

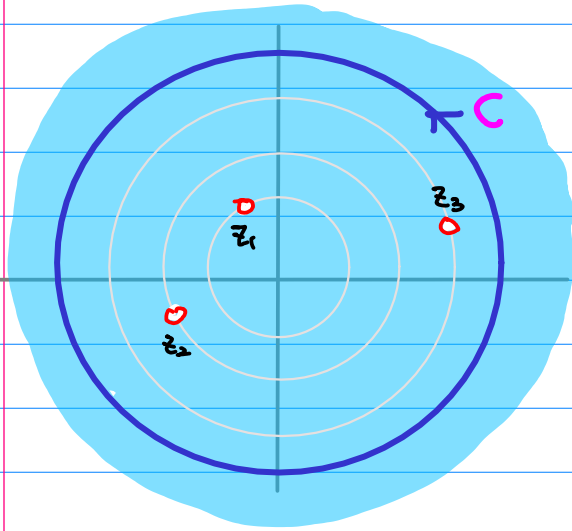
Laurent Series and Geometric Series

20170802

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Series Expansion at $z=0$



$$f(z) = \sum_{n=\eta_1}^{\infty} a_n^{(mf)} z^n$$

$$a_n^{(mf)} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz$$
$$= \sum_k \operatorname{Res} \left(\frac{f(z)}{z^{n+1}}, z_k \right)$$

Poles z_k

$$n \geq 0 \quad z_1, z_2, z_3, \circ$$

$$n < 0 \quad z_1, z_2, z_3$$

* General Series Expansion at $z=0$

$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n$$

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz \\ &= \sum_k \operatorname{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right) \end{aligned}$$

* z -transform

$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

$$\begin{aligned} x_n &= \frac{1}{2\pi i} \oint_C X(z) z^{n+1} dz \\ &= \sum_k \operatorname{Res}(X(z) z^{n+1}, z_k) \end{aligned}$$

z -Transform $X(z)$
Laurent Series $f(z)$

z -Transform $X(z) \longleftrightarrow x_n$
Laurent Series $f(z) \longleftrightarrow a_n$

$$X(z) = f(z^{-1}) \longleftrightarrow x_n = a_n$$

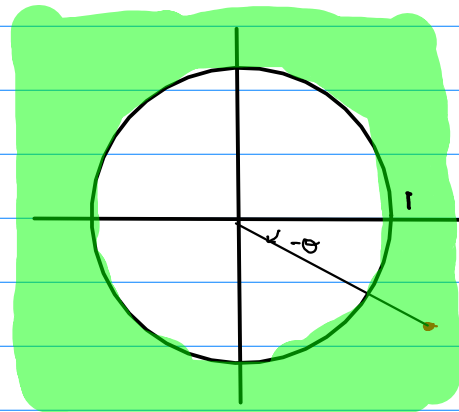
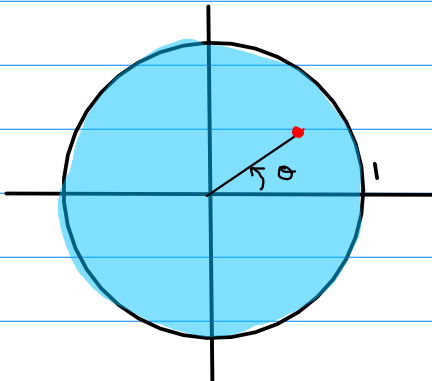
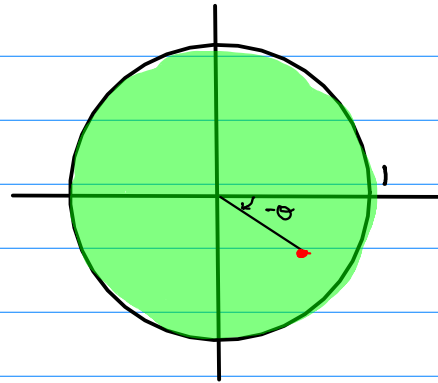
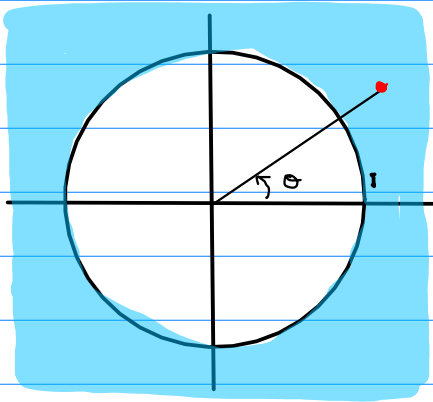
z -Transform $X(z) \longleftrightarrow x_n$
Laurent Series $f(z) \longleftrightarrow a_n$

$$X(z) = f(z) \longleftrightarrow x_n = a_{-n}$$

Mapping $w = \frac{1}{z}$

$$z = \rho e^{j\theta}$$

$$z^{-1} = \frac{1}{\rho} e^{-j\theta}$$



- inverse magnitude
- negative phase

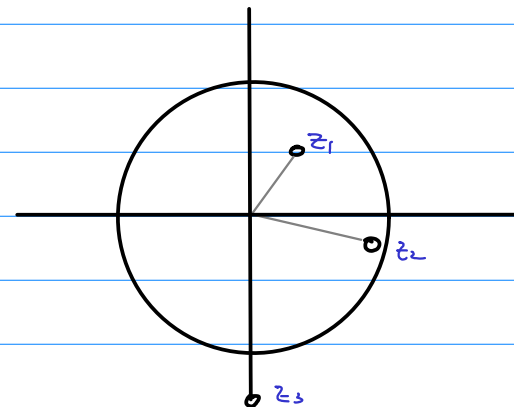
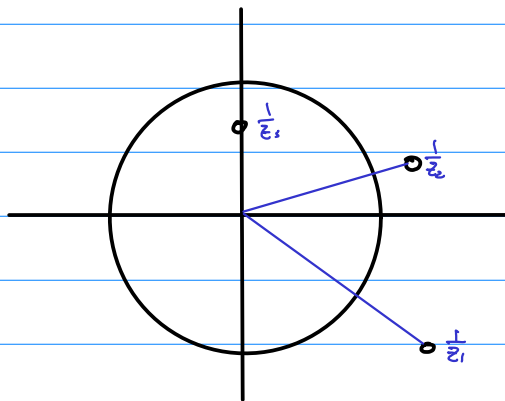
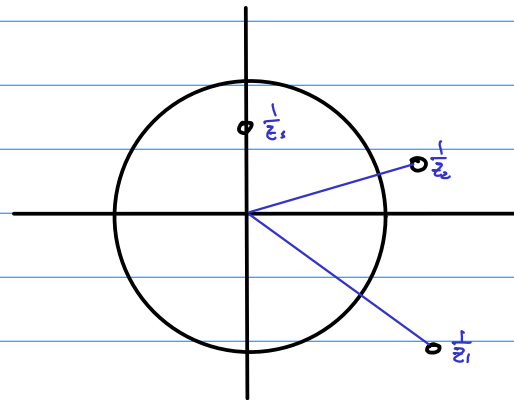
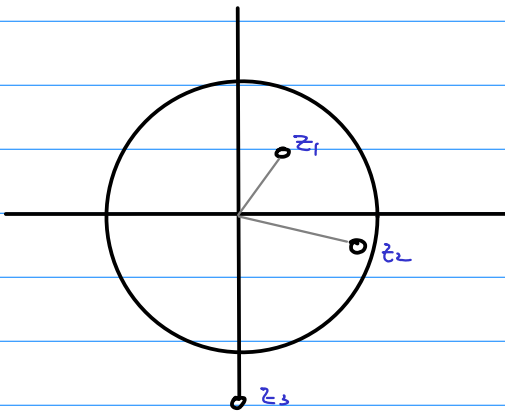
$$f(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)(z - p_3)}$$

$$f(z^{-1}) = \frac{(\frac{1}{z} - z_1)(\frac{1}{z} - z_2)}{(\frac{1}{z} - p_1)(\frac{1}{z} - p_2)(\frac{1}{z} - p_3)}$$

$$= \frac{(1 - z_1 z)(1 - z_2 z)}{(1 - p_1 z)(1 - p_2 z)(1 - p_3 z)}$$

$$\frac{1}{z_1}, \frac{1}{z_2}$$

$$\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}$$



$g(z)$ with a simple pole
 $b > 0$ assumed

$$g(z) = \frac{1}{1-bz} = \frac{b^{-1}}{b^{-1}-z} \quad |bz| < 1 \quad \text{green circle} \quad |z| < \frac{1}{b}$$

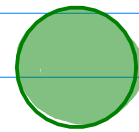
$$h(z) = \frac{1}{1-\frac{b}{z}} = \frac{z}{z-b} \quad \left|\frac{b}{z}\right| < 1 \quad \text{orange square} \quad |z| > b$$

$$g(z^{-1}) = \frac{b^{-1}}{b^{-1}-z^{-1}} = \frac{z}{z-b} = h(z)$$

$$h(z^{-1}) = \frac{z^{-1}}{z^{-1}-b} = \frac{b^{-1}}{b^{-1}-z} = g(z)$$

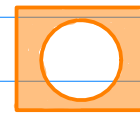
$$\boxed{\begin{aligned} g(z^{-1}) &= h(z) \\ h(z^{-1}) &= g(z) \end{aligned}}$$

$$g(z) = \frac{b^{-1}}{b^{-1} - z} = \frac{\circ}{\circ - z}$$



$|z| < \frac{1}{b}$

$$h(z) = \frac{z}{z - b} = \frac{z}{z - \square}$$



$|z| > b$

$$g(z^{-1}) = \frac{b^{-1}}{b^{-1} - z^{-1}} = \frac{z}{z - b} = h(z)$$

$$\frac{\circ}{\circ - z^{-1}} = \frac{z}{z - \square}$$

$$h(z^{-1}) = \frac{z^{-1}}{z^{-1} - b} = \frac{b^{-1}}{b^{-1} - z} = g(z)$$

$$\frac{z^{-1}}{z^{-1} - \square} = \frac{\circ}{\circ - z}$$

$\circ = b^{-1}$

$\square = b$

$\circ^{-1} = \square$

$\square^{-1} = \circ$

Infinite Sum of G.P.

