Z Transform (H.1)
Definition

20170715

Based on
Complex Analysis for Mathematics and Engineering
J. Mathews

z - Transform

$$X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}$$

$$z = |r|\, e^{j2\pi F}$$
$$= |r|\, e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One Sided z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k]\, z^{-k}$$

Inverse z-Transform

$$X(z) = \mathcal{Z}\left[\{x_n\}_{n=0}^{\infty}\right] = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$x[n] \rightarrow X(z)$

$$x_n = x[n] = \mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi i}\int_C X(z) z^{n-1}\,dz$$

$x[n] \leftarrow X(z)$

Admissible Form of z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$X(z)$: admissible z-transform

if $X(z)$ is a rational function

$$\boxed{X(z) = \frac{P(z)}{Q(z)}} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \cdots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \cdots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$: a polynomial of degree p

$Q(z)$: a polynomial of degree q

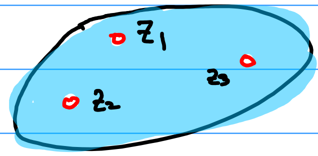
Integration of a function of a complex var.

$$\boxed{\oint_C f(z)\,dz} = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

finite number k of

singular points z_k

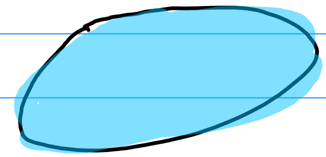
residue theorem



$$\boxed{\oint_C f(z)\,dz} = 0$$

if $f(z)$ is analytic within and on C

no singularity



$$\boxed{\oint_C f(z)\,dz} = 0$$

if $\boxed{f(z) = F'(z)}$ on C

: $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

Thomas J. Cavicchi
Digital Signal Processing, Wiley, 2000

Series Expansion

can expand $f(z)$ about **any point** z_m
over powers of $(z - z_m)$

whether or not $f(z)$ is singular at z_m
or at other points between z and z_m

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

① **Laurent Series** Expansion of $f(z)$ at z_m
general n_1 — depend on $f(z)$ and z_m

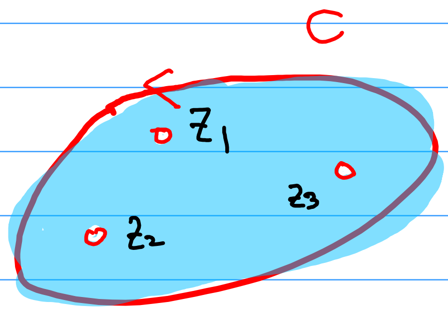
② **z-transform** of $a_n^{\{m\}}$
general n_1 — depend on $f(z)$
$\boxed{z_m = 0}$

③ **Taylor Series** Expansion of $f(z)$ at z_m
positive n_1 — depend on $f(z)$ and z_m $(n_1 \geq 0)$

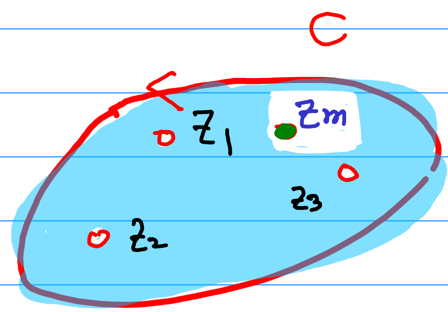
④ **MacLaurin Series** Expansion of $f(z)$ at z_m
positive n_1 — depend on $f(z)$ $(n_1 \geq 0)$
$\boxed{z_m = 0}$

Series Expansion at z_m

no annular region



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$



$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

Let z_1, z_2, z_3 poles of $f(z)$

Then the poles of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

$n \geq 0$	$z_1, z_2, z_3,$	z_m
$n < 0$	z_1, z_2, z_3	

https://en.wikiversity.org/wiki/Complex_Analysis_in_plain_view

Laurent Series with Annular Region

expanded at each pole of $f(z)$

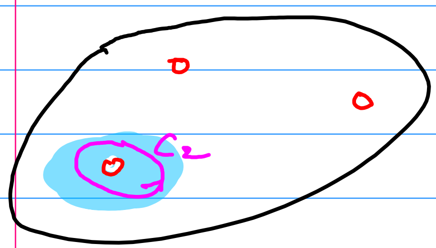
$\boxed{z_1}$ Laurent series expansion at z_1

$$\boxed{f(z) = \sum_{n=-\infty}^{+\infty} \tilde{a}_n^{\{1\}} (z - z_1)^n}$$

$$\tilde{a}_{-1}^{\{1\}} = \mathbf{Res}(f(z), z_1) = \frac{1}{2\pi i} \oint_{C_1} f(z)\, dz$$

$\boxed{z_2}$ Laurent series expansion at z_2

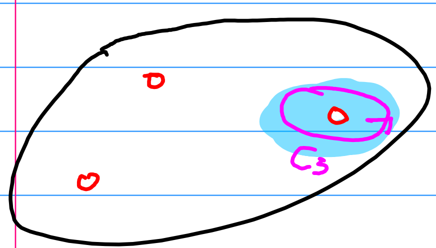
$$\boxed{f(z) = \sum_{n=-\infty}^{+\infty} \tilde{a}_n^{\{2\}} (z - z_2)^n}$$



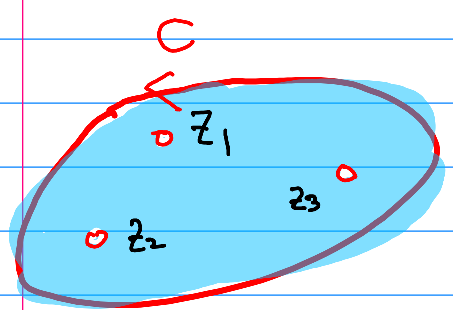
$$\tilde{a}_{-1}^{\{2\}} = \mathbf{Res}(f(z), z_2) = \frac{1}{2\pi i} \oint_{C_2} f(z)\, dz$$

$\boxed{z_3}$ Laurent series expansion at z_3

$$\boxed{f(z) = \sum_{n=-\infty}^{+\infty} \tilde{a}_n^{\{3\}} (z - z_3)^n}$$

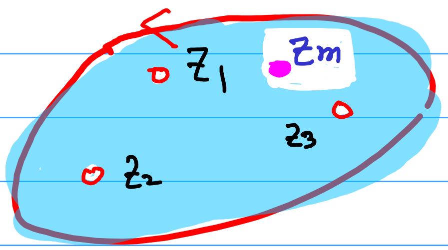


$$\tilde{a}_{-1}^{\{3\}} = \mathbf{Res}(f(z), z_3) = \frac{1}{2\pi i} \oint_{C_3} f(z)\, dz$$



