Systems of Linear Differential Equations (H.1)

20150620

Systems of Linear Equations

eq 1 $\quad a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$

eq 2 $\quad a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

eq n $\quad a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

$A x = b$

?

if A^{-1} exists, $x = A^{-1} b$ unique solution

if does not, $\begin{cases} \text{many solutions} \\ \text{no solution} \end{cases}$

$A x = 0$

if A^{-1} exists, $x = A^{-1} 0 = 0$ unique solution

if does not, $\begin{cases} \text{many solutions} \\ \text{no solution} \end{cases}$

1st Order differential Equation

$$\frac{dy}{dt} = g(t, y) \qquad \text{find } y(t)$$

$$\frac{dy}{dx} = g(x, y) \qquad \text{find } y(x)$$

1st Order Linear Differential Equation

$$a_0(t)\boxed{\frac{dy}{dt}} + a_1(t)\boxed{y(t)} = g(t)$$

Systems of Linear Differential Equations

Find $x_1(t),\ x_2(t),\ \ldots,\ x_n(t)$ — n functions of t

1st Order System

$$\frac{dx_1}{dt} = g_1(t, x_1, x_2, \ldots, x_n)$$

$$\frac{dx_2}{dt} = g_2(t, x_1, x_2, \ldots, x_n)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{dx_n}{dt} = g_n(t, x_1, x_2, \ldots, x_n)$$

Linear 1st Order System

known fn.

$$\frac{dx_1}{dt} = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t)$$

$$\frac{dx_2}{dt} = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t)$$

Linear 1st Order System

$$\frac{dx_1}{dt} = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t)$$
$$\frac{dx_2}{dt} = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$\frac{dx_n}{dt} = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t)$$

Linear 1st Order System in a matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$$\dot{x}' = a \cdot x + f \qquad x'(t) = a \cdot x(t) + f(t)$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

$$X' = A \qquad X + F$$

$$\begin{cases} X' = AX + F & \text{non-homogeneous eq.} \\ X' = AX & \text{homogeneous eq.} \end{cases}$$

$$\begin{cases} X' - AX = F \\ X' - AX = 0 \end{cases}$$

Homogeneous Eq

$$\boxed{\dot{X} = \underbrace{A}_{n\times n} X} \leftarrow \begin{array}{l} X_1 : \text{a solution} \\ X_2 : \text{another solution} \\ \vdots \\ X_n : \text{yet another solution} \end{array}$$

linear combition

$$\boxed{X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n}$$

$\rightarrow$ general

$$X_i \Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$x_1(t)$: a function of t

$x_2(t)$: a function of t

$x_n(t)$: a function of t

Non-homogeneous Solution

$$\boxed{\dot{X} = AX + F} \longleftarrow X_p$$

$$X = X_c + X_p$$

$$= \underbrace{\boxed{c_1 X_1 + c_2 X_2 + \cdots + c_n X_n}}_{(= X_c)} + \boxed{X_p}$$

Homogeneous Eq

$$\boxed{\dot{X} = AX}$$

$$\boxed{X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n}$$

Linear Independent ? $x_1, x_2, \dots, x_n$

$$\mathbf{X}_1 = \begin{pmatrix} x_{11}(t) \\ x_{12}(t) \\ \\ x_{1n}(t) \end{pmatrix} \quad \mathbf{X}_2 = \begin{pmatrix} x_{21}(t) \\ x_{22}(t) \\ \\ x_{2n}(t) \end{pmatrix} \quad \mathbf{X}_n = \begin{pmatrix} x_{n1}(t) \\ x_{n2}(t) \\ \\ x_{nn}(t) \end{pmatrix}$$

Wronskian

determinants

$$W(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) = \begin{vmatrix} x_{11}(t) & x_{21}(t) & \cdots & x_{n1}(t) \\ x_{12}(t) & x_{22}(t) & \cdots & x_{n2}(t) \\ \vdots & \vdots & & \vdots \\ x_{1n}(t) & x_{2n}(t) & \cdots & x_{nn}(t) \end{vmatrix}$$

$\neq 0 \Rightarrow$ linearly Independent

Homogeneous Linear Systems

$$\boxed{\dot{X} = AX}$$

$$\boxed{X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n}$$

$y'' + ay' + by = 0$

assume $y = e^{mx}$

$m^2 + am + b = 0$

m_1 , m_2

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

assumption

$$x_1(t) = k_1 e^{\lambda t}$$
$$x_2(t) = k_2 e^{\lambda t}$$
$$\vdots \qquad \vdots$$
$$x_n(t) = k_n e^{\lambda t}$$

the same λ

different k_i

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} k_1 e^{\lambda t} \\ k_2 e^{\lambda t} \\ \vdots \\ k_n e^{\lambda t} \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = K e^{\lambda t}$$

$$\boxed{X = K e^{\lambda t}}$$

Assumption

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} k_1 e^{\lambda t} \\ k_2 e^{\lambda t} \\ \vdots \\ k_n e^{\lambda t} \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = K e^{\lambda t}$$

$$\overset{\text{substitute}}{\downarrow} \qquad \overset{\text{substitute}}{\downarrow}$$

$$\dot{X} = AX$$

$$\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \lambda e^{\lambda t} = \lambda I \underbrace{\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}}_{= K} e^{\lambda t} = A \underbrace{\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}}_{= K} e^{\lambda t}$$