Simple pole \Rightarrow

$$\frac{\text{cloud}}{z - \square} \Rightarrow \frac{z}{z - \square} \Rightarrow \frac{1}{1 - \frac{\square}{z}} \quad \text{infinite sum of G.P.}$$

$$\frac{\text{cloud}}{\Delta - z} \Rightarrow \frac{\circ}{\circ - z} \Rightarrow \frac{1}{1 - \frac{z}{\circ}} \quad \text{infinite sum of G.P.}$$

Convergence Condition

$$\frac{b^{-1}}{b^{-1} - z} \Rightarrow \text{think this way} \quad b^{-1} - z > 0 \quad |z| > |b^{-1}| \quad \text{pole } b^{-1}$$

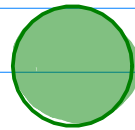
$$\frac{b^{-1}}{b^{-1} - z} = \frac{\circ}{\circ - z} \quad \text{green circle} \quad |z| < \frac{1}{b} \quad |z| < p$$

$$\frac{z}{z - b} \Rightarrow \text{think this way} \quad z - b > 0 \quad |z| > b \quad \text{pole } b$$

$$\frac{z}{z - b} = \frac{z}{z - \square} \quad \text{orange square} \quad |z| > b \quad |z| > p$$

Two Sequences are involved (causal, anti-causal)

$$\frac{b^{-1}}{b^{-1} - z} = \frac{\circ}{\circ - z}$$



$|z| < b \quad |z| < p$

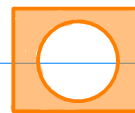
positive seq

① $(n \geq 0) \quad (bz)^0 + (bz)^1 + (bz)^2 + \dots = \sum_{n=0}^{\infty} b^n z^n \quad \text{L.S.}$

negative seq

② $(n \leq 0) \quad (b^{-1}z^{-1})^0 + (b^{-1}z^{-1})^1 + (b^{-1}z^{-1})^2 + \dots = \sum_{n=0}^{-\infty} b^{-n} z^{-n} \quad \text{z.T.}$

$$\frac{z}{z - b} = \frac{z}{z - \square}$$



$|z| > b \quad |z| > p$

positive seq

① $(n \geq 0) \quad (bz^{-1})^0 + (bz^{-1})^1 + (bz^{-1})^2 + \dots = \sum_{n=0}^{\infty} b^n z^{-n} \quad \text{z.T.}$

negative seq

② $(n \leq 0) \quad (b^{-1}z)^0 + (b^{-1}z)^1 + (b^{-1}z)^2 + \dots = \sum_{n=0}^{-\infty} b^{-n} z^n \quad \text{L.S.}$

$$\boxed{n \geq 0} \quad \boxed{n \leq 0}$$

$$\boxed{\text{L.S.}} \quad \boxed{\text{z.T.}}$$

$$\binom{}{}^0 + \binom{}{}^1 + \binom{}{}^2 + \dots \longrightarrow (n \geq 0)$$

$$\binom{}{}^0 + \binom{}{}^{-1} + \binom{}{}^{-2} + \dots \longrightarrow (n \leq 0)$$

$$\sum \text{☁} \boxed{z^n} \longrightarrow \text{L.S.}$$

$$\sum \text{☁} \boxed{z^{-n}} \longrightarrow \text{z.T.}$$

$$(b z)^{-1} = (b^{-1} z^{-1})^{-1}$$

$(n \geq 0)$
$(b z)^n$
$b^n z^n$
$n=1, 2, \dots$

L.S.

$(n \leq 0)$
$(b^{-1} z^{-1})^n$
$b^{-n} z^{-n}$
$n=-1, -2, \dots$

z.T.

$$(b z^{-1})^{-1} = (b^{-1} z)^{-1}$$

$(n \geq 0)$
$(b z^{-1})^n$
$b^n z^{-n}$
$n=1, 2, \dots$

z.T.

$(n \leq 0)$
$(b^{-1} z)^n$
$b^{-n} z^n$
$n=-1, -2, \dots$

L.S.

$$(bz) = (b^{-1}z^{-1})^{-1}$$

	$(n \geq 0)$	=	$(n \leq 0)$	
L.S.	$\sum_{n=1, 2, \dots} b^n z^n$	=	$\sum_{n=-1, -2, \dots} b^{-n} z^{-n}$	Z.T.
	$ bz \leq 1$	=	$ b^{-1}z^{-1} \leq 1$	

$$(bz^{-1}) = (b^{-1}z)^{-1}$$

	$(n \geq 0)$	=	$(n \leq 0)$	
Z.T.	$\sum_{n=1, 2, \dots} b^n z^{-n}$	=	$\sum_{n=-1, -2, \dots} b^{-n} z^n$	L.S.
	$ bz^{-1} \leq 1$	=	$ b^{-1}z \leq 1$	

$(n \geq 0)$

$$b z^n$$

$$b^n z^n$$

$$\left| \frac{z}{b} \right| < 1$$

$$\boxed{\frac{b^{-1}}{b^{-1} - z}} \quad \text{●}$$

$(n \leq 0)$

$$\left(b^{-1} z^{-1} \right)^{-1}$$

$$b^{-n} z^{-n}$$

$$\left| \frac{z}{b} \right| < 1$$

$$\boxed{\frac{b^{-1}}{b^{-1} - z}} \quad \text{●}$$

$k \geq 0$

$n \leq 0$

$(n \geq 0)$

$$b z^{-1}$$

$$b^n z^{-n}$$

$$\left| \frac{b}{z} \right| < 1$$

$$\boxed{\frac{z}{z - b}} \quad \text{○}$$

$(n \leq 0)$

$$\left(b^{-1} z \right)^{-1}$$

$$b^{-n} z^n$$

$$\left| \frac{b}{z} \right| < 1$$

$$\boxed{\frac{z}{z - b}} \quad \text{○}$$

$k \geq 0$

$n \leq 0$

	$\frac{0}{0 - z}$	$\frac{z}{z - a}$
	pole $p = 0$	pole $p = a$
	c.r $\left(\frac{z}{0}\right)$	c.r $\left(\frac{a}{z}\right)$
	r.o.c $ z < 0$	r.o.c $ z > a$
$(n \geq 0)$	$\sum_{n=0}^{\infty} \left(\frac{z}{0}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{0}\right)^n z^n$	$\sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \sum_{n=0}^{\infty} a^n z^{-n}$
$(n \leq 0)$	$\sum_{n=0}^{-\infty} \left(\frac{z}{0}\right)^{-n} = \sum_{n=0}^{-\infty} 0^n z^{-n}$	$\sum_{n=0}^{-\infty} \left(\frac{a}{z}\right)^{-n} = \sum_{n=0}^{-\infty} \left(\frac{1}{a}\right)^n z^n$
	L.S: $b^n z^n \quad (n \geq 0)$	Z.T: $b^n z^{-n} \quad (n \geq 0)$
	Z.T: $b^{-n} z^{-n} \quad (n \leq 0)$	L.S: $b^{-n} z^n \quad (n \leq 0)$
	$p = 0 = b^{-1}$	$p = a = b$

$$\sum_{n=0}^{\infty} ()^n = \sum_{n=0}^{-\infty} ()^{-n}$$

<p style="text-align: right;">L.S.</p> $\sum_{n=0}^{\infty} \left(\frac{z}{0}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{0}\right)^n z^n$ <p style="text-align: center;"> $(n \geq 0)$ $b^n z^n$ </p>	<p style="text-align: right;">Z.T</p> $\sum_{n=0}^{\infty} \left(\frac{\square}{z}\right)^n = \sum_{n=0}^{\infty} \square^n z^{-n}$ <p style="text-align: center;"> $(n \geq 0)$ $b^n z^{-n}$ </p>
<p style="text-align: right;">Z.T</p> $\sum_{n=0}^{-\infty} \left(\frac{z}{0}\right)^{-n} = \sum_{n=0}^{-\infty} 0^n z^{-n}$ <p style="text-align: center;"> $(n \leq 0)$ $b^{-n} z^{-n}$ </p>	<p style="text-align: right;">L.S.</p> $\sum_{n=0}^{-\infty} \left(\frac{\square}{z}\right)^{-n} = \sum_{n=0}^{-\infty} \left(\frac{1}{\square}\right)^n z^n$ <p style="text-align: center;"> $(n \leq 0)$ $b^{-n} z^n$ </p>
<p style="text-align: center;">$p = 0 = b^{-1}$</p>	<p style="text-align: center;">$p = \square = b$</p>

$$n \geq 0$$

$$b^n$$

&

$$n \leq 0$$

$$b^{-n}$$

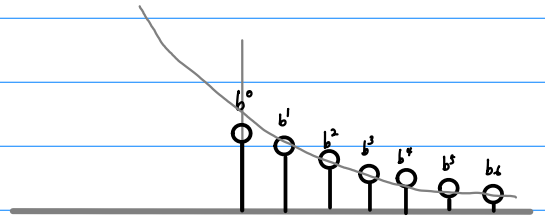
assumed

	$\frac{0}{0 - z}$		$\frac{z}{z - \square}$
$(n \geq 0)$	$\left(\frac{z}{0}\right)^n \leftrightarrow b^n z^n$ <p style="text-align: right;">L.S. (n ≥ 0)</p>		$\left(\frac{\square}{z}\right)^n \leftrightarrow b^n z^{-n}$ <p style="text-align: right;">Z.T. (n ≥ 0)</p>
$(n \leq 0)$	$\left(\frac{z}{0}\right)^{-n} \leftrightarrow b^{-n} z^{-n}$ <p style="text-align: right;">Z.T. (n < 0)</p>		$\left(\frac{\square}{z}\right)^{-n} \leftrightarrow b^{-n} z^n$ <p style="text-align: right;">L.S. (n < 0)</p>
	$p = 0 = b^1$		$p = \square = b$

$$(n \geq 0)$$

$$n$$

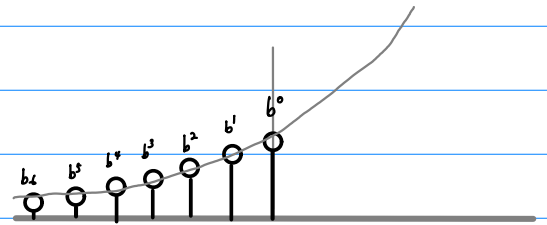
$$b^n$$



$$(n \leq 0)$$

$$-n$$

$$b^{-n}$$



$$\left(\frac{z}{0}\right)^n, \left(\frac{z}{0}\right)^{-n}, \left(\frac{\square}{z}\right)^n, \left(\frac{\square}{z}\right)^{-n}$$