$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

Residue Theorem + Laurent Series



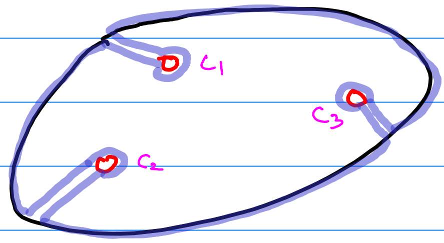
* whether z_m is singular or not

$$f(z) = \sum_{n=n_1}^{\infty} \boxed{a_n^{\{m\}}} (z - z_m)^n$$

$$\boxed{a_n^{\{m\}}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz$$

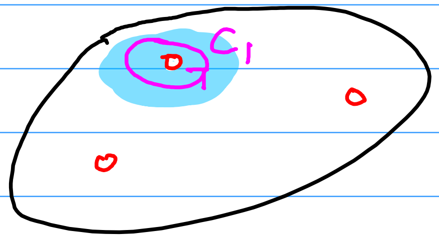
$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

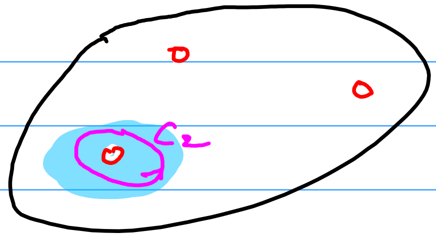
$n=-1$

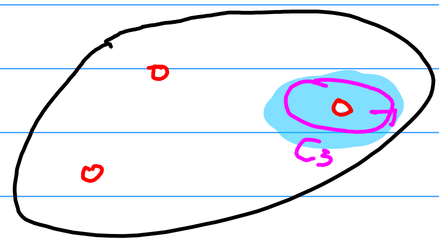


$$a_{-1}^{\{m\}} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

$$= \sum_k \mathrm{Res}\left(f(z), z_k\right)$$







$\tilde{a}_{-1}^{\{1\}} = \mathbf{Res}(f(z), z_1)$ $\tilde{a}_{-1}^{\{2\}} = \mathbf{Res}(f(z), z_2)$ $\tilde{a}_{-1}^{\{3\}} = \mathbf{Res}(f(z), z_3)$

$$\boxed{a_{-1}^{\{m\}} = \tilde{a}_{-1}^{\{1\}} + \tilde{a}_{-1}^{\{2\}} + \tilde{a}_{-1}^{\{3\}}}$$

$$a_{-1}^{\{m\}} = \mathbf{Res}(f(z), z_1) + \mathbf{Res}(f(z), z_2) + \mathbf{Res}(f(z), z_3)$$

Laurent Series coefficient $\tilde{a}_{-1}^{\{i\}}$

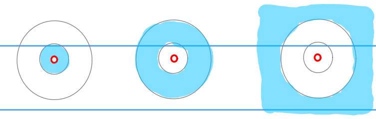
- singular center z_i
- punctured open disk

We do not say this $a_{-1}^{\{m\}}$ is a residue
because it is not
isolated singular center nor punctured open disk ROC

Laurent Series

Annular Region of Convergence

no singularity in this region

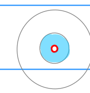


can be expanded at a singular / non-singular point

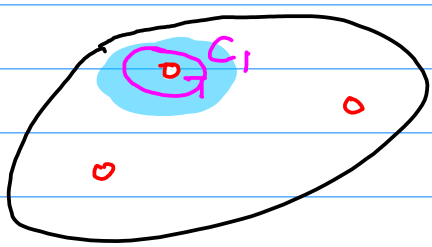
this point need not be in the Convergence region

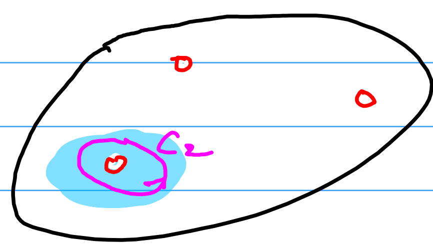
Residue

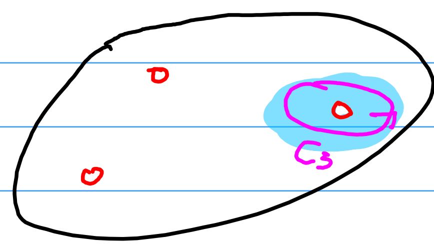
a punctured open disk and thus annular region



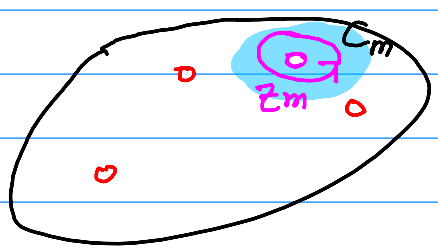
must expanded at a pole (a singular point)

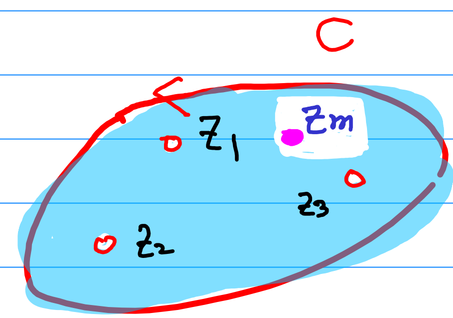






$\tilde{a}_{-1}^{\{1\}} = \mathbf{Res}(f(z), z_1)$ $\tilde{a}_{-1}^{\{2\}} = \mathbf{Res}(f(z), z_2)$ $\tilde{a}_{-1}^{\{3\}} = \mathbf{Res}(f(z), z_3)$





$\tilde{a}_{-1}^{\{m\}} = \mathbf{Res}(f(z), z_m)$

when z_m is a pole

$\cancel{\tilde{a}_{-1}^{\{m\}} = Res(f(z), z_m)}$

when z_m is <u>not</u> a pole

Computing $a_n^{\{m\}}$ (general formula)

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$\boxed{n \leftarrow k}$

$$f(z) = \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^k$$

for a given n

$$\frac{f(z)}{(z - z_m)^{n+1}} = \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^{k-n-1}$$

k : index variable
n : fixed value

$$\boxed{\oint_C \frac{f(z)}{(z - z_m)^{n+1}} dz} = \oint_C \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^{k-n-1} dz$$

$$= \sum_{k=n_1}^{\infty} \boxed{\oint_C a_k^{\{m\}} (z - z_m)^{k-n-1} dz}$$

$\boxed{k = n}$

$$\boxed{\oint_C \frac{f(z)}{(z - z_m)^{n+1}} dz} = \boxed{\oint_C a_n^{\{m\}} \frac{1}{(z - z_m)} dz} = \boxed{2\pi i \cdot a_n^{\{m\}}}$$

$$\boxed{a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} dz}$$

$$\oint_C \underbrace{\boxed{\cdots (z - z_m)^{-3} + (z - z_m)^{-2}}}_{\to 0} + \boxed{\frac{1}{(z - z_m)}} + \underbrace{\boxed{1 + (z - z_m) + (z - z_m)^2 + \cdots}}_{\to 0} dz$$