Condition

$$(A - \lambda I)K = 0$$

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \lambda I e^{\lambda t} = \lambda I \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} e^{\lambda t}$$

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

$$\boxed{KI = K}$$

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \lambda$$

$$\boxed{\lambda K I = \lambda K}$$

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \lambda e^{\lambda t} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \lambda I e^{\lambda t} = A \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} e^{\lambda t}$$

$$\therefore I K e^{\lambda t} = A K e^{\lambda t}$$

$a_0 y' + a_1 y =$

$(a_0 m + a_1) = 0$

$a_0 y'' + a_1 y' + a_2 y = x$

$(a_0 m^2 + a_1 m + a_2) = 0$

$$(A - \lambda I) K e^{\lambda t} = 0$$

$$(A - \lambda I) K = 0$$

matrix A의 eigenvalue　　matrix A의 eigenvector

$m = m_1, m_2$

$y = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

aux eq　　char. eq

$m^2 + am + b = 0$　　$(A - \lambda I) K = 0$　　eigen value

m_i →	λ_i
$e^{m_i t}$ →	$K e^{\lambda_i t}$
1.n Linear ODE	system of Linear ODE (n×n)

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

$$X' = A X + F$$

1 Linear Diff Eq	n Linear Diff Eq
$y' + ay' = 0$	$X' - AX = 0$
$y = e^{mx}$ assume	$X = K \cdot e^{\lambda t}$
aux eq	char eq
$m + a = 0$	$(A - \lambda I)K = 0$
	$\det(A - \lambda I) = 0$
m_1	$\lambda_1, \lambda_2, \ldots, \lambda_n$
	$K_1, K_2, \ldots, K_n$
$e^{m_1 t}$	$K_i e^{\lambda_i t}$

Homogeneous Solution

$$\dot{X} = AX$$

assume $X = Ke^{\lambda t}$

$$\begin{bmatrix} k \\ k_2 \\ \\ k_n \end{bmatrix} \lambda e^{\lambda t} = \begin{bmatrix} k_1 \\ k_2 \\ \\ k_n \end{bmatrix} \lambda I e^{\lambda t} = A \begin{bmatrix} k_1 \\ k_2 \\ \\ k_n \end{bmatrix} e^{\lambda t}$$

$$(A - \lambda I)K = 0$$

eigenvalue

eigenvectors

Distinct Eigenvalues

$$\boxed{\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 + \cdots + c_n \mathbf{X}_n}$$

$$\mathbf{X} = c_1 \mathbf{K}_1 e^{\lambda_1 t} + c_2 \mathbf{K}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{K}_n e^{\lambda_n t}$$

$$\mathbf{X}_i = \mathbf{K}_i e^{\lambda_i t}$$

$$\mathbf{X}_i = \begin{pmatrix} x_{i1}(t) \\ x_{i2}(t) \\ \vdots \\ x_{in}(t) \end{pmatrix} = \begin{pmatrix} k_{i1} e^{\lambda_i t} \\ k_{i2} e^{\lambda_i t} \\ \vdots \\ k_{in} e^{\lambda_i t} \end{pmatrix} = \begin{pmatrix} k_{i1} \\ k_{i2} \\ \vdots \\ k_{in} \end{pmatrix} e^{\lambda_i t} = \mathbf{K}_i e^{\lambda_i t}$$

i-th eigenvector

i-th eigenvalue

Repeated Eigenvalues

eigenvalue of multiplicity m

Charateristic eq $= (\lambda-\lambda_1)^m(\lambda-\lambda_2)\cdots$

$$\left.\begin{array}{l}(\lambda-\lambda_1)\\(\lambda-\lambda_1)^2\\ \vdots\\(\lambda-\lambda_1)^m\end{array}\right\}\text{factors}$$

$(\lambda-\lambda_1)^{m+1} \longrightarrow$ ~~factor~~

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assume $\mathbf{X} = \mathbf{K}e^{\lambda t}$

$$(\mathbf{A}-\lambda \mathbf{I})\mathbf{K} = \mathbf{0}$$

to find eigenvalues

$(\mathbf{A}-\lambda\mathbf{I})^{-1}$ should not exists

$\det(\mathbf{A}-\lambda\mathbf{I}) = 0$ ⋯ λ의 n차 polynomial

Char. eq

Repeated Eigenvalues $(\lambda - \lambda_1)^{m}$

$$\mathbf{X}_1 = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ \\ x_{1n}(t) \end{bmatrix} = \begin{bmatrix} k_{11} e^{\lambda_1 t} \\ k_{12} e^{\lambda_1 t} \\ \\ k_{1n} e^{\lambda_1 t} \end{bmatrix} = \begin{bmatrix} k_{11} \\ k_{12} \\ \\ k_{1n} \end{bmatrix} e^{\lambda_1 t}$$

$$\mathbf{X}_1 = \mathbf{K}_{11} e^{\lambda_1 t}$$

$$\mathbf{X}_2 = t\,\mathbf{K}_{21} e^{\lambda_1 t} + \mathbf{K}_{22} e^{\lambda_1 t}$$

$$\mathbf{X}_3 = \frac{t^2}{2!}\mathbf{K}_{31} e^{\lambda_1 t} + t\,\mathbf{K}_{32} e^{\lambda_1 t} + \mathbf{K}_{33} e^{\lambda_1 t}$$