p

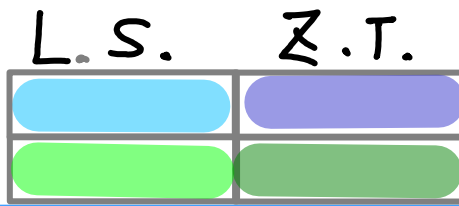
$\frac{z^+}{0^-} \quad \frac{z}{p} \rightarrow \text{L.S.}$ <p style="text-align: center;">↕</p> $(n \geq 0) \quad p^{-n} z^n \text{ L.S.}$ $(n < 0) \rightarrow b^n z^n \text{ L.S.}$ $p = 0 = b^{-1}$	$\frac{\square^+}{z^-} \quad \frac{p}{z} \rightarrow \text{z.T.}$ <p style="text-align: center;">↕</p> $(n \geq 0) \quad p^n z^{-n} \text{ z.T.}$ $(n < 0) \rightarrow b^n z^{-n} \text{ z.T.}$ $p = \square = b$
$\frac{0^+}{z^-} \quad \frac{p}{z} \rightarrow \text{z.T.}$ <p style="text-align: center;">↕</p> $(n \leq 0) \quad p^n z^{-n} \text{ z.T.}$ $(n < 0) \rightarrow b^{-n} z^{-n} \text{ z.T.}$ $p = 0 = b^{-1}$	$\frac{z^+}{\square^-} \quad \frac{z}{p} \rightarrow \text{L.S.}$ <p style="text-align: center;">↕</p> $(n \leq 0) \quad p^{-n} z^n \text{ L.S.}$ $(n < 0) \rightarrow b^{-n} z^n \text{ L.S.}$ $p = \square = b$

$\frac{z^+}{p^-}$	$\frac{p^+}{z^-}$
-------------------	-------------------

L.S.
z.T.

$p = 0 = \square^{-1}$	$p = \square = 0^{-1}$
------------------------	------------------------

$(n \geq 0)$
 $(n \leq 0)$



L.S.

Z.T.

$$\frac{z^+}{O^-}$$

$$\frac{z}{p}$$

$$\frac{\square^+}{z^-}$$

$$\frac{p}{z}$$

$(n \geq 0)$

$(n \geq 0)$
 $(n \geq 0)$

$p^{-n} z^n$ L.S.
 $b^n z^n$ L.S.

$$p = O = b^{-1}$$

$(n \geq 0)$
 $(n \geq 0)$

$p^n z^{-n}$ Z.T.
 $b^n z^{-n}$ Z.T.

$$p = \square = b$$

$$\frac{z^+}{\square^-}$$

$$\frac{z}{p}$$

$$\frac{O^+}{z^-}$$

$$\frac{p}{z}$$

$(n \leq 0)$

$(n \leq 0)$
 $(n \leq 0)$

$p^{-n} z^n$ L.S.
 $b^{-n} z^n$ L.S.

$$p = \square = b$$

$(n \leq 0)$
 $(n \leq 0)$

$p^n z^{-n}$ Z.T.
 $b^{-n} z^{-n}$ Z.T.

$$p = O = b^{-1}$$

L.S.

 $p^{-n} z^n$

Z.T.

 $p^n z^{-n}$ $(n \geq 0)$

$$\left(\frac{z}{\circ}\right)^n$$

$$\left(\frac{1}{\circ}\right)^n z^n$$

$$\left(\frac{\square}{z}\right)^n$$

$$\square^n z^{-n}$$

$$\frac{\circ}{\circ - z}$$

$$p^{-n} z^n$$

$$\frac{z}{z - \square}$$

$$p^n z^{-n}$$

 z^0, z^1, z^2, \dots $z^0, z^{-1}, z^{-2}, \dots$ $(n \leq 0)$

$$\left(\frac{\square}{z}\right)^{-n}$$

$$\left(\frac{1}{\square}\right)^n z^n$$

$$\left(\frac{z}{\circ}\right)^{-n}$$

$$\circ^n z^{-n}$$

$$\frac{z}{z - \square}$$

$$p^{-n} z^n$$

$$\frac{\circ}{\circ - z}$$

$$p^n z^{-n}$$

 $z^0, z^{-1}, z^{-2}, \dots$ z^0, z^1, z^2, \dots $(n \geq 0)$

$$p = \circ = b^{-1}$$

 $(n \geq 0)$

$$p = \square = b$$

 $(n \leq 0)$

$$p = \square = b$$

 $(n \leq 0)$

$$p = \circ = b^{-1}$$

L.S. $\circ = \square^{-1} p^{-n} z^n$

Z.T. $\circ = \square^{-1} p^n z^{-n}$

$(n \geq 0)$

$$\left(\frac{z}{\square^{-1}}\right)^n$$

$$\left(\frac{1}{\square^{-1}}\right)^n z^n$$

$$\left(\frac{\square^{-1}}{z}\right)^n$$

$$\square^{-n} z^{-n}$$

$$\frac{\square^{-1}}{\square^{-1} - z}$$

$$p^{-n} z^n$$

$$\frac{z}{z - \square^{-1}}$$

$$p^n z^{-n}$$

$(n \leq 0)$

$$\left(\frac{\square^{-1}}{z}\right)^{-n}$$

$$\left(\frac{1}{\square^{-1}}\right)^{-n} z^{-n}$$

$$\left(\frac{z}{\square^{-1}}\right)^{-n}$$

$$\square^{-n} z^{-n}$$

$$\frac{z}{z - \square^{-1}}$$

$$p^{-n} z^n$$

$$\frac{\square^{-1}}{\square^{-1} - z}$$

$$p^n z^{-n}$$

$(n \geq 0)$

$$p = \circ = b^{-1}$$

$(n \geq 0)$

$$p = \square = b$$

$(n \leq 0)$

$$p = \square = b$$

$(n \leq 0)$

$$p = \circ = b^{-1}$$

b & p with \sum notations

L.S.

Z.T.

$(n \geq 0)$	$\sum_{n=0}^{\infty} \left(\frac{z}{\circ}\right)^n = \sum_{n=0}^{\infty} p^{-n} z^n$ $= \sum_{n=0}^{\infty} b^n z^n$	$\sum_{n=0}^{\infty} \left(\frac{\square}{z}\right)^n = \sum_{n=0}^{\infty} p^n z^{-n}$ $= \sum_{n=0}^{\infty} b^n z^{-n}$
$(n \leq 0)$	$\sum_{n=0}^{-\infty} \left(\frac{\square}{z}\right)^{-n} = \sum_{n=0}^{-\infty} p^{-n} z^n$ $= \sum_{n=0}^{-\infty} b^{-n} z^n$	$\sum_{n=0}^{-\infty} \left(\frac{z}{\circ}\right)^{-n} = \sum_{n=0}^{-\infty} p^n z^{-n}$ $= \sum_{n=0}^{-\infty} b^{-n} z^{-n}$
$(n \geq 0)$ $(n \leq 0)$	$p = \circ = b^{-1}$ $p = \square = b$	$(n \geq 0)$ $(n \leq 0)$

L.S.	Z.T.
L.S.	Z.T.

$(n \geq 0)$	$(n \geq 0)$
$(n \leq 0)$	$(n \leq 0)$

p^{-n}	p^n
p^{-n}	p^n

b^n	b^n
b^{-n}	b^{-n}

$p = \bigcirc$	$p = \square$
$p = \square$	$p = \bigcirc$

$\bigcirc = b^{-1}$	$\square = b$
$\square = b$	$\bigcirc = b^{-1}$

\square^n	\square^n
\square^{-n}	\square^{-n}

\square^n	\square^n
\square^{-n}	\square^{-n}

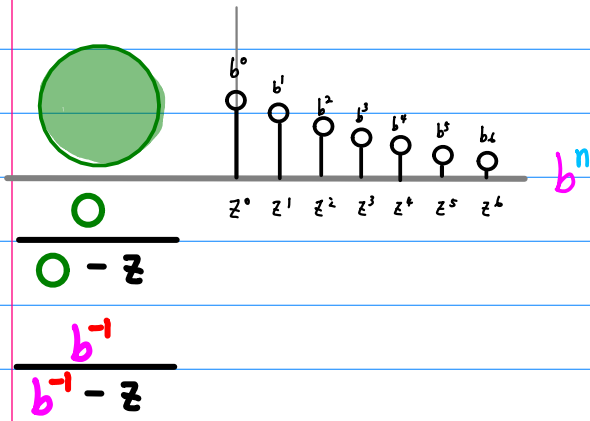
$\frac{\bigcirc}{\bigcirc - z}$	$\frac{z}{z - \square}$
$\frac{z}{z - \square}$	$\frac{\bigcirc}{\bigcirc - z}$

$\frac{\square^{-1}}{\square^{-1} - z}$	$\frac{z}{z - \square}$
$\frac{z}{z - \square}$	$\frac{\square^{-1}}{\square^{-1} - z}$