$$= \oint_C \boxed{\frac{1}{(z - z_m)}} dz = 2\pi i$$

Computing $a_n^{\{m\}}$ using Residues

expansion at z_m

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)\end{aligned}}$$

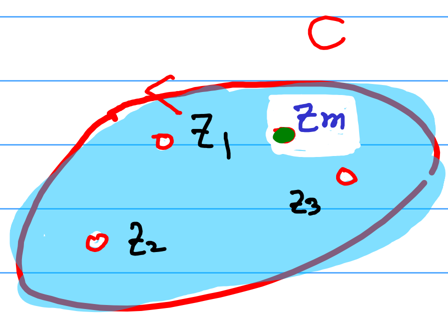
$\boxed{n=-1}$ $n+1=0$ $(z-z_m)^{n+1}=1$

$$\boxed{\begin{aligned} a_{-1}^{\{m\}} &= \frac{1}{2\pi i}\oint_C f(z)\,dz \\ &= \sum_k \mathrm{Res}\left(f(z), z_k\right)\end{aligned}}$$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_k \mathrm{Res}\left(f(z), z_k\right)$$

$$a_{-1}^{\{m\}} = \cancel{\mathrm{Res}\left(f(z), z_m\right)}$$

we do not say this is a residue



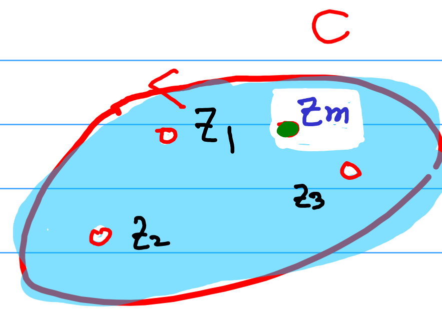
$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}}(z-z_m)^n}$$

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)\end{aligned}}$$

Residue → Laurent series → annular region) a punctured
→ expanded at a pole ✲ open disk

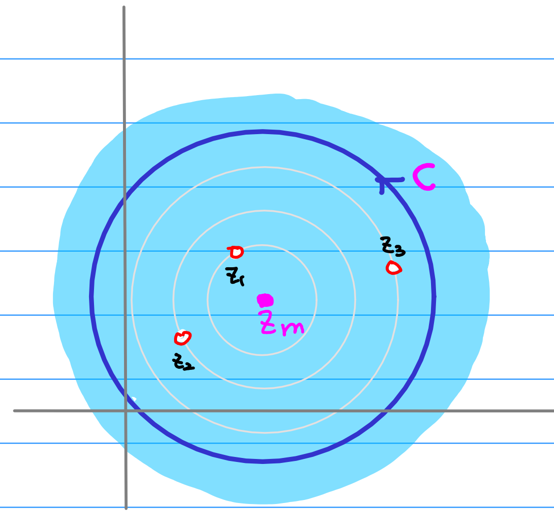
$$\dots,\ a^{\{m\}}_{-2},\ a^{\{m\}}_{-1},\ a^{\{m\}}_{0},\ a^{\{m\}}_{+1},\ a^{\{m\}}_{+2},\ \dots$$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a^{\{m\}}_n (z - z_m)^n}$$



$$a^{\{m\}}_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

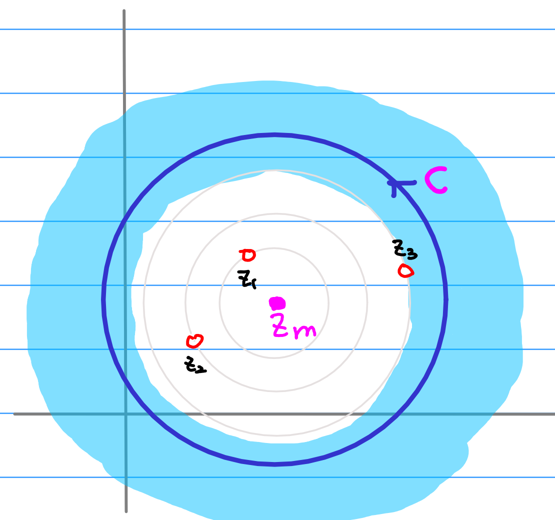


$$a^{\{m\}}_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

$$= \sum_k \mathrm{Res}\left(f(z), z_k\right)$$

$$\vdots$$

$$a^{\{m\}}_{-3} = \sum_k \mathrm{Res}\left(f(z)(z-z_m)^2, z_k\right)$$

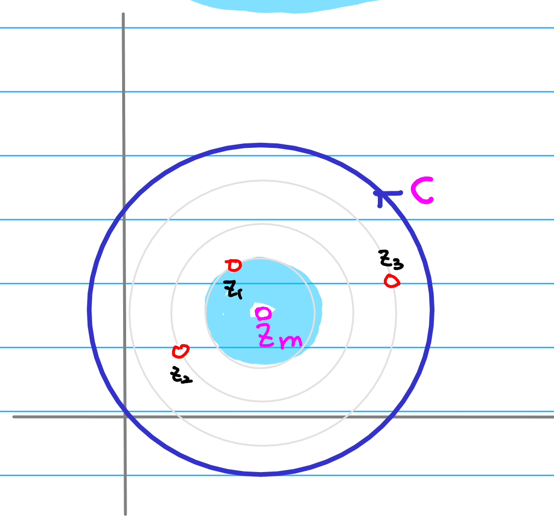


$$a^{\{m\}}_{-2} = \sum_k \mathrm{Res}\left(f(z)(z-z_m)^1, z_k\right)$$

$$a^{\{m\}}_{-1} = \sum_k \mathrm{Res}\left(f(z), z_k\right)$$

$$a^{\{m\}}_{0} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^1}, z_k\right)$$

$$a^{\{m\}}_{1} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^2}, z_k\right)$$



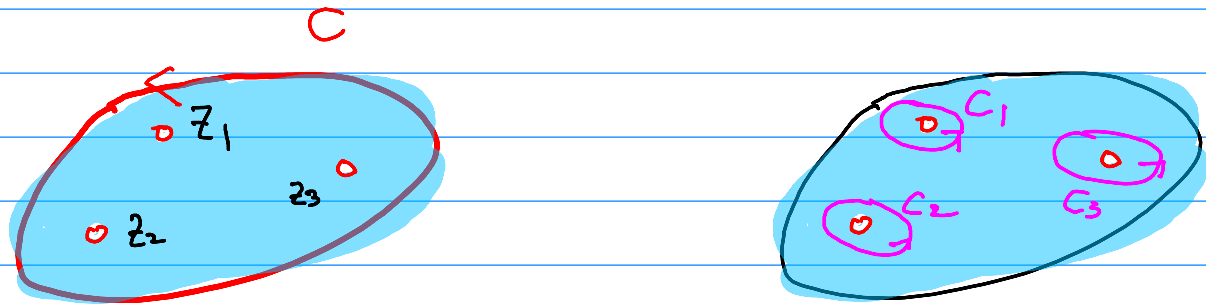
$$a^{\{m\}}_{2} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^3}, z_k\right)$$

$$\vdots$$

	annular region			non-annular region
	punctured	ring	outside circle	
singular center Z_m	$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$			
non-singular center Z_m	$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} dz' = \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$			

	annular region			non-annular region
	punctured	ring	outside circle	
singular center Z_m	Laurent Series			×
non-singular center Z_m	Laurent Series			×