$$\mathbf{X}_m = \frac{t^{m-1}}{(m-1)!}\mathbf{K}_{m1} e^{\lambda_1 t} + \frac{t^{m-2}}{(m-2)!}\mathbf{K}_{m2} e^{\lambda_1 t} + \cdots + t\,\mathbf{K}_{m\,m-1} e^{\lambda_1 t} + \mathbf{K}_{mm} e^{\lambda_1 t}$$

always m linearly independent eigenvectors

$(\lambda - \lambda_1)^m$

$$\boxed{X_2 = t K_{21} e^{\lambda_1 t} + K_{22} e^{\lambda_1 t}}$$

$$X_2 = K t e^{\lambda_1 t} + P e^{\lambda_1 t}$$

$$X_2' = K e^{\lambda_1 t} + \lambda_1 K t e^{\lambda_1 t} + \lambda_1 P e^{\lambda_1 t}$$

$$A X_2 = AK t e^{\lambda_1 t} + AP e^{\lambda_1 t}$$

$$(AK - \lambda_1 K) t e^{\lambda_1 t} + (AP - \lambda_1 P - K) e^{\lambda_1 t} = 0$$

$$AK - \lambda_1 K = 0 \qquad (A - \lambda_1 I) K = 0$$

$$(AP - \lambda_1 P - K) = 0 \qquad (A - \lambda_1 I) P = K$$

$$\boxed{\mathbf{X}_3 = \frac{t^2}{2!}\mathbf{K}_{31}\, e^{\lambda_1 t} + t\,\mathbf{K}_{32}\, e^{\lambda_1 t} + \mathbf{K}_{33}\, e^{\lambda_1 t}}$$

$$X_3 = K\frac{t^2}{2!}e^{\lambda_1 t} + P\,t\,e^{\lambda_1 t} + Q\,e^{\lambda_1 t}$$

$$X_3' = K t\, e^{\lambda_1 t} + \lambda_1 K \frac{t^2}{2!} e^{\lambda_1 t} + P\, e^{\lambda_1 t} + \lambda_1 P\, t\, e^{\lambda_1 t} + \lambda_1 Q\, e^{\lambda_1 t}$$

$$AX_3 = AK\frac{t^2}{2!}e^{\lambda_1 t} + AP\,t\,e^{\lambda_1 t} + AQ\,e^{\lambda_1 t}$$

$$(AK - \lambda_1 K)\frac{t^2}{2!}e^{\lambda_1 t} + (AP - \lambda_1 P - K)\,t\,e^{\lambda_1 t} + (AQ - \lambda_1 Q - P)e^{\lambda_1 t} = 0$$

$$\begin{aligned} AK - \lambda_1 K &= 0 \qquad & (A - \lambda_1 I)K &= 0 \\ AP - \lambda_1 P &= K \qquad & (A - \lambda_1 I)P &= K \\ AQ - \lambda_1 Q &= P \qquad & (A - \lambda_1 I)Q &= P \end{aligned}$$

Repeated Eigenvalues

sometimes m linearly independent eigenvectors can be found

$(\lambda - \lambda_1)^m$

$$c_1 \mathbf{K}_1 e^{\lambda_1 t} + c_2 \mathbf{K}_2 e^{\lambda_1 t} + \cdots + c_m \mathbf{K}_m e^{\lambda_1 t}$$

sometimes

$\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m$ eigen value λ_1 에 대한 m 개의 linearly indep. eigenvectors

Zill & Wright 10.2 Example 3

ex)

$$X' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} X$$

eigenvalues $(\lambda+1)^2(\lambda-5)=0$

$\lambda = -1$

$$(A-\lambda I)p = \begin{pmatrix} 1+1 & -2 & 2 \\ -2 & 1+1 & -2 \\ 2 & -2 & 1+1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} 2a & -2b & 2c \\ -2a & 2b & -2c \\ 2a & -2b & 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{2a - 2b + 2c = 0}$$
$$-2a + 2b - 2c = 0$$
$$2a - 2b + 2c = 0$$

the same eq → only one eq

$(\lambda+1)^2$

$$2a - 2b + 2c = 0$$

a	b	c	
1 ←	1	0	choose 2 free variables
0 ←	1	1	choose 2 free variables

$(\lambda+1)^2$

$$c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

the only c_1 & c_2

$c_1 = 0$, $c_2 = 0$

⇒ linearly independent

$\lambda = 5$

$$(A - \lambda I)p = \begin{pmatrix} 1-5 & -2 & 2 \\ -2 & 1-5 & -2 \\ 2 & -2 & 1-5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} -4a & -2b & 2c \\ -2a & -4b & -2c \\ 2a & -2b & -4c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -4a \quad -2b \quad +2c &= 0 \\ -2a \quad -4b \quad -2c &= 0 \\ 2a \quad -2b \quad -4c &= 0 \end{aligned}$$

$$\boxed{\begin{aligned} -6a + 6c &= 0 \\ -6b - 6c &= 0 \end{aligned}}$$ 2 eq's

$(\lambda+5)$ ①

$$\begin{aligned} -a \quad\quad + c &= 0 \\ -b - c &= 0 \end{aligned}$$

$1 \quad -1 \quad \leftarrow 1$ choose 1 free variable $(\lambda+5)$ ①

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\overbrace{\lambda = -1 \qquad \lambda = -1}^{\text{multiplicity of 2}} \qquad \lambda = 5$$

$$\underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \qquad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