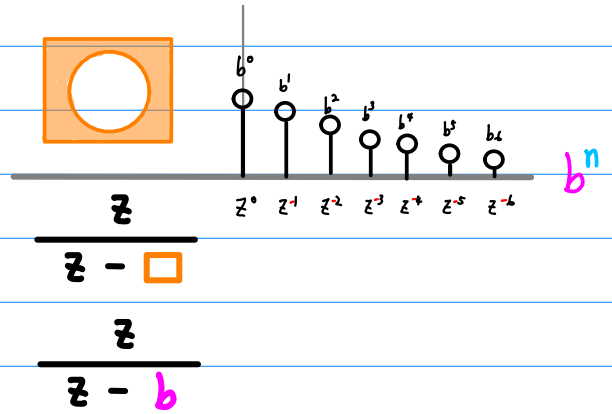
L.S.: $a_n z^n$

Z.T.: $x_n z^{-n}$

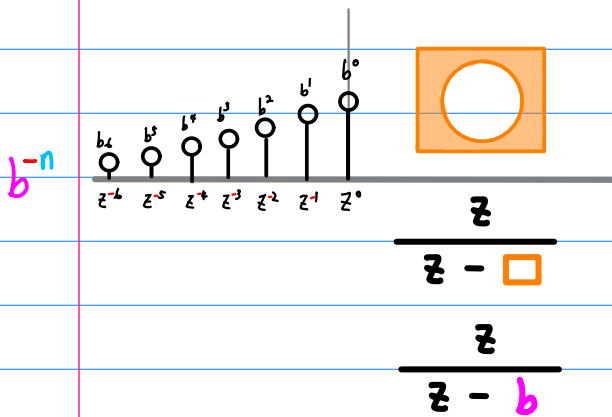
$p^{-n} z^n$



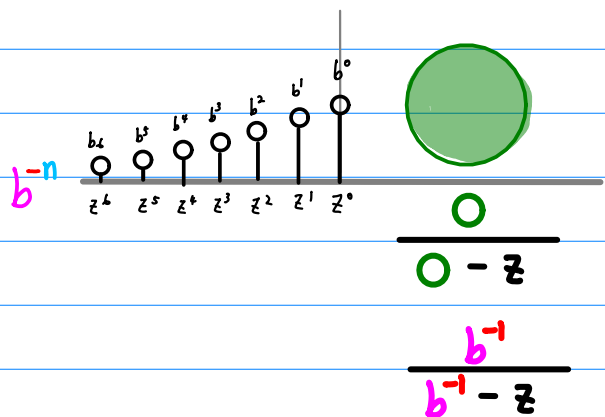
$p^n z^{-n}$



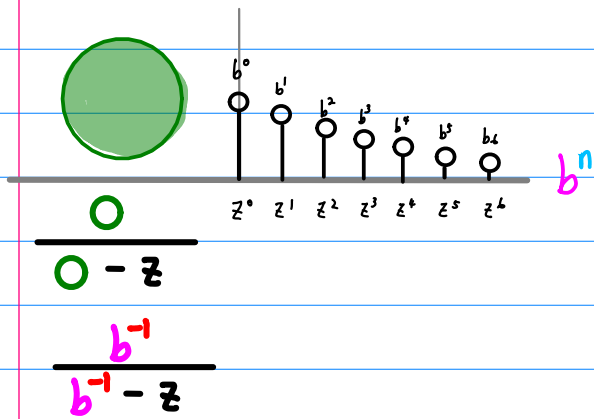
$p^{-n} z^n$



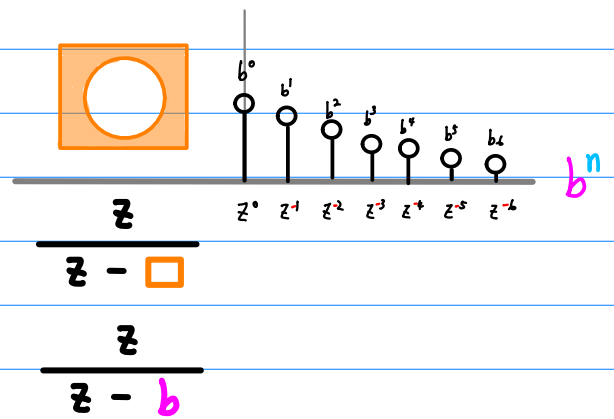
$p^n z^{-n}$



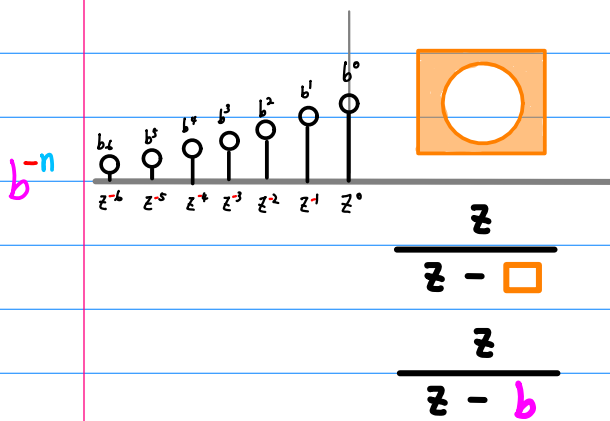
$$b^n z^n$$



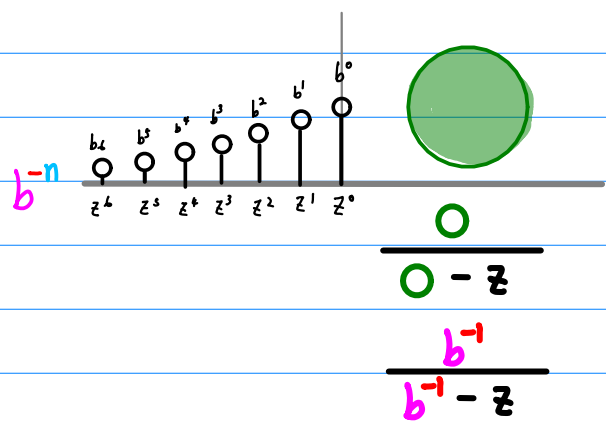
$$b^n z^{-n}$$



$$b^{-n} z^n$$



$$b^{-n} z^{-n}$$



b^n & b^{-n}

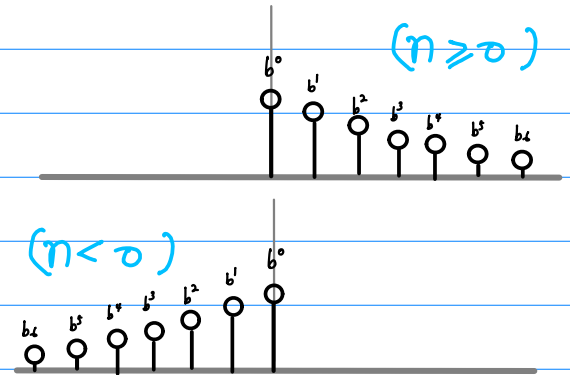
$0 < b < 1$ assumed

$$(n \geq 0) \quad a_n = x_n = b^n$$

$$(n < 0) \quad a_n = x_n = b^{-n}$$

$$a_n = p^{-n} \quad (n \geq 0, n < 0)$$

$$x_n = p^n \quad (n \geq 0, n < 0)$$



a_n Laurent Series Coefficient

x_n input to z -Transform

$$\begin{cases} \text{causal signal} & \begin{cases} x_n = 0 & n < 0 \\ x_n \neq 0 & n \geq 0 \end{cases} \\ \text{anti-causal signal} & \begin{cases} x_n \neq 0 & n < 0 \\ x_n = 0 & n \geq 0 \end{cases} \end{cases}$$

p the simple pole of $f(z)$ or $X(z)$

$$\frac{b^{-1}}{b^{-1} - z}$$

$$p = b^{-1}$$

$$\frac{z}{z - b}$$

$$p = b$$

$$Z.T.: x_n z^{-n}$$

$$L.S.: a_n z^n$$

Z.T.	$z^{-n} \quad (n \geq 0)$	$b^n z^{-n} \quad (n \geq 0)$	+ -
	$z^{-n} \quad (n < 0)$	$b^{-n} z^{-n} \quad (n < 0)$	- -
L.S.	$z^n \quad (n \geq 0)$	$b^n z^n \quad (n \geq 0)$	+ +
	$z^n \quad (n < 0)$	$b^{-n} z^n \quad (n < 0)$	- +

$(n \geq 0)$	Z.T. z^{-n}	Z.T. $b^n z^{-n}$	+ -
	L.S. z^n	L.S. $b^n z^n$	+ +
$(n < 0)$	Z.T. z^{-n}	Z.T. $b^{-n} z^{-n}$	- -
	L.S. z^n	L.S. $b^{-n} z^n$	- +

Laurent Series

$$a_n$$

z-Transform

$$x_n$$

$$a_n = b^n \quad (n \geq 0)$$

$$bz = \frac{z}{p} \quad p = b^{-1}$$

$$\left| \frac{z}{p} \right| < 1 \quad |z| < p$$

$$a_n = p^{-n}$$

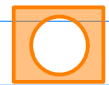


$$x_n = b^n \quad (n \geq 0)$$

$$bz^{-1} = \frac{p}{z} \quad p = b$$

$$\left| \frac{p}{z} \right| < 1 \quad |z| > p$$

$$x_n = p^n$$



$$a_n = b^{-n} \quad (n < 0)$$

$$b^{-1}z = \frac{z}{p} \quad p = b$$

$$\left| \frac{z}{p} \right|^{-1} < 1 \quad |z| > p$$

$$a_n = p^{-n}$$



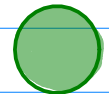
$$(n < 0) \rightarrow (k > 0)$$

$$x_n = b^{-n} \quad (n < 0)$$

$$b^{-1}z^{-1} = \frac{p}{z} \quad p = b^{-1}$$

$$\left| \frac{p}{z} \right|^{-1} < 1 \quad |z| < p$$

$$x_n = p^n$$

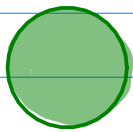


$$(n < 0) \rightarrow (k > 0)$$

Converging Geometric Series

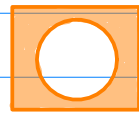
$$\left| \frac{z}{p} \right| < 1$$

$$|z| < p$$



$$\left| \frac{p}{z} \right| < 1$$

$$p < |z|$$



$$\left| \frac{p}{z} \right| > 1$$

$$\frac{1}{1 - \frac{z}{p}} = \frac{p}{p - z}$$

think $p - z > 0$

$$p > |z|$$

$$\left| \frac{z}{p} \right| > 1$$

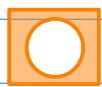
$$\frac{1}{1 - \frac{p}{z}} = \frac{z}{z - p}$$

think $z - p > 0$

$$|z| > p$$

z -Transform

$$\boxed{\frac{p}{z}} = \boxed{b z^{-1}}$$



causal

$$(n \geq 0) \quad p = b$$

$$\boxed{\frac{p}{z}} = \boxed{b^{-1} z^{-1}}$$



anticausal

$$(n < 0) \quad p = b^{-1}$$



$$\frac{z}{p} = b z$$

Laurent Series

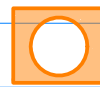
$$\boxed{\frac{z}{p}} = \boxed{b z}$$



$$(n \geq 0)$$

$$p = b^{-1}$$

$$\boxed{\frac{z}{p}} = \boxed{b^{-1} z}$$



$$(n < 0)$$



$$\frac{p}{z} = b z^{-1}$$

$$p = b$$

Z.T.

