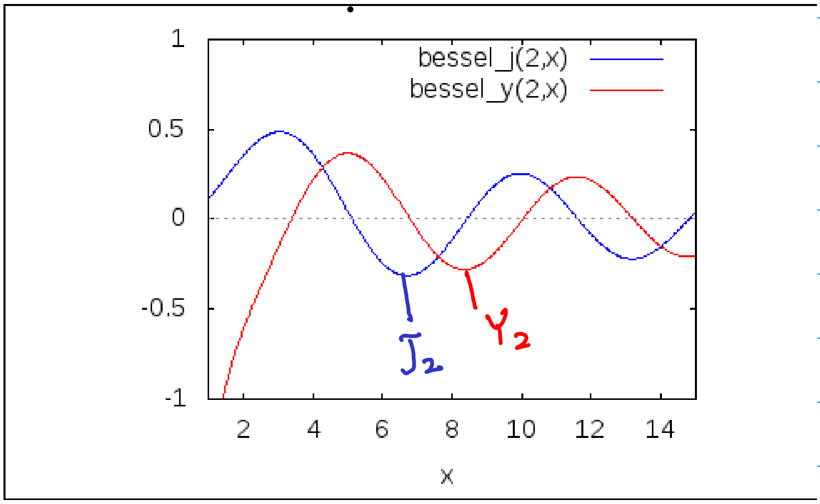


$\nu = 2 =$ integer $\quad x^2 y'' + x\, y' + (x^2 - 4)\, y = 0$

$$y = C_1 J_2(x) + C_2 Y_2(x)$$

```
(%i61) wxplot2d([bessel_j(2,x), bessel_y(-2,x)],
       [x,1,15], [y,-1, +1])$
plot2d: some values were clipped.
```

(%t61)



Bessel's Equation

order ν

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$\boxed{y = \sum_{n=0}^{\infty} c_n x^{n+r}}$$

$r = \pm \nu$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \qquad (r = \nu)$$

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

Fractional order ν $\qquad y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$

* Integer order
Fractional order ν $\qquad y = c_1 J_\nu(x) + c_2 Y_{-\nu}(x)$

all values of ν