2 distinct eigenvalues .... not always

$$X = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}$$

$$X' = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} X$$

$\downarrow$

multiplicity 3 eigenvalue $\quad (\lambda - \lambda_1)^3$

$$(A - \lambda_1 I) K = 0$$
$$(A - \lambda_1 I) P = K$$
$$(A - \lambda_1 I) Q = P$$

$$X_1 = K e^{\lambda_1 t}$$
$$X_2 = K t e^{\lambda_1 t} + P e^{\lambda_1 t}$$
$$X_3 = K \frac{t^2}{2} e^{\lambda_1 t} + P t e^{\lambda_1 t} + Q e^{\lambda_1 t}$$

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$
$$= c_1 \left(K e^{\lambda_1 t}\right) + c_3 \left(K t e^{\lambda_1 t} + P e^{\lambda_1 t}\right)$$
$$+ c_3 \left(K \frac{t^2}{2} e^{\lambda_1 t} + P t e^{\lambda_1 t} + Q e^{\lambda_1 t}\right)$$

Complex Eigenvalues

real entries (coefficient)

$$\dot{X} = AX$$

$$X_i = K_i e^{\lambda_i t}$$ assumption

$$\downarrow$$

$$(A - \lambda I)K = 0$$ condition

※ complex number λ_1 인 경우,
complex number K_1 로 생각

$\lambda_1 = \alpha + j\beta$ $\bar{\lambda}_1 = \alpha - j\beta$

$$K_1 e^{\lambda_1 t} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} e^{\lambda_1 t}.$$

$$\bar{K}_1 e^{\bar{\lambda}_1 t} = \begin{bmatrix} \bar{k}_1 \\ \bar{k}_2 \\ \vdots \\ \bar{k}_n \end{bmatrix} e^{\bar{\lambda}_1 t}.$$

if this is a solution then this also is a solution

$$\frac{dx}{dt} = 6x - y$$
$$\frac{dy}{dt} = 5x + 4y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\dot{\mathbf{x}} = A \quad \mathbf{x}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix}$$

$$\lambda = +5 \pm \sqrt{25-29} = +5 \pm 2i \qquad = \lambda^2 - 10\lambda + 29 = 0$$

eigenvalues	λ_1	λ_2
eigenvectors	$\mathbf{p}_1$	$\mathbf{p}_2$

$$\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = c_1 \mathbf{p}_1 e^{\lambda_1 t} + c_2 \mathbf{p}_2 e^{\lambda_2 t}$$
$$= c_1 \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(+5+2i)t} + c_2 \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(+5-2i)t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \quad e^{(+5+2i)t} + c_2 \quad e^{(+5-2i)t} \\ c_1(1-2i)e^{(+5+2i)t} + c_2(1+2i)e^{(+5-2i)t} \end{pmatrix}$$

Complex eigenvalue

$\lambda_1 = 5 + 2i$

Complex eigenvector $\mathbf{p}_1$

$$A - \lambda I = \begin{bmatrix} 6-(5+2i) & -1 \\ 5 & 4-(5+2i) \end{bmatrix}$$

$$= \begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix}$$

$$(A - \lambda_1 I)\mathbf{p}_1 = 0$$

$$\begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(1-2i)a - b = 0$

$5a - (1+2i)b = 0$

$b = (1-2i)a$

$a = 1$

$b = (1-2i)$

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$$

$$\overline{\mathbf{p}}_1 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}$$

Complex eigenvalue conjugate

$\lambda_2 = 5 - 2i$

Complex eigenvector $\mathbf{p}_2$

$$A - \lambda I = \begin{bmatrix} 6-(5-2i) & -1 \\ 5 & 4-(5-2i) \end{bmatrix}$$

$$= \begin{bmatrix} 1+2i & -1 \\ 5 & -1+2i \end{bmatrix}$$

$$(A - \lambda_2 I)\mathbf{p}_2 = 0$$

$$\begin{bmatrix} 1+2i & -1 \\ 5 & -1+2i \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(1+2i)c - d = 0$

$5c - (1-2i)d = 0$

$d = (1+2i)c$

$c = 1$

$d = (1+2i)$

$$\mathbf{p}_2 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}$$

$$\overline{\mathbf{p}}_2 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$$

$\lambda_1 = +5 + 2i$ $\qquad$ $\lambda_2 = +5 - 2i$

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} \qquad \mathbf{p}_2 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}$$

$$\mathbf{x}_1 = \mathbf{p}_1 e^{\lambda_1 t} \qquad \mathbf{x}_2 = \mathbf{p}_2 e^{\lambda_2 t}$$

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(+5+2i)t} \qquad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(+5-2i)t}$$

$$\mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = c_1 \mathbf{p}_1 e^{\lambda_1 t} + c_2 \mathbf{p}_2 e^{\lambda_2 t}$$

$$= c_1 \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(+5+2i)t} + c_2 \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(+5-2i)t}$$

$$c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t}$$
$$= e^{\alpha t}\left(c_3 \cos \beta t + c_4 \sin \beta t\right)$$

$\left\{ e^{(\alpha+i\beta)t},\ e^{(\alpha-i\beta)t} \right\}$ linearly indep.