$$p z^{-1} = b z^{-1} \quad (n \geq 0)$$

$$\sum_k (b z^{-1})^k = \frac{z}{z-b}$$

$$(k = n \geq 0)$$

$$|b z^{-1}| < 1 \quad \square$$

$$p = b$$

$$p z^{-1} = b^{-1} z^{-1} \quad (n < 0)$$

$$\sum_k (b z)^k = \frac{b^{-1}}{b^{-1}-z}$$

$$(k = -n \geq 0)$$

$$|b z| < 1 \quad \bullet$$

$$p = b^{-1}$$

L.S.

$$p^{-1} z = b z \quad (n \geq 0)$$

$$\sum_k (b z)^k = \frac{b^{-1}}{b^{-1}-z}$$

$$(k = n \geq 0)$$

$$|b z| < 1 \quad \bullet$$

$$p = b^{-1}$$

$$p^{-1} z = b^{-1} z \quad (n < 0)$$

$$\sum_k (b z^{-1})^k = \frac{z}{z-b}$$

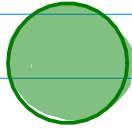
$$(k = -n \geq 0)$$

$$|b z^{-1}| < 1 \quad \square$$

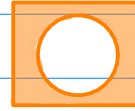
$$p = b$$

L.S.

Z.T.



$$\left| \frac{z}{p} \right| < 1$$



$$\left| \frac{p}{z} \right| < 1$$

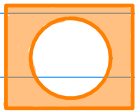
$$n \geq 0$$

$\frac{p}{p-z}$	$\frac{0}{0-z}$	$\frac{b^{-1}}{b^{-1}-z}$
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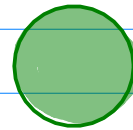
$\frac{z}{z-p}$	$\frac{z}{z-\square}$	$\frac{z}{z-b}$
-----------------	-----------------------	-----------------

$$a_n = p^{-n} = b^n$$

$$x_n = p^n = b^{-n}$$



$$\left| \frac{p}{z} \right| < 1$$



$$\left| \frac{z}{p} \right| < 1$$

$$n < 0$$

$\frac{z}{z-p}$	$\frac{z}{z-\square}$	$\frac{z}{z-b}$
-----------------	-----------------------	-----------------

$\frac{p}{p-z}$	$\frac{0}{0-z}$	$\frac{b^{-1}}{b^{-1}-z}$
-----------------	-----------------	---------------------------

$$a_n = p^{-n} = b^{-n}$$

$$x_n = p^n = b^{-n}$$

$$(n \geq 0)$$

$$a_n = x_n = b^n$$

$$(n < 0)$$

$$a_n = x_n = b^{-n}$$

Laurent Series

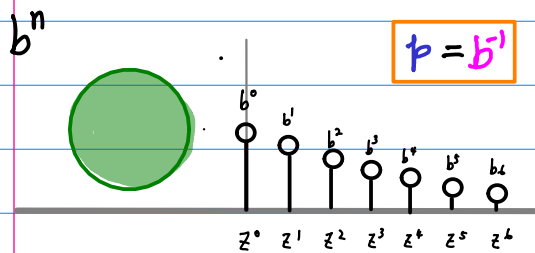
$$a_n = p^{-n} \quad (n \geq 0, n < 0)$$

z-Transform

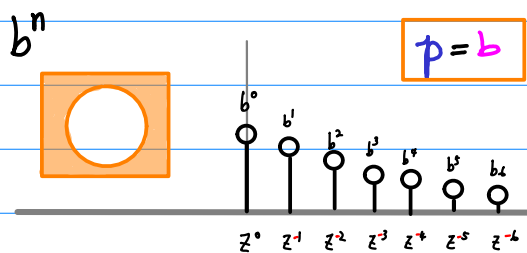
$$x_n = p^n \quad (n \geq 0, n < 0)$$

L.S.

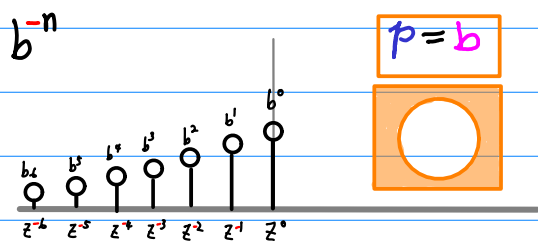
Z.T.



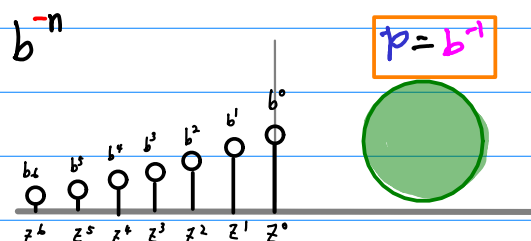
$$a_n = p^{-n} \quad (n \geq 0)$$



$$x_n = p^n \quad (n \geq 0)$$



$$a_n = p^{-n} \quad (n \leq 0)$$



$$x_n = p^n \quad (n \leq 0)$$

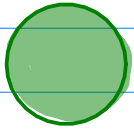
$$\begin{array}{ll} (n \geq 0) & a_n = x_n = b^n \\ (n < 0) & a_n = x_n = b^{-n} \end{array}$$

$$\begin{array}{ll} \text{Laurent Series} & a_n = p^{-n} \quad (n \geq 0, n < 0) \\ \text{z-Transform} & x_n = p^n \quad (n \geq 0, n < 0) \end{array}$$

Laurent Series

Z - Transform

$n \geq 0$



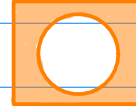
$$|z| < p$$

$$\left| \frac{z}{p} \right| < 1$$

$$\textcircled{1} z^1, z^2, z^3, \dots$$

$$\textcircled{2} \frac{1}{1 - \frac{z}{p}} = \frac{p}{p - z}$$

$$\textcircled{3} a_n = p^{-n} = b^n \quad (p = b^{-1})$$



$$|z| > p$$

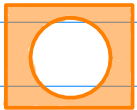
$$\left| \frac{p}{z} \right| < 1$$

$$\textcircled{1} z^{-1}, z^{-2}, z^{-3}, \dots$$

$$\textcircled{2} \frac{1}{1 - \frac{p}{z}} = \frac{z}{z - p}$$

$$\textcircled{3} x_n = p^n = b^n \quad (p = b)$$

$n \leq 0$



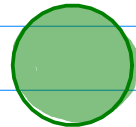
$$|z| > p$$

$$\left| \frac{p}{z} \right| < 1$$

$$\textcircled{1} z^1, z^2, z^3, \dots$$

$$\textcircled{2} \frac{1}{1 - \frac{p}{z}} = \frac{z}{z - p}$$

$$\textcircled{3} a_n = p^{-n} = b^n \quad (p = b)$$



$$|z| < p$$

$$\left| \frac{z}{p} \right| < 1$$

$$\textcircled{1} z^1, z^2, z^3, \dots$$

anti-causal

$$\textcircled{2} \frac{1}{1 - \frac{z}{p}} = \frac{p}{p - z}$$

$$\textcircled{3} x_n = p^n = b^{-n} \quad (p = b^{-1})$$

L. S.

Z. T.

causal

$(n \geq 0)$

$$f(z) = \frac{p}{p-z} = \sum_{n=0}^{\infty} p^{-n} z^n$$

$$= \sum_{n=0}^{\infty} b^n z^n$$

$p = b^{-1}$

$$X(z) = \frac{z}{z-p} = \sum_{n=0}^{\infty} p^n z^{-n}$$

$$= \sum_{n=0}^{\infty} b^n z^{-n}$$

$p = b$

anticausal

$(n \leq 0)$

$$f(z) = \frac{z}{z-p} = \sum_{n=0}^{\infty} p^n z^{-n}$$

$$= \sum_{n=0}^{-\infty} p^{-n} z^n = \sum_{n=0}^{-\infty} b^{-n} z^n$$

$p = b$

$$X(z) = \frac{p}{p-z} = \sum_{n=0}^{\infty} p^{-n} z^n$$

$$= \sum_{n=0}^{-\infty} p^n z^{-n} = \sum_{n=0}^{-\infty} b^{-n} z^{-n}$$

$p = b^{-1}$



$$\begin{aligned}
 a_n &= \left(\frac{1}{2}\right)^n \quad (n \geq 0) \\
 &= p^{-n} \quad (n \geq 0) \quad p=2 \\
 f(z) &= \frac{2}{2-z}
 \end{aligned}$$

$$\begin{aligned}
 x_n &= \left(\frac{1}{2}\right)^n \quad (n \geq 0) \\
 &= p^n \quad (n \geq 0) \quad p=\frac{1}{2} \\
 X(z) &= \frac{z}{z-0.5}
 \end{aligned}$$

$$\begin{aligned}
 a_n &= \left(\frac{1}{2}\right)^{-n} \quad (n \leq 0) \\
 &= p^{-n} \quad (n \leq 0) \quad p=\frac{1}{2} \\
 f(z) &= \frac{z}{z-0.5}
 \end{aligned}$$

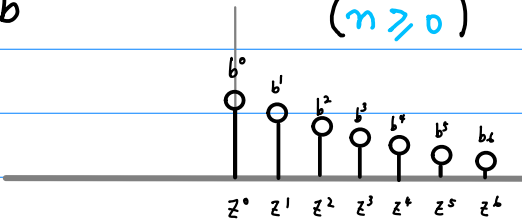
$$\begin{aligned}
 x_n &= \left(\frac{1}{2}\right)^{-n} \quad (n \leq 0) \\
 &= p^n \quad (n \leq 0) \quad p=2 \\
 X(z) &= \frac{2}{2-z}
 \end{aligned}$$

$$\begin{aligned}
 a_n &= b^n \quad (n \geq 0) \\
 &= p^{-n} \quad (n \geq 0) \quad p=b^{-1} \\
 f(z) &= \frac{b^{-1}}{b^{-1}-z}
 \end{aligned}$$

$$\begin{aligned}
 x_n &= b^n \quad (n \geq 0) \\
 &= p^n \quad (n \geq 0) \quad p=b \\
 X(z) &= \frac{z}{z-b}
 \end{aligned}$$

$$\begin{aligned}
 a_n &= b^{-n} \quad (n \leq 0) \\
 &= p^{-n} \quad (n \leq 0) \quad p=b \\
 f(z) &= \frac{z}{z-b}
 \end{aligned}$$