	annular region			non-annular region
	punctured	ring	utside circle	
singular center Z_m	residue	×	×	×
non-singular center Z_m	×	×	×	×



$$\boxed{\oint_C f(z)\,dz = 2\pi j \sum_{k=1}^{M} \tilde{a}_{-1}^{\{k\}} = 2\pi j \sum_{k=1}^{M} \mathrm{Res}(f(z), z_k)}$$

residue theorem

$$\boxed{a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)}$$

Laurent coefficient

C encloses k poles

C_k encloses only the k-th pole

$\tilde{a}_{-1}^{\{k\}}$ the residue of the k-th pole enclosed by C, z_k

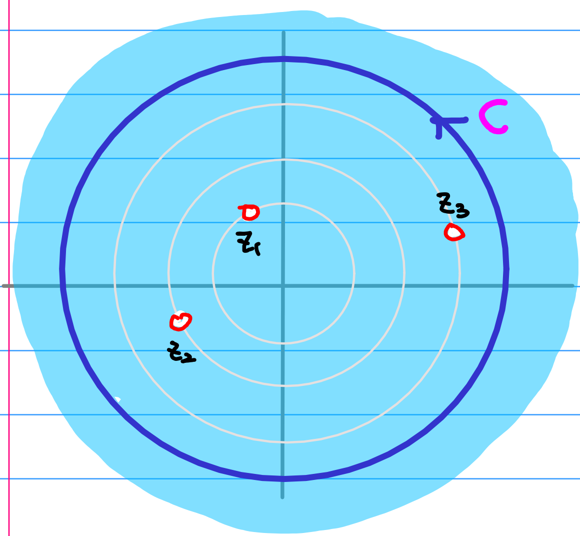
Residues

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s)\, ds \Rightarrow \oint_C f(s)\, ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s)\, ds = Res(f(z), z_0)$$

$$= \begin{cases} \lim\limits_{z \to z_0} (z - z_0) f(z) & (\text{simple}) \\ \frac{1}{(n-1)!} \lim\limits_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & (\text{order } n) \end{cases}$$

Series Expansion at $z=0$



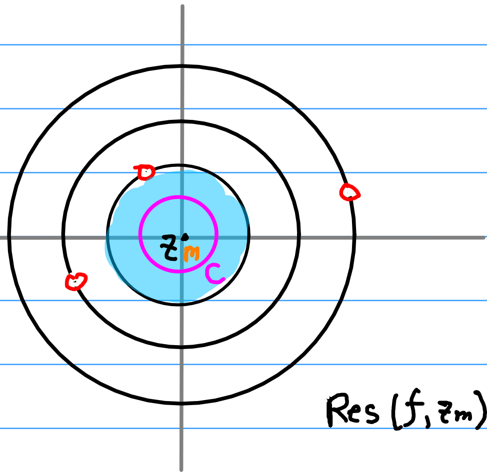
$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} z^n}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\, dz$$

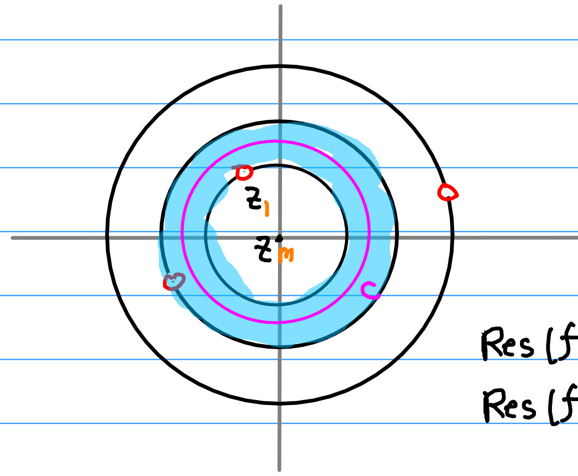
$$= \sum_k \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

Poles z_k

$n \geq 0$	$z_1, z_2, z_3, 0$
$n < 0$	z_1, z_2, z_3

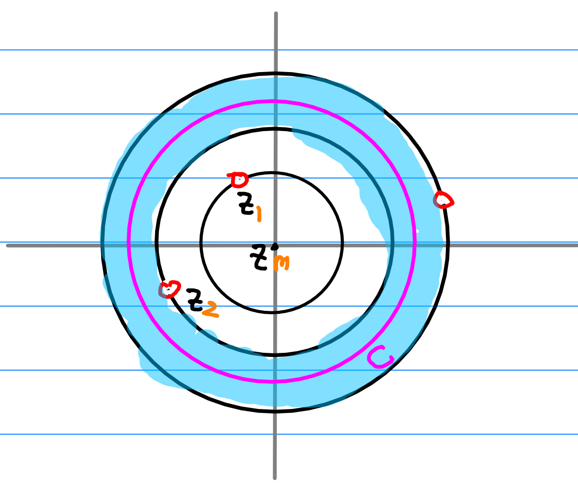


$\text{Res}(f, z_m)$

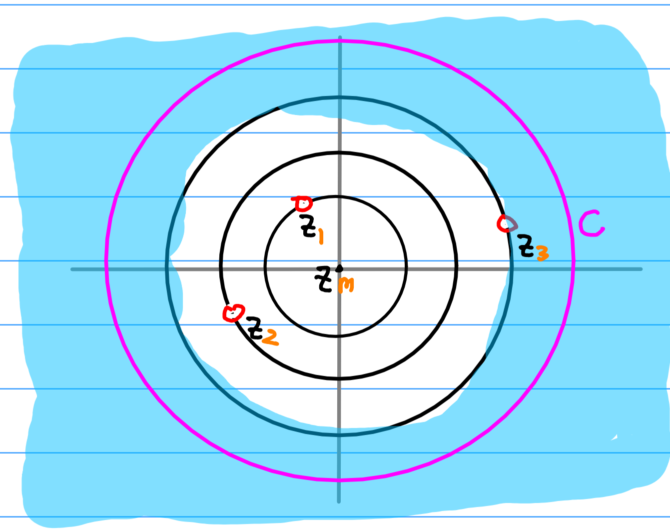


$\text{Res}(f, z_1)$

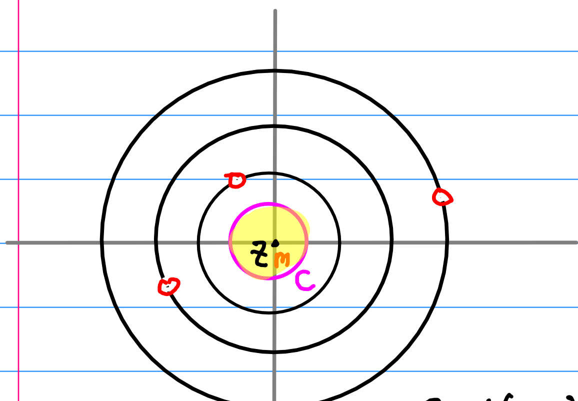
$\text{Res}(f, z_m)$



$\text{Res}(f, z_1) + \text{Res}(f, z_1) + \text{Res}(f, z_m)$

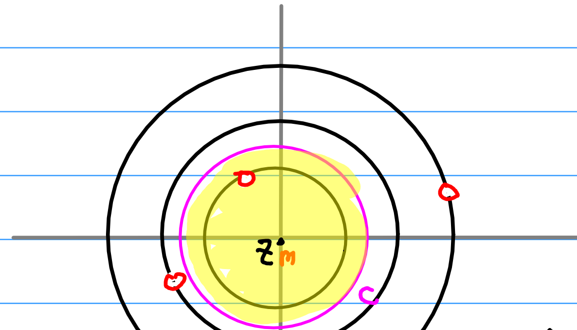


$\text{Res}(f, z_1) + \text{Res}(f, z_1) + \text{Res}(f, z_3)$
$+ \text{Res}(f, z_m)$