$\left\{ e^{\alpha t} \cos \beta t,\ e^{\alpha t} \sin \beta t \right\}$ linearly indep.

$$\boldsymbol{x} = c_1 \boldsymbol{x}_1 + c_2 \boldsymbol{x}_2 = c_1 \boldsymbol{p}_1 e^{\lambda_1 t} + c_2 \boldsymbol{p}_2 e^{\lambda_2 t}$$

$$= c_1 \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(+5+2i)t} + c_2 \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(+5-2i)t}$$

$$e^{(+5+2i)t} = e^{5t} e^{+2it} = e^{5t} (\cos 2t + i \sin 2t)$$
$$e^{(+5-2i)t} = e^{5t} e^{-2it} = e^{5t} (\cos 2t - i \sin 2t)$$

$$\begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(+5+2i)t} = \begin{pmatrix} \cos 2t + i \sin 2t \\ \cos 2t + i \sin 2t - 2i(\cos 2t + i \sin 2t) \end{pmatrix} e^{+5t}$$

$$\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(+5-2i)t} = \begin{pmatrix} \cos 2t - i \sin 2t \\ \cos 2t - i \sin 2t + 2i(\cos 2t - i \sin 2t) \end{pmatrix} e^{+5t}$$

euler formula

$$y'' + ay' + by = 0$$

$$y(t) = c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t}$$
$$= e^{\alpha t} (c_3 \cos \beta t + c_4 \sin \beta t)$$

$\{ e^{(\alpha + i\beta)t}, e^{(\alpha - i\beta)t} \}$ linearly independent

$\{ e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t \}$ linearly independent

$$\begin{pmatrix}1\\1-2i\end{pmatrix}e^{(+5+2i)t} = \begin{pmatrix}\cos 2t + i\sin 2t\\ \cos 2t + i\sin 2t - 2i(\cos 2t + i\sin 2t)\end{pmatrix}e^{+5t}$$

$$\begin{pmatrix}1\\1+2i\end{pmatrix}e^{(+5-2i)t} = \begin{pmatrix}\cos 2t - i\sin 2t\\ \cos 2t - i\sin 2t + 2i(\cos 2t - i\sin 2t)\end{pmatrix}e^{+5t}$$

$+$

$$2\begin{pmatrix}\cos 2t\\ \cos 2t + 2\sin 2t\end{pmatrix}e^{+5t}$$

$$\begin{pmatrix}1\\1-2i\end{pmatrix}e^{(+5+2i)t} = \begin{pmatrix}\cos 2t + i\sin 2t\\ \cos 2t + i\sin 2t - 2i(\cos 2t + i\sin 2t)\end{pmatrix}e^{+5t}$$

$$\begin{pmatrix}1\\1+2i\end{pmatrix}e^{(+5-2i)t} = \begin{pmatrix}\cos 2t - i\sin 2t\\ \cos 2t - i\sin 2t + 2i(\cos 2t - i\sin 2t)\end{pmatrix}e^{+5t}$$

$-$

$$2i\begin{pmatrix}\sin 2t\\ \sin 2t - 2\cos 2t\end{pmatrix}e^{+5t}$$

$$\boldsymbol{x} = c_1 \boldsymbol{x}_1 + c_2 \boldsymbol{x}_2 = c_1 \boldsymbol{p}_1 e^{\lambda_1 t} + c_2 \boldsymbol{p}_2 e^{\lambda_2 t}$$

$$= c_1 \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(+5+2i)t} + c_2 \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} e^{(+5-2i)t}$$

Or $\boldsymbol{x} = \hat{c}_3 \boldsymbol{x}_1 + \hat{c}_4 \boldsymbol{x}_2$

$$= \hat{c}_3 \begin{pmatrix} \cos 2t \\ \cos 2t + 2 \sin 2t \end{pmatrix} e^{+5t} + \hat{c}_4 \begin{pmatrix} \sin 2t \\ \sin 2t - 2 \cos 2t \end{pmatrix} e^{+5t}$$

this can be easily obtained

by finding b_1, b_2

$$\boldsymbol{b}_1 = \frac{1}{2}(\boldsymbol{p}_1 + \boldsymbol{p}_2) = \frac{1}{2}\begin{pmatrix} 1+1 \\ 1-2i+1+2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{b}_2 = \frac{i}{2}(-\boldsymbol{p}_1 + \boldsymbol{p}_2) = \frac{i}{2}\begin{pmatrix} -1+1 \\ -1+2i+1+2i \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$b_1 = \frac{1}{2}(p_1 + p_2) \qquad b_2 = \frac{i}{2}(-p_1 + p_2)$$

$$x_1 = (b_1 \cos\beta t - b_2 \sin\beta t)e^{\alpha t}$$

$$x_2 = (b_2 \cos\beta t + b_1 \sin\beta t)e^{\alpha t}$$

$$x_1 = \left[\begin{pmatrix}1\\1\end{pmatrix}\cos 2t - \begin{pmatrix}0\\-2\end{pmatrix}\sin 2t\right]e^{+5t}$$

$$x_2 = \left[\begin{pmatrix}0\\-2\end{pmatrix}\cos 2t + \begin{pmatrix}1\\1\end{pmatrix}\sin 2t\right]e^{+5t}$$