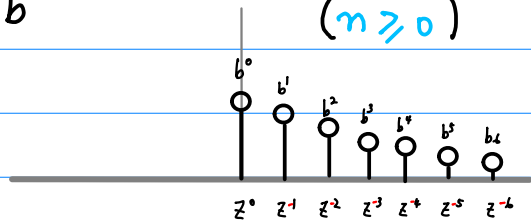
$$\begin{aligned}
 x_n &= b^{-n} \quad (n \leq 0) \\
 &= p^n \quad (n \leq 0) \quad p=b^{-1} \\
 X(z) &= \frac{b^{-1}}{b^{-1}-z}
 \end{aligned}$$

b^n $(n \geq 0)$ 

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - 0.5} \quad |z| < 2$$

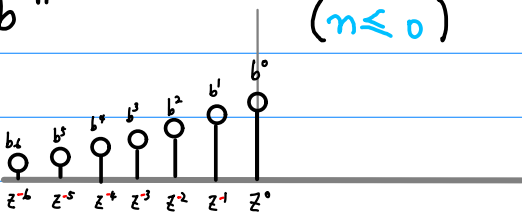
$$f(z) = \frac{z}{2-z} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n$$

$$a_n = \left(\frac{1}{2}\right)^n \\ = p^{-n} \quad \boxed{p=2}$$

 b^n $(n \geq 0)$ 

$$X(z) = \frac{z}{z - 0.5} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} \quad |z| > \frac{1}{2}$$

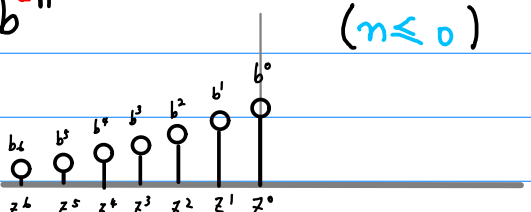
$$x_n = \left(\frac{1}{2}\right)^n \\ = p^n \quad \boxed{p=\frac{1}{2}}$$

 b^{-n} $(n \leq 0)$ 

$$X(z^{-1}) = \frac{z}{2-z^{-1}} \quad |z| > \frac{1}{2}$$

$$f(z) = \frac{z}{z - 0.5} = \sum_{n=-\infty}^0 \left(\frac{1}{2}\right)^{-n} z^n \\ = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

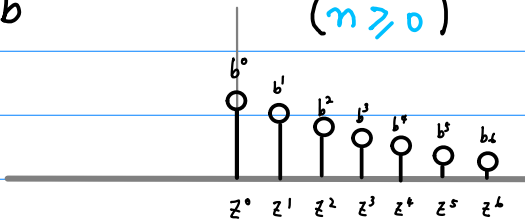
$$a_n = \left(\frac{1}{2}\right)^{-n} \\ = p^{-n} \quad \boxed{p=\frac{1}{2}}$$

 b^{-n} $(n \leq 0)$ 

$$X(z) = \frac{z}{2-z} = \sum_{n=-\infty}^0 \left(\frac{1}{2}\right)^{-n} z^{-n} \\ = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n \quad |z| < 2$$

$$x_n = \left(\frac{1}{2}\right)^{-n} \\ = p^n \quad \boxed{p=2}$$

$$b^n \quad (n \geq 0)$$

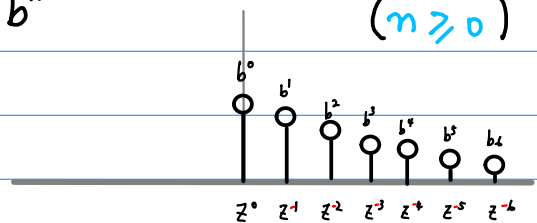


$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - b} \quad |z| < b^{-1}$$

$$f(z) = \frac{b^{-1}}{b^{-1} - z} = \sum_{n=0}^{\infty} b^n z^n$$

$$a_n = b^n \\ = p^{-n} \quad \boxed{p = b^{-1}}$$

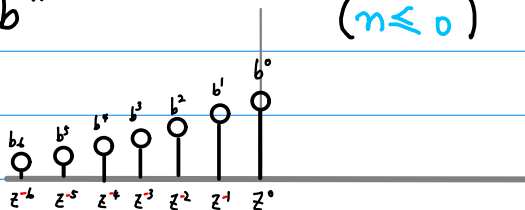
$$b^n \quad (n \geq 0)$$



$$X(z) = \frac{z}{z - b} = \sum_{n=0}^{\infty} b^n z^{-n} \quad |z| > b$$

$$x_n = b^n \\ = p^n \quad \boxed{p = b}$$

$$b^{-n} \quad (n \leq 0)$$

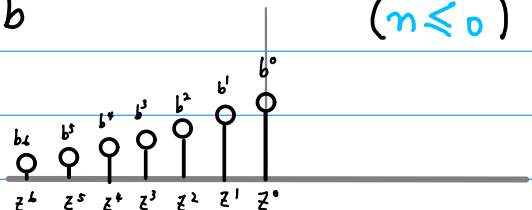


$$X(z^{-1}) = \frac{b^{-1}}{b^{-1} - z^{-1}} \quad |z| > b$$

$$f(z) = \frac{z}{z - b} = \sum_{n=-\infty}^0 b^{-n} z^n \\ = \sum_{n=0}^{\infty} b^n z^{-n}$$

$$a_n = b^{-n} \\ = p^{-n} \quad \boxed{p = b}$$

$$b^{-n} \quad (n \leq 0)$$



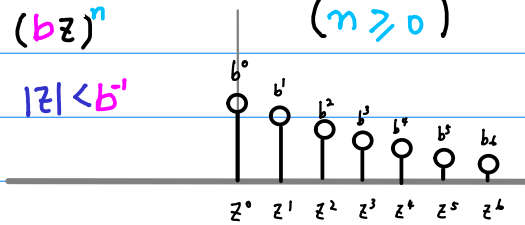
$$X(z) = \frac{b^{-1}}{b^{-1} - z} = \sum_{n=-\infty}^0 b^{-n} z^{-n} \quad |z| < b^{-1}$$

$$= \sum_{n=0}^{\infty} b^n z^n$$

$$x_n = b^{-n} \\ = p^n \quad \boxed{p = b^{-1}}$$

$$(bz)^n \quad (n \geq 0)$$

$$|z| < b^{-1}$$



$$f(z) = \frac{1}{1-bz} = \frac{b^{-1}}{b^{-1}-z}$$

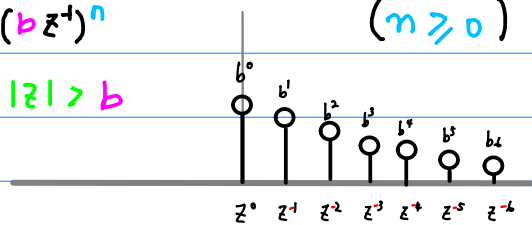
$$a_n = b^n$$

$$= p^{-n}$$

$$p = b^{-1}$$

$$(bz^{-1})^n \quad (n \geq 0)$$

$$|z| > b$$



$$X(z) = \frac{1}{1-b/z} = \frac{z}{z-b}$$

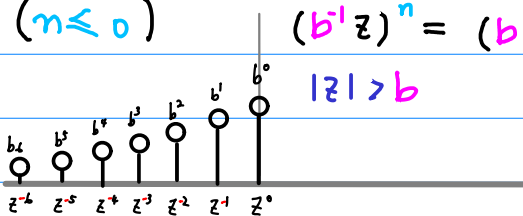
$$x_n = b^n$$

$$= p^n$$

$$p = b$$

$$(n \leq 0) \quad (b^{-1}z)^n = (bz^{-1})^{-n}$$

$$|z| > b$$



$$f(z) = \frac{1}{1-(bz^{-1})} = \frac{z}{z-b}$$

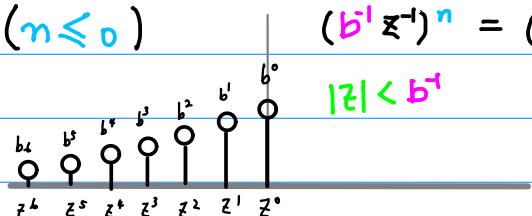
$$a_n = b^{-n}$$

$$= p^{-n}$$

$$p = b$$

$$(n \leq 0) \quad (b^{-1}z^{-1})^n = (bz)^{-n}$$

$$|z| < b^{-1}$$

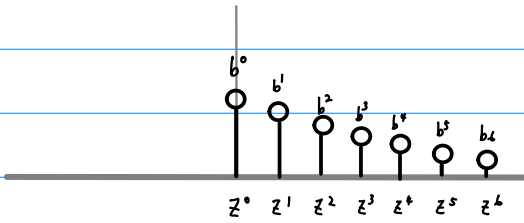


$$X(z) = \frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1}-z}$$

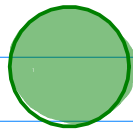
$$x_n = b^{-n}$$

$$= p^n$$

$$p = b^{-1}$$

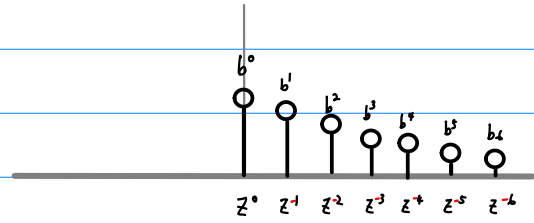


$$f(z) = \sum_{n=0}^{\infty} (bz)^n \quad |bz| < 1$$

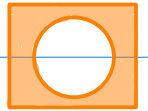


$$a_n = b^n \\ = p^{-n}$$

$$p = b^{-1}$$

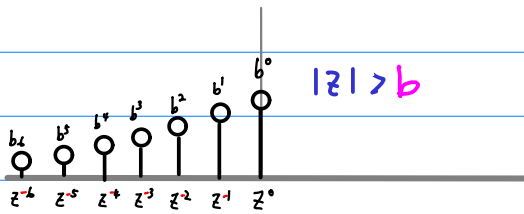


$$X(z) = \sum_{n=0}^{\infty} (bz^{-1})^n \quad |bz^{-1}| < 1$$



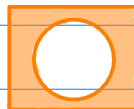
$$x_n = b^n \\ = p^{-n}$$

$$p = b$$



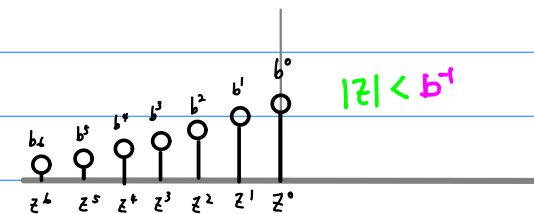
$$f(z) = \sum_{n=-\infty}^0 (bz^{-1})^{-n} \quad |bz^{-1}| < 1$$