$\mathrm{Res}(f, z_m)$

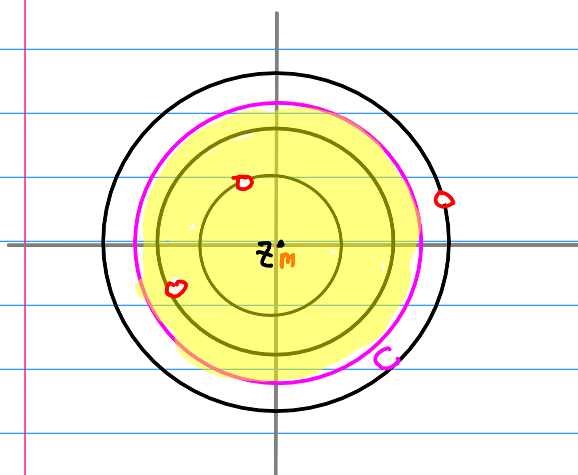
$(n \geq 0)$



$\mathrm{Res}(f, z_1)$

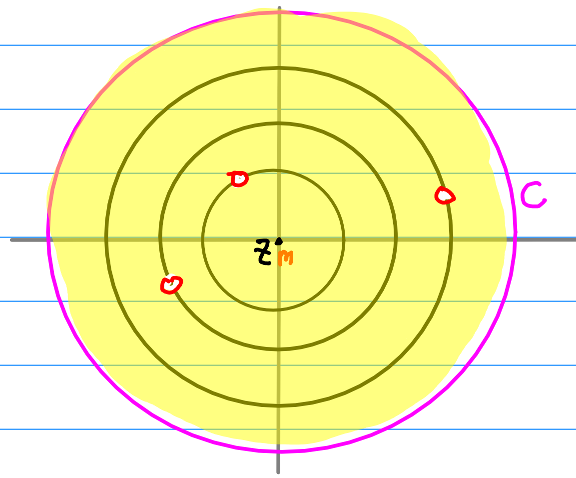
$\mathrm{Res}(f, z_m)$

$(n \geq 0)$



$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_m)$

$(n \geq 0)$



$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_3)$

$+ \mathrm{Res}(f, z_m)$

$(n \geq 0)$

Inverse z-Transform $x[n] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$

$$X(z) = \sum_{k=0}^{\infty} x_k\, z^{-k}$$

$$z^{n-1} X(z) = \left(\sum_{k=0}^{\infty} x_k\, z^{-k}\right) z^{n-1}$$

$$\boxed{\int z^{n-1}\, \text{LHS}\, dz = \int \text{RHS}\, z^{n-1}\, dz}$$

$$= \sum_{k=0}^{\infty} x_k\, z^{-k+n-1}$$

$$\boxed{[0,\infty) = [0, n-1] \cup [n] \cup [n+1, \infty)}$$

$$= \sum_{k=0}^{n-1} x_k\, z^{-k+n-1} + \sum_{k=n}^{n} x_k\, z^{-k+n-1} + \sum_{k=n+1}^{\infty} x_k\, z^{-k+n-1}$$

$$= \sum_{k=0}^{n-1} x_k\, z^{-k+n-1} + \frac{x_n}{z^1} + \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}$$

$$\int_c X(z)\, z^{n-1}\, dz = \int_c \sum_{k=0}^{n-1} x_k\, z^{-k+n-1}\, dz + \int_c \frac{x_n}{z^1}\, dz + \int_c \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}\, dz$$

$$= \sum_{k=0}^{n-1} x_k \int_c z^{-k+n-1}\, dz + x_n \int_c \frac{1}{z^1}\, dz + \sum_{k=n+1}^{\infty} x_k \int_c \frac{1}{z^{k-n+1}}\, dz$$

$$= \sum_{k=0}^{n-1} x_k \cdot 0 + x_n \cdot 2\pi i + \sum_{k=n+1}^{\infty} x_k \cdot 0$$

$$x[n] = \frac{1}{2\pi i}\int_c X(z)\, z^{n-1}\, dz$$

Z - transform

$\boxed{z_m = 0}$

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi i} \oint_C f(z)\, z^{n-1}\, dz \\ &= \sum_k \mathrm{Res}\left(f(z)\, z^{n-1}, z_k\right) \end{aligned}}$$