$$x = \hat{C}_3 x_1 + \hat{C}_4 x_2$$

$$= \hat{C}_3\begin{pmatrix}\cos 2t\\ \cos 2t + 2\sin 2t\end{pmatrix}e^{+5t} + \hat{C}_4\begin{pmatrix}\sin 2t\\ \sin 2t - 2\cos 2t\end{pmatrix}e^{+5t}$$

$$\lambda_1 = \alpha + j\beta$$
$$\bar{\lambda}_1 = \alpha - j\beta$$

$$K_1 e^{\lambda_1 t} = K_1 e^{(\alpha + j\beta)t} = K_1 e^{\alpha t}(\cos\beta t + j\sin\beta t)$$

$$\bar{K}_1 e^{\bar{\lambda}_1 t} = \bar{K}_1 e^{(\alpha - j\beta)t} = \bar{K}_1 e^{\alpha t}(\cos\beta t - j\sin\beta t)$$

$$\frac{1}{2}\left(K_1 e^{\lambda_1 t} + \bar{K}_1 e^{\bar{\lambda}_1 t}\right) = \frac{1}{2}(K_1 + \bar{K}_1)e^{\alpha t}\cos\beta t$$
$$-\frac{j}{2}(-K_1 + \bar{K}_1)e^{\alpha t}\sin\beta t$$

$$-j K_1 e^{\lambda_1 t} = -j K_1 e^{(\alpha + j\beta)t} = K_1 e^{\alpha t}(-j\cos\beta t + \sin\beta t)$$

$$j\bar{K}_1 e^{\bar{\lambda}_1 t} = j\bar{K}_1 e^{(\alpha - j\beta)t} = \bar{K}_1 e^{\alpha t}(j\cos\beta t + \sin\beta t)$$

$$\frac{j}{2}\left(-K_1 e^{\lambda_1 t} + \bar{K}_1 e^{\bar{\lambda}_1 t}\right) = \frac{j}{2}(-K_1 + \bar{K}_1)e^{\alpha t}\cos\beta t$$
$$+\frac{1}{2}(K_1 + \bar{K}_1)e^{\alpha t}\sin\beta t$$

real entries (coefficient)

$$\dot{X} = AX$$

$$K_1 e^{\lambda_1 t} = K_1 e^{(\alpha + j\beta)t} = K_1 e^{\alpha t}(\cos\beta t + j\sin\beta t) \quad \text{a sol}$$

$$\bar{K}_1 e^{\bar{\lambda}_1 t} = \bar{K}_1 e^{(\alpha - j\beta)t} = \bar{K}_1 e^{\alpha t}(\cos\beta t - j\sin\beta t) \quad \text{a sol}$$

$$\frac{1}{2}\left(K_1 e^{\lambda_1 t} + \bar{K}_1 e^{\bar{\lambda}_1 t}\right) = \frac{1}{2}(K_1 + \bar{K}_1)e^{\alpha t}\cos\beta t - \frac{j}{2}(-K_1 + \bar{K}_1)e^{\alpha t}\sin\beta t \quad \rightarrow \text{sol}$$

$$\frac{j}{2}\left(-K_1 e^{\lambda_1 t} + \bar{K}_1 e^{\bar{\lambda}_1 t}\right) = \frac{j}{2}(-K_1 + \bar{K}_1)e^{\alpha t}\cos\beta t + \frac{1}{2}(K_1 + \bar{K}_1)e^{\alpha t}\sin\beta t \quad \rightarrow \text{sol}$$

$$B_1 = \boxed{\frac{1}{2}(K_1 + \bar{K}_1)}$$

$$B_2 = \boxed{\frac{1}{2i}(K_1 - \bar{K}_1)} = \frac{-i}{2}(K_1 - \bar{K}_1)$$

eigenvector

$$B_1 = \boxed{\frac{1}{2}\left(K_1 + \bar{K}_1\right)}$$

$$B_2 = \boxed{\frac{1}{2i}\left(K_1 - \bar{K}_1\right)} = \frac{-i}{2}\left(K_1 - \bar{K}_1\right)$$

$$X_1 = \left(B_1 \cos\beta t - B_2 \sin\beta t\right) e^{\alpha t} \rightarrow \text{sol}$$

$$X_2 = \left(B_2 \cos\beta t + B_1 \sin\beta t\right) e^{\alpha t} \rightarrow \text{sol}$$

Diagonalization

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbb{X}' = A\,\mathbb{X}$$ coupled system

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{bmatrix} b_{11}(t) & & 0 \\ & b_{22}(t) & \\ 0 & & b_{nn}(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$Y' = \Lambda Y$$ uncoupled system

$$y_1' = b_{11}(t)\, y_1$$

$$y_2' = b_{11}(t)\, y_2$$

$$y_n' = b_{11}(t)\, y_n$$

$$\boxed{AP = P\Lambda \qquad \begin{cases} A = P\Lambda P^{-1} \\ \Lambda = P^{-1}AP \end{cases}}$$

$$\underbrace{A}_{n \times n}$$

$\lambda_1, \lambda_2, \cdots, \lambda_n$; n개의 distinct eigen value

$$\lambda_1 \downarrow \; p_1, \quad \lambda_2 \downarrow \; p_2, \quad \cdots, \quad \lambda_n \downarrow \; p_n$$

eigenvector.