$$= \sum_{k=0}^{\infty} (bz^{-1})^k$$



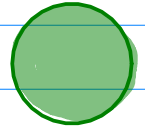
$$a_n = b^{-n} \\ = p^{-n}$$

$$p = b$$



$$X(z) = \sum_{n=-\infty}^0 (bz)^{-n} \quad |bz| < 1$$

$$= \sum_{k=0}^{\infty} (bz)^k$$

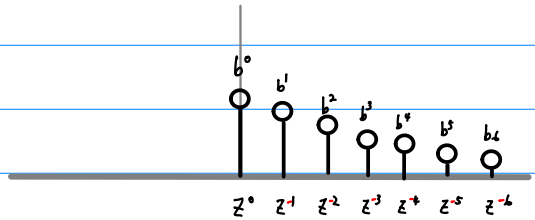
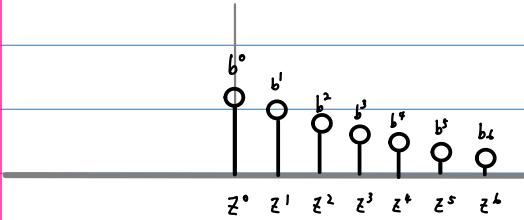


$$x_n = b^{-n} \\ = p^{-n}$$

$$p = b^{-1}$$

$$\frac{1}{1 - (bz)} = \frac{b^{-1}}{z - b^{-1}}$$

$$\frac{1}{1 - (bz^{-1})} = \frac{z}{z - b}$$



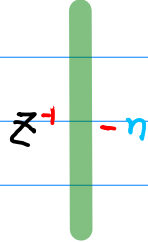
$$p = b^{-1}$$

$$p^{-n}$$

$$b^{-n}$$

$$|bz| < 1$$

$$(bz)^n$$

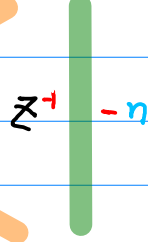


$$z^{-1}$$

$$p^{-n}$$

$$b^{-n}$$

$$(bz^{-1})^n$$



$$p = b$$

$$|bz^{-1}| < 1$$

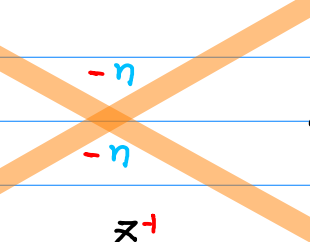
$$|bz^{-1}| < 1$$

$$(bz^{-1})^{-n}$$

$$b^{-n}$$

$$p = b$$

$$p^{-n}$$



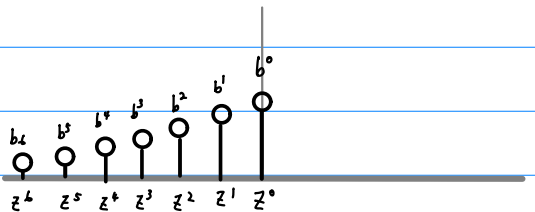
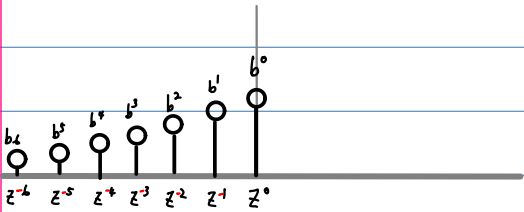
$$(bz)^{-n}$$

$$b^{-n}$$

$$p^{-n}$$

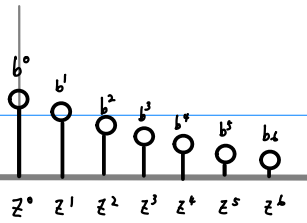
$$|bz| < 1$$

$$p = b^{-1}$$

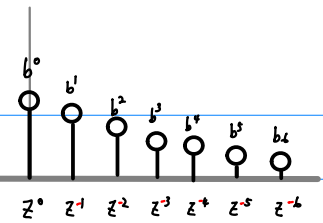


$$\frac{1}{1 - (bz^{-1})} = \frac{z}{z - b}$$

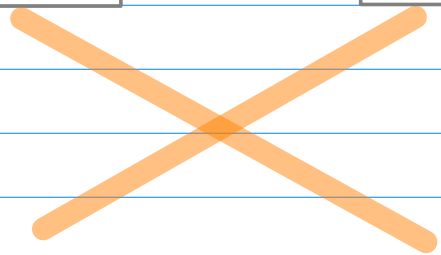
$$\frac{1}{1 - (bz)} = \frac{b^{-1}}{z - b^{-1}}$$



$$\frac{1}{1-(bz)} = \frac{b^{-1}}{z - b}$$

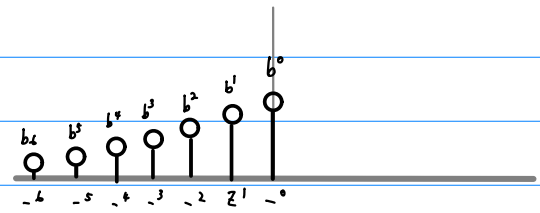
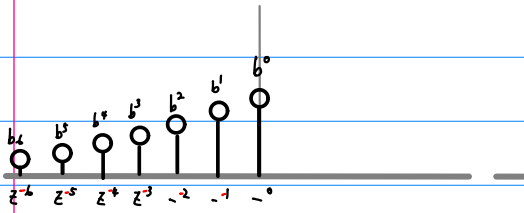


$$\frac{1}{1-(bz^{-1})} = \frac{z}{z - b}$$



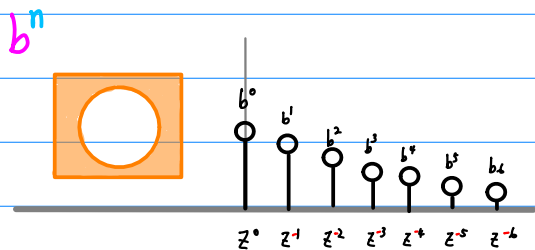
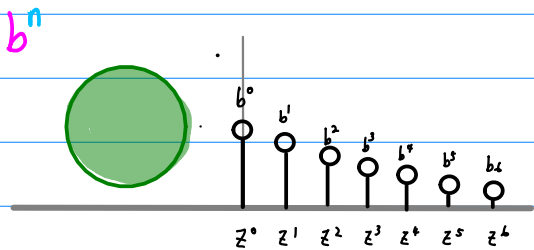
$$\frac{1}{1-(bz^{-1})} = \frac{z}{z - b}$$

$$\frac{1}{1-(bz)} = \frac{b^{-1}}{z - b}$$



$$x_n = a_{-n}$$

$$X(z) = f(z)$$



$$f(z) = \frac{1}{1-(bz)} \quad |z| < b$$

$$X(z) = \frac{1}{1-(b/z)} \quad |z| > b$$

$$a_n = b^n \quad (n \geq 0)$$

$$= p^{-n} \quad (p=b^{-1})$$

$$x_n = b^n \quad (n \geq 0)$$

$$= p^n \quad (p=b)$$

$$f(z) = \frac{1}{1-(b/z)} \quad |z| > b$$

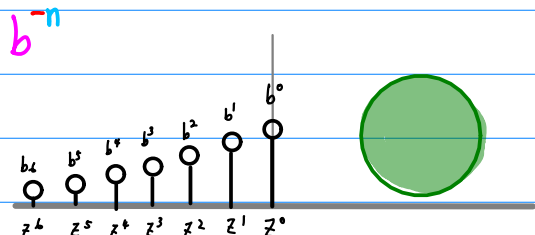
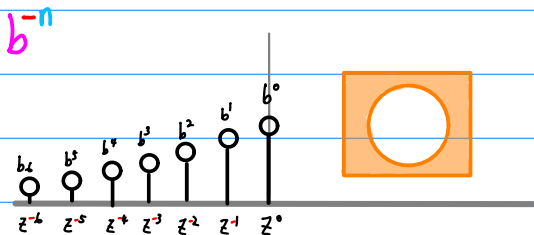
$$X(z) = \frac{1}{1-(bz)} \quad |z| < b$$

$$a_n = b^{-n} \quad (n \leq 0)$$

$$= p^{-n} \quad (p=b)$$

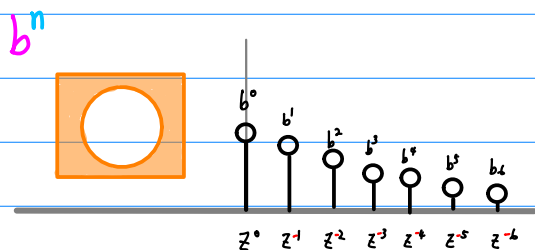
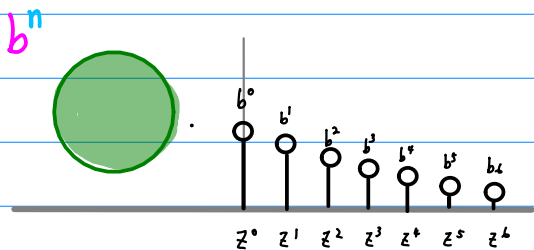
$$x_n = b^{-n} \quad (n \leq 0)$$