$n > 0$ z_k : poles of $f(z)$

$n = 0$ z_k : poles of $f(z)$ + $z = 0$

$z^{n-1} = z^{-1} = \frac{1}{z}$

$x[n]$ includes $u[n]$ $\rightarrow$ $X[z]$ contains z on its numerator

Also, think about modified partial fraction $\frac{X[z]}{z}$

Laurent Expansion

expansion at z_m

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k\right) \end{aligned}}$$

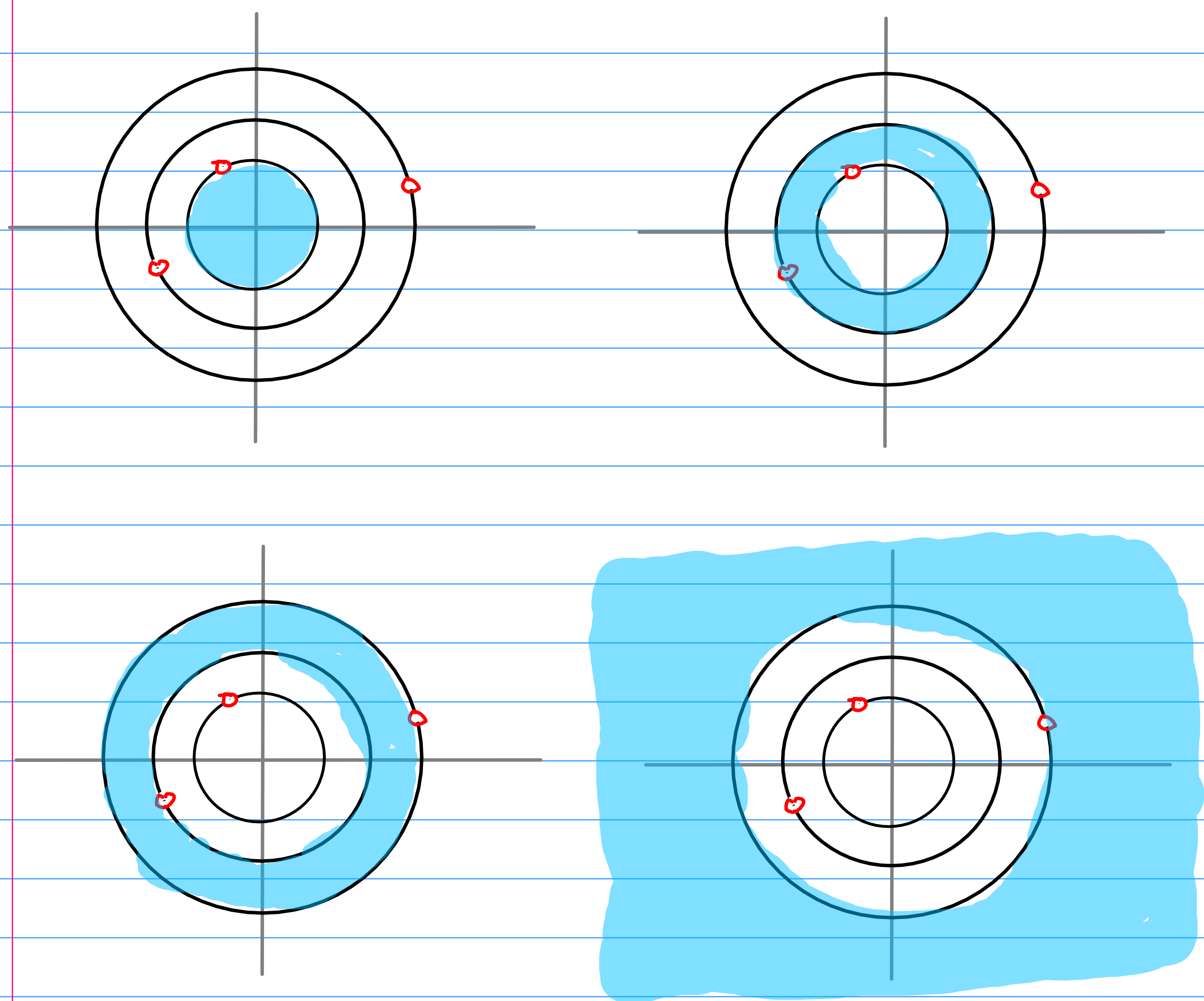
$\boxed{z_m = 0}$

$$\boxed{\begin{aligned} a_n^{\{0\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right) \end{aligned}}$$

$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i} \oint_C f(z)\, z^{n-1}\, dz \\ &= \sum_k \mathrm{Res}\left(f(z)\, z^{n-1}, z_k\right) \end{aligned}$$

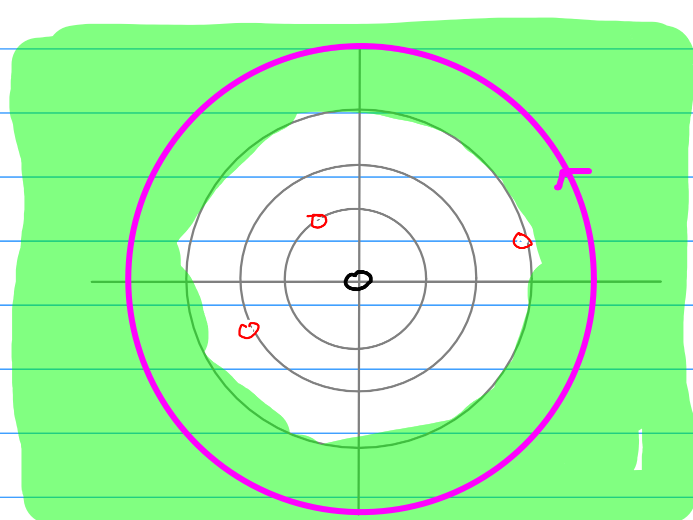
$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{-n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{-n+1}}, z_k\right) \end{aligned}$$

Different D, Different Laurent Series



$$x[n] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$$

$$= \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right)$$



z-transform

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

Complex Variables and Ap
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

$D_1 : \ |z| < 1$

$D_2 : \ 1 < |z| < 2$

$D_3 : \ 2 < |z|$

① D_1 $\quad |z| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(2^{-n-1} - 1\right) z^n \quad |z| < 1$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

③ D_3 $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

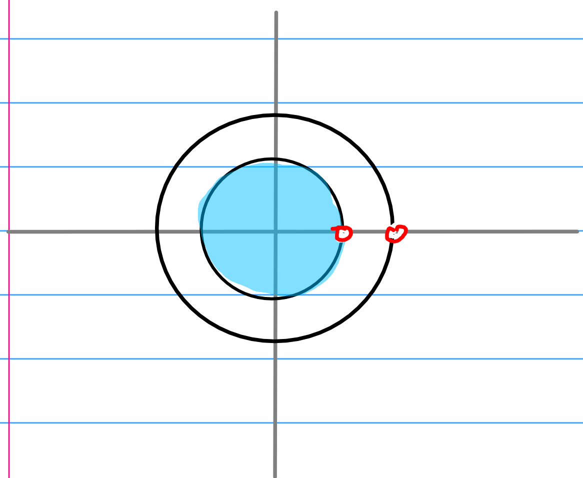
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

① D_1 $|z|<1$, $\left|\frac{z}{2}\right|<1$

$$\frac{f(z)}{z^{n+1}} = \frac{-1}{(z-1)(z-2)z^{n+1}}$$

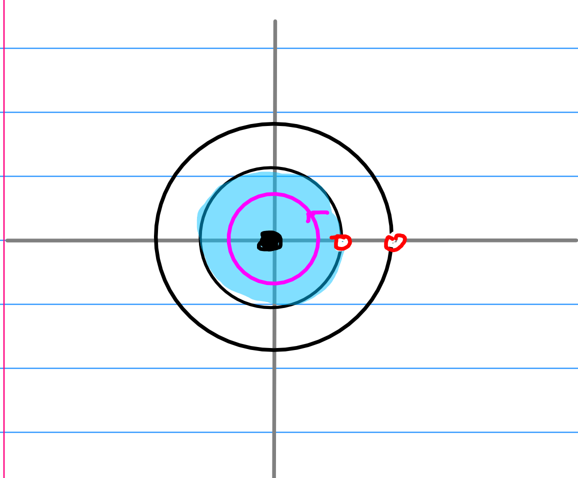


$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \quad |z|<1$$

$$a_n = \frac{f(z)}{z^{n+1}} = \frac{1}{(z-1)(z-2)z^{n+1}} \qquad \frac{1}{z-1} - \frac{1}{z-2}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_n)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$n \geq 0$ then the pole $z=0$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{d}{dz}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)\left((z-1)^{-2} - (z-2)^{-2}\right)$$

$$\frac{d^2}{dz^2}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)\left((z-1)^{-3} - (z-2)^{-3}\right)$$

$$\frac{d^3}{dz^3}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)(-3)\left((z-1)^{-4} - (z-2)^{-4}\right)$$

$$\frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n n! \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