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} = P \qquad \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix} = \Lambda$$

$$\Lambda \leftarrow A \qquad\qquad A \leftarrow \Lambda$$

$$\Lambda = P^{-1}AP \qquad\qquad A = P\Lambda P^{-1}$$

$$\Lambda^k = P^{-1}A^kP \qquad\qquad A^k = P\Lambda^k P^{-1}$$

$$e^{\Lambda} = \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} \qquad\qquad e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$$e^{\Lambda} = P^{-1}e^{A}P \qquad\qquad e^{A} = Pe^{\Lambda}P^{-1}$$

$$X' = AX$$ coupled system

$$Y' = \Lambda Y$$ uncoupled system

$$P^{-1}AP = \Lambda$$ ..... λ_i e-val로 채워진 대각행렬

$$X = PY$$

$$X' = AX$$

$$PY' = APY$$

$$Y' = P^{-1}APY$$

$$Y' = \Lambda Y$$

$$\Downarrow$$

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$X' = A X$$ coupled system

$$Y' = \Lambda Y$$ uncoupled system

$\lambda_1, \lambda_2, \cdots, \lambda_n$; n개의 distinct eigenvalue

$$P = \begin{bmatrix} \boxed{p_1} & \boxed{p_2} & \cdots & \boxed{p_n} \end{bmatrix}$$ eigenvector.

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\begin{cases} y_1' = \lambda_1 y_1 \Rightarrow y_1 = c_1 e^{\lambda_1 t} \\ y_2' = \lambda_2 y_2 \Rightarrow y_2 = c_2 e^{\lambda_2 t} \\ \vdots \\ y_n' = \lambda_n y_n \Rightarrow y_n = c_n e^{\lambda_n t} \end{cases}$$

$$X = PY - \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Non-homogeneous Linear Systems

① Undetermined Coefficients

② Variation of Parameters

Undetermined Coefficients

$$X' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} -8 \\ 3 \end{pmatrix} \quad \text{non-homogeneous}$$

$$\Downarrow$$

$$\boxed{X_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}}$$

$$X' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X \quad \text{homogeneous}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (-1-\lambda)(+1-\lambda) + 2 = -1 + \lambda^2 + 2 = \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$\begin{pmatrix} -1-i & 2 \\ -1 & +1-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-(1+i)a + 2b = 0$$

$$-a + (1-i)b = 0$$

$$a = (1-i)b$$

$$a = 1$$

$$b = 1-i$$

$$\lambda_1 = +i \qquad p_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$\lambda_2 = -i \qquad p_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \qquad \lambda_1 = +i \qquad \lambda_2 = -i$$

$\lambda_1 = +i$ eigenvector p_1

$$\begin{pmatrix} -1-i & 2 \\ -1 & +1-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-(1+i)a + 2b = 0$ ...... ①

$-a + (1-i)b = 0$ ...... ②

②

$a = (1-i)b$

$a = (1-i)$

$b = 1$

$$P_1 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

①

$b = \frac{1}{2}(1+i)a$

$a = 1$

$b = \frac{1}{2}(1+i)$

$$P_1 = \begin{pmatrix} 1 \\ \frac{1}{2}(1+i) \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 \\ \frac{1}{2}(1-i) \end{pmatrix}$$

$-1 \quad +1 \qquad\qquad 0$

$\frac{1}{2}(\cancel{1}-i+\cancel{1}-i) \qquad -i$

$$P_1 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

$$b_1 = \frac{1}{2}(p_1 + p_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$b_2 = \frac{i}{2}(-p_1 + p_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 \\ \frac{1}{2}(1+i) \end{pmatrix} \qquad \frac{-i\,i}{2}$$

$$P_2 = \begin{pmatrix} 1 \\ \frac{1}{2}(1-i) \end{pmatrix}$$

$$b_1 = \frac{1}{2}(p_1 + p_2) = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$

$$b_2 = \frac{i}{2}(-p_1 + p_2) = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

$$X_1 = (B_1 \cos\beta t - B_2 \sin\beta t)\, e^{\alpha t}$$

$$X_2 = (B_2 \cos\beta t + B_1 \sin\beta t)\, e^{\alpha t}$$

$$X_1 = (B_1 \cos\beta t - B_2 \sin\beta t)\, e^{\alpha t}$$

$$X_2 = (B_2 \cos\beta t + B_1 \sin\beta t)\, e^{\alpha t}$$

$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad b_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t = \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t = \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}$$

$$x(t) = c_1 \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}$$

$$b_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \qquad b_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \sin t = \begin{pmatrix} \cos t \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \sin t = \begin{pmatrix} \sin t \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t \end{pmatrix}$$

$$x(t) = c_1 \begin{pmatrix} \cos t \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t \end{pmatrix}$$

$$x' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} x + \begin{pmatrix} -8 \\ 3 \end{pmatrix}$$

$$x_p = \begin{pmatrix} a \\ b \end{pmatrix} \quad x_p' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0 = -a + 2b - 8$$
$$0 = -a + b + 3$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 14 \\ 11 \end{pmatrix}$$

$$x = x_c + x_p$$

$$x(t) = c_1 \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix} + \begin{pmatrix} 14 \\ 11 \end{pmatrix}$$

$$x = x_c + x_p$$

$$x(t) = c_1 \begin{pmatrix} \cos t \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t \end{pmatrix} + \begin{pmatrix} 14 \\ 11 \end{pmatrix}$$