$$= p^n \quad (p=b^{-1})$$



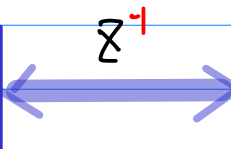
$$x_n = a_n$$

$$X(z) = f(z^{-1})$$



$$f(z) = \frac{1}{1-(bz)} \quad |z| < b^{-1}$$

$$X(z) = \frac{1}{1-(b/z)} \quad |z| > b$$

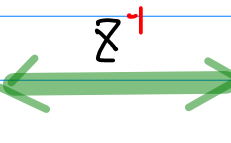


$$a_n = b^n \quad (n \geq 0)$$

$$x_n = b^n \quad (n \geq 0)$$

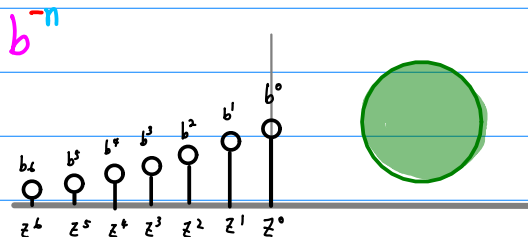
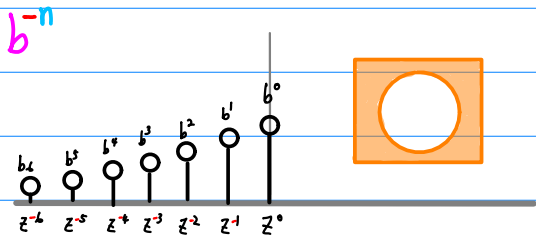
$$f(z) = \frac{1}{1-(b/z)} \quad |z| > b$$

$$X(z) = \frac{1}{1-(bz)} \quad |z| < b^{-1}$$



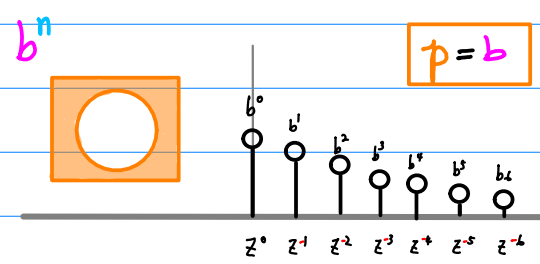
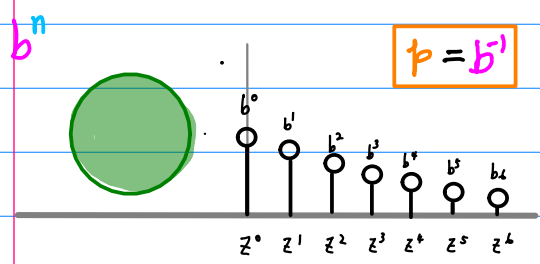
$$a_n = b^{-n} \quad (n \leq 0)$$

$$x_n = b^{-n} \quad (n \leq 0)$$



$$a_n = p^{-n}$$

$$x_n = p^n$$



$$f(z) = \frac{1}{1 - (b/z)} \quad |z| < b^{-1}$$

$$X(z) = \frac{1}{1 - (b/z)} \quad |z| > b$$

$$a_n = p^{-n} \quad (n \geq 0)$$

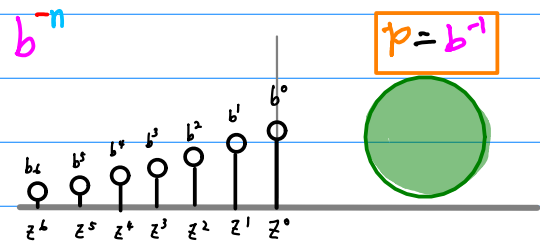
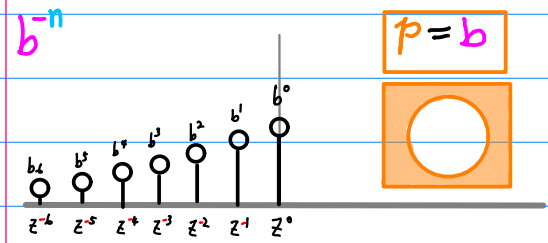
$$x_n = p^n \quad (n \geq 0)$$

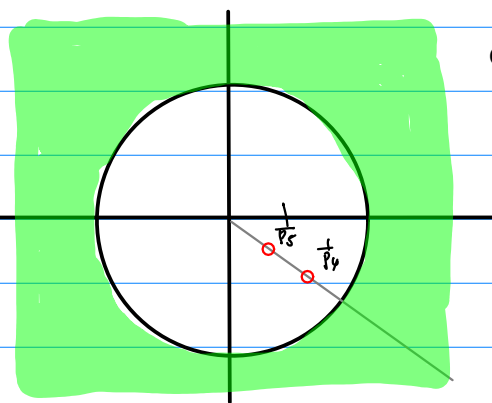
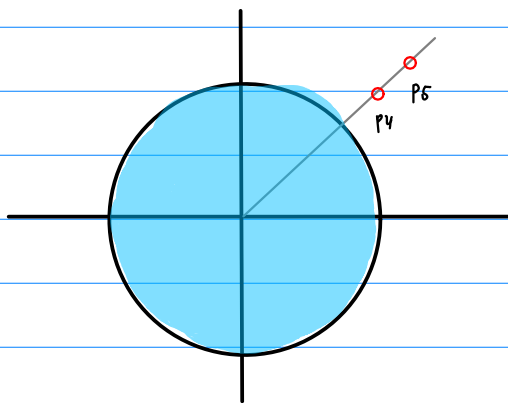
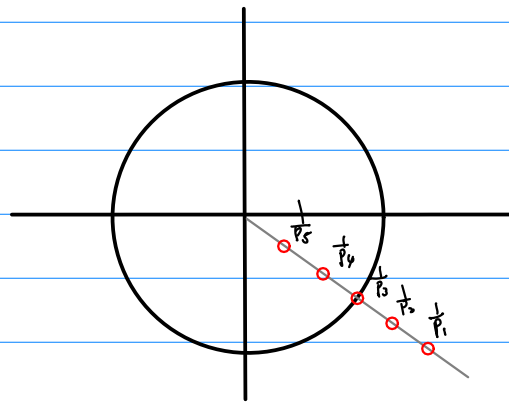
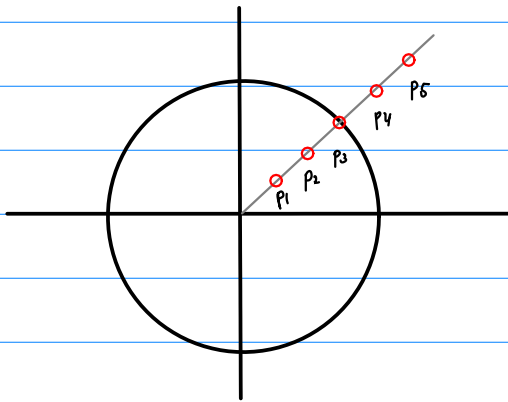
$$f(z) = \frac{1}{1 - (b/z)} \quad |z| > b$$

$$X(z) = \frac{1}{1 - (b/z)} \quad |z| < b^{-1}$$

$$a_n = p^{-n} \quad (n \leq 0)$$

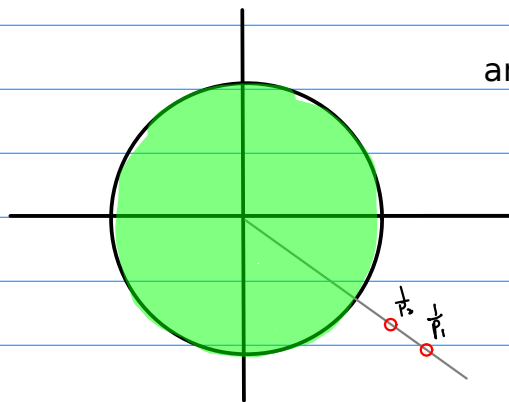
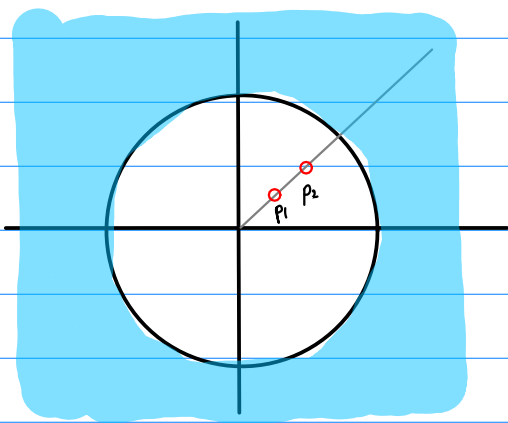
$$x_n = p^n \quad (n \leq 0)$$





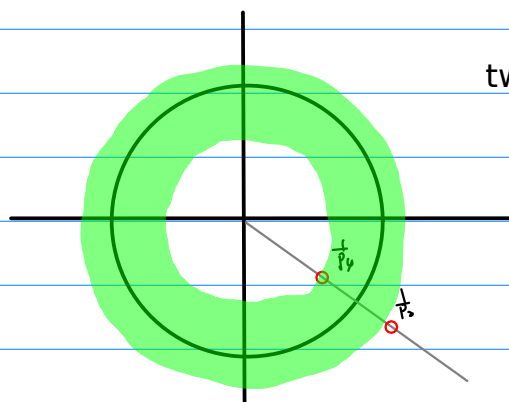
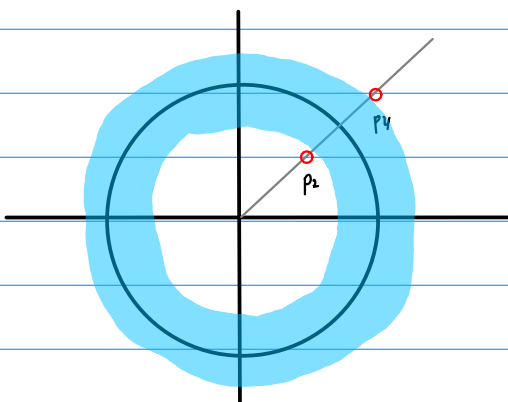
causal

$n \geq 0$



anticausal

$n < 0$



two-sided