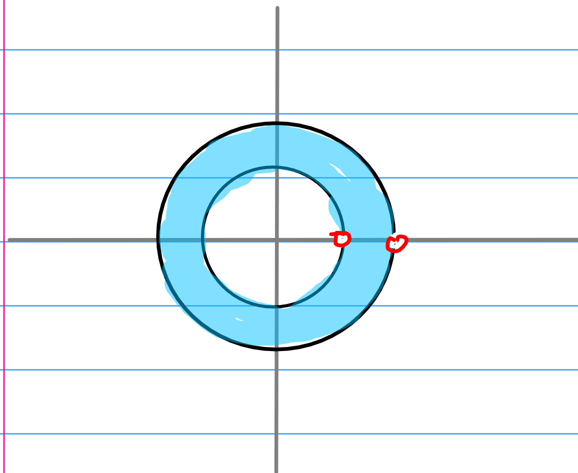
$$= -1 + 2^{-n-1}$$

$$a_n = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$



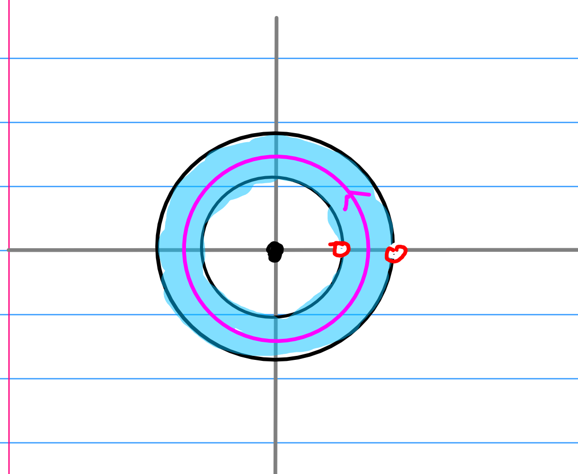
$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$a_n = \sum_{k=1}^{M} \operatorname{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$
$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$\frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{1}{n!} \lim_{z\to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z\to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$
$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$
$$= -1 + 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z\to 1} (z-1)\frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

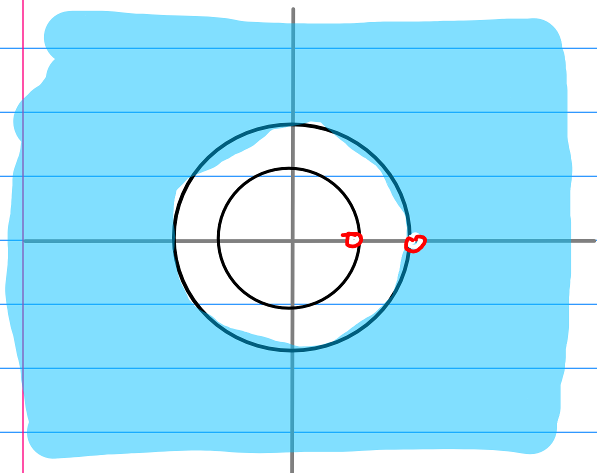
$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
1	1	1	2^{-1}	2^{-2}	2^{-3}	

$$\begin{cases} a_n = 2^{-n-1} & n \geq 0 \\ a_n = 1 & n < 0 \end{cases} \qquad \begin{cases} 2^{-n-1} z^n \\ z^{-n} \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

③ D_3 $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$



$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\,\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\,\frac{1}{1-\left(\frac{2}{z}\right)}$$

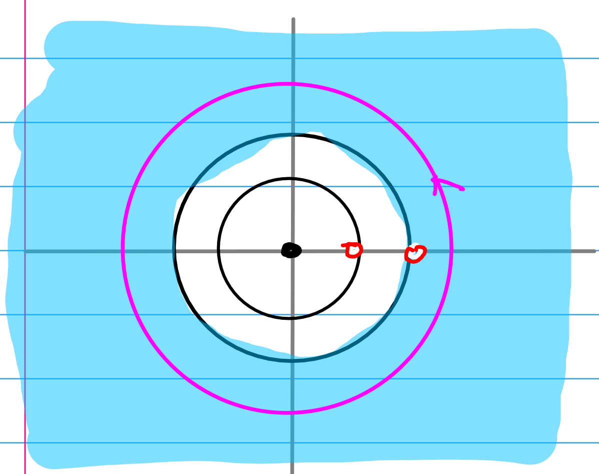
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$a_n = \sum_{k=1}^{M} \operatorname{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}},\ z_k\right) = \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}},\ 0\right)$$

$$+ \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}},\ 1\right)$$

$$+ \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}},\ 2\right)$$



$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z\to 1} (z-1)\frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right) = \lim_{z\to 2} (z-2)\frac{-1}{(z-1)(z-2)z^{n+1}} = -\frac{1}{2^{n+1}}$$

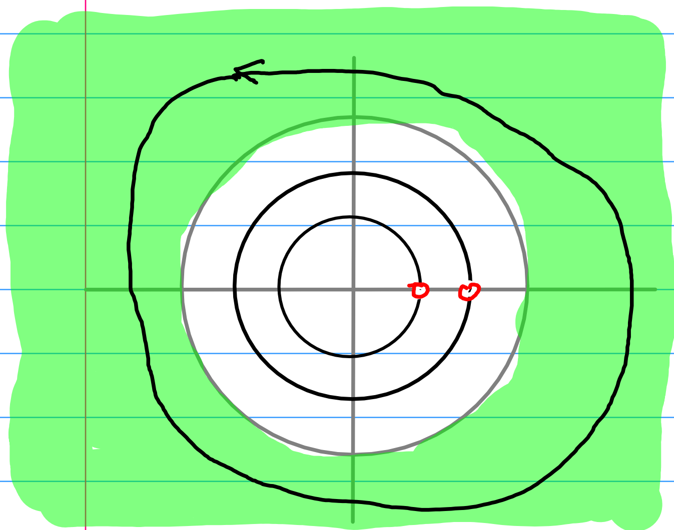
$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
-2^2	-2	-1	-2^{-1}	-2^{-2}	-2^{-3}	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 2\right)$
$1-2^2$	$1-2$	0	0	0	0	

$$a_n = 1 - 2^{-n+1} \quad n < 0$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \sum_{n=-1}^{-\infty} (1-2^{-n+1})z^n = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



$x[n]$

$$= \frac{1}{2\pi i} \int_C \boxed{X(z)\, z^{n-1}}\, dz$$

$$= \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, z_j\right)$$

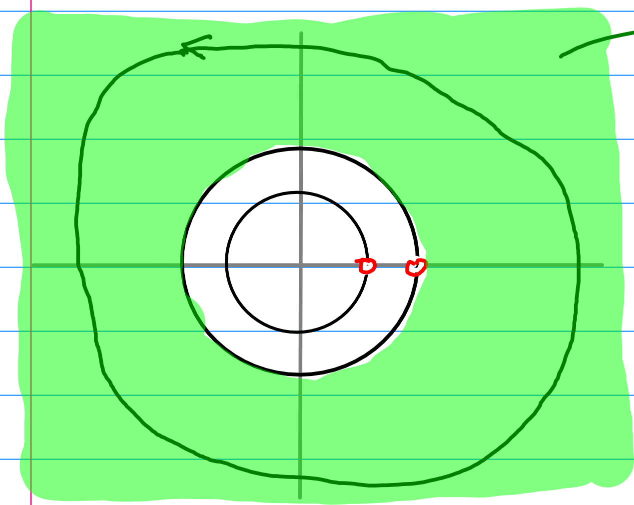
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z)\, z^{n-1} = \frac{-1}{(z-1)(z-2)}\, z^{n-1}$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 1\right) = (z-1)\frac{-1}{(z-1)(z-2)}\, z^{n-1}\Big|_{z=1} = 1$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 2\right) = (z-2)\frac{-1}{(z-1)(z-2)}\, z^{n-1}\Big|_{z=2} = -2^{n-1}$$

$$x[n] = 1 - 2^{n-1}$$



ROC (Region of Convergence)

$$|z|>2 \Rightarrow \frac{2}{|z|}<1$$

$$\left(\frac{2}{z}\right)^0+\left(\frac{2}{z}\right)^1+\left(\frac{2}{z}\right)^2+\cdots \xrightarrow[\text{Converge}]{} \frac{1}{1-\frac{2}{z}}$$

$$|z|>2 \Rightarrow \frac{1}{|z|}<1$$

$$\left(\frac{1}{z}\right)^0+\left(\frac{1}{z}\right)^1+\left(\frac{1}{z}\right)^2+\cdots \xrightarrow[\text{Converge}]{} \frac{1}{1-\frac{1}{z}}$$

$$f(z)=\frac{1}{z-1}-\frac{1}{z-2}=\frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)}-\frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}=\sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}}$$

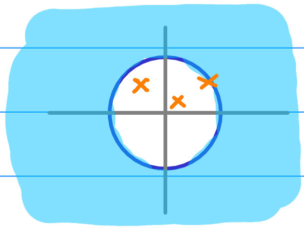
$$=\sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}$$

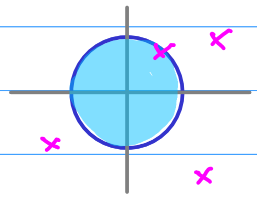
$$\begin{aligned}&\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\\ &+\frac{1}{2}\left\{\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right\}\end{aligned} \xrightarrow[\text{Converge}]{} \frac{1}{z-1}-\frac{1}{z-2}=\frac{-1}{(z-1)(z-2)}$$

$$(1-2^0)z^{-1}+(1-2^1)z^{-2}+(1-2^2)z^{-3}+\cdots \xrightarrow[\text{Converge}]{} \frac{-1}{(z-1)(z-2)} \quad (|z|>2)$$

$$x[n]=1-2^{n-1} \longleftrightarrow X(z)=\frac{-1}{(z-1)(z-2)} \quad (|z|>2)$$

causal $x[n]=0$ $(n<0)$ anti-causal $x[n]=0$ $(n>0)$

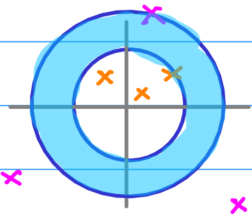




ROC: outside a circle ROC: inside a circle

bi-causal $x[n]$

Overlapped ROC



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n \qquad z_m = 0 \qquad a_n^{\{0\}} \rightarrow a_n$$

Laurent Series at $z=0$

$$f(z) = \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots$$

Z-transform

$\boxed{\text{Bi-causal}}$ $$X(z) = \cdots + x[-2] z^2 + x[-1] z^1 + x[0] z^0 + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3} + \cdots$$

$\boxed{\text{Causal}}$ $$X(z) = x[0] z^0 + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3} + \cdots$$

$\boxed{\text{Anti-causal}}$ $$X(z) = \cdots + x[-2] z^2 + x[-1] z^1 + x[0] z^0$$

$$a_n \longleftrightarrow x[-n]$$

$$a_{-n} \longleftrightarrow x[n]$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_k \right)$$

analytic at z_m

$n_1 \geq 0$ — Taylor Series

general n_1 $\quad z_m = 0$ — MacLaurin Series

singular at z_m

general n_1 — Laurent Series

general n_1 $\quad z_m = 0$ — z - Transform

$$\boxed{\begin{aligned} f(z) &= \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n \\ a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} \, dz' \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_k \right) \end{aligned}}$$

$$\boxed{z_m = 0} \quad \boxed{a_{-n}^{\{0\}} = h(n)} \quad \boxed{n \to -n}$$

$$\boxed{\begin{aligned} H(z) &= \sum_{n=n_1}^{\infty} h(-n) \, z^n \\ h(n) &= \frac{1}{2\pi i} \oint_c \frac{H(z')}{z'^{\,n+1}} \, dz' \\ &= \sum_k \mathrm{Res}\left(\frac{H(z)}{z^{n+1}} , z_k \right) \end{aligned}}$$

$$\boxed{\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n) \, z^{-n} \\ h(n) &= \frac{1}{2\pi i} \oint_c H(z') \, z'^{\,n-1} \, dz' \\ &= \sum_k \mathrm{Res}\left(H(z) \, z^{n-1} , z_k \right) \end{aligned}}$$

C is in the same region of analyticity of $f(z)$

typically a circle centered on z_m

z_k within C : singularities of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

C is in the same region of analyticity of $H(z)$

typically a circle centered on z_m

generally a circle centered on the origin

may enclose any or all singularities of $H(z)$

often the unit circle

z_k within C : singularities of $H(z)\, z^{n-1}$

$$\boxed{\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} && z \in \text{R.O.C} \\ h(n) &= \frac{1}{2\pi i}\oint_C H(z')\, z'^{\,n-1}\, dz' && C \text{ in R.O.C.} \\ &= \sum_k \operatorname{Res}\left(H(z)\, z^{n-1}, z_k\right) \end{aligned}}$$

① a power series representation
of a function $f(z)$ of a complex variable z

② a transform $H(z)$ of a sequence of 1

$$X(z) = \frac{z}{z - \frac{1}{2}} \qquad \text{pole } z_0 = \frac{1}{2}$$

$$x[n] = \mathrm{Res}\left(X(z) z^{n-1},\ z_0\right) = \mathrm{Res}\left(\frac{z}{z-\frac{1}{2}} z^{n-1},\ \frac{1}{2}\right)$$

$$= \mathrm{Res}\left(\frac{z^n}{z-\frac{1}{2}},\ \frac{1}{2}\right) = \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{z^n}{z - \frac{1}{2}} = \left(\frac{1}{2}\right)^n$$

$$x[n] = \frac{1}{2^n} \qquad n \geq 0$$

$$\left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \left(\frac{1}{2}\right)^3 z^{-3} + \cdots = \frac{1}{1 - \left(\frac{1}{2} z^{-1}\right)}$$

$$= \frac{z}{z - \frac{1}{2}}$$