$$X' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} X + \underline{\begin{pmatrix} 6t \\ 40t + 4 \end{pmatrix}}$$

$$\Downarrow$$

$$X_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

Variation of Parameters

$$\boxed{\dot{X} = AX}$$

$$\boxed{X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n}$$

$$= c_1 \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} + c_2 \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 x_{11} + c_2 x_{12} + \cdots + c_n x_{1n} \\ c_1 x_{21} + c_2 x_{22} + \cdots + c_n x_{2n} \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ c_1 x_{n1} + c_2 x_{n2} + \cdots + c_n x_{nn} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} x_{11} & x_{12} & & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{nn} \end{pmatrix}} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$X = \boxed{\phi(t)} C$$

$$\boxed{\dot{X} = AX}$$

$$\boxed{X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n}$$

$$= \begin{pmatrix} x_{11} & x_{12} & & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} X_1 & X_2 & & X_n \end{pmatrix}}_{n \times n} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$X = \boxed{\phi(t)}\, C$$

linear combination coefficients

fundamenta matrix

Fundamental Matrix $\phi(t)$

$$\boxed{\dot{X} = AX}$$

$$\boxed{X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n}$$

$$= \begin{pmatrix} x_{11} & x_{12} & & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & x_2 & & x_n \end{pmatrix} C$$

$$X = \phi(t) C$$

① non-singular (invertible)

② $\phi'(t) = A\phi(t)$

$$\det(\phi(t)) = W(\underbrace{x_1, x_2, \cdots, x_n}_{\text{Sol. Vectors}})$$

Wronskian

$\det \neq 0 \Leftarrow$ linear independent

$\phi^{-1}(t)$ exists

$$X_h = \boxed{\phi(t)}\,\boxed{C} \qquad C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

assumption

$$X_p = \boxed{\phi(t)}\,\boxed{u(t)} \qquad u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$$

$$\boxed{\dot{X} = AX + F(t)}$$

$$X_p' = [\phi(t)u(t)]' = \boxed{\phi'(t)}u(t) + \phi(t)u'(t)$$

$$= \boxed{A\phi(t)}u(t) + \boxed{\phi(t)u'(t)}$$

$$AX + F(t) = A\phi(t)u(t) + \boxed{F(t)}$$

condition

$$\boxed{\phi(t)u'(t) = F(t)}$$

$$u'(t) = \phi^{-1}(t)F(t) \qquad u(t) = \int \phi^{-1}(t)F(t)\,dt$$

$$X = \boxed{\phi(t)}\boxed{C}$$ homogeneous sol

$$X = \boxed{\phi(t)}\boxed{U(t)}$$ particular sol.

$$X = \underbrace{\phi(t)\,C}_{X_c} + \underbrace{\phi(t)\int \phi^{-1}(t)\,F(t)\,dt}_{X_p}$$

Zill & Write Example 4 in sec 3.12

$$\boxed{\begin{aligned} x_1'' + 10x_1 \quad & -4x_2 = 0 \\ -4x_1 + x_2'' & + 4x_2 = 0 \end{aligned}}
\qquad
\boxed{\begin{aligned} x_1(0) = 0 \quad & x_2(0) = 0 \\ x_1'(0) = 1 \quad & x_2'(0) = -1 \end{aligned}}$$

$$(D^2+10)x_1 \qquad -4x_2 = 0 \quad \cdots (1)$$
$$-4x_1 \quad + (D^2+4)x_2 = 0 \quad \cdots (2)$$

$(2) \to \quad x_1 = \frac{1}{4}(D^2+4)x_2$

$\to (1) \quad (D^2+10)\frac{1}{4}(D^2+4)x_2 - 4x_2 = 0$

$$\left((D^2+10)(D^2+4) - 16\right)x_2 = 0$$
$$(D^4 + 14D^2 + 40 - 16)x_2 = 0$$
$$((D^2)^2 + 14D^2 + 24)x_2 = 0$$
$$(D^2+2)(D^2+12)x_2 = 0$$

$$(D^2+2)(D^2+12)x_1 = 0 \qquad\qquad (D^2+2)(D^2+12)x_2 = 0$$

$(m^2+2)(m^2+12) = 0$ $\qquad$ $(m^2+2)(m^2+12) = 0$

$m^2+2 = 0 \quad m = \pm\sqrt{2}i$ $\qquad$ $m^2+2=0 \quad m = \pm\sqrt{2}i$

$m^2+12 = 0 \quad m = \pm\sqrt{12}i$ $\qquad$ $m^2+12=0 \quad m = \pm\sqrt{12}i$

$+\sqrt{2}i, \ -\sqrt{2}i, \ +2\sqrt{3}i, \ -2\sqrt{3}i$ $\qquad$ $+\sqrt{2}i, \ -\sqrt{2}i, \ +2\sqrt{3}i, \ -2\sqrt{3}i$

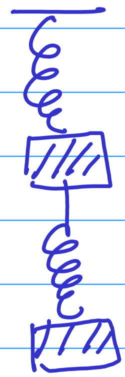
$\alpha=0$ $\qquad e^{+\sqrt{2}i} \quad e^{-\sqrt{2}i} \quad e^{+2\sqrt{3}i} \quad e^{-2\sqrt{3}i}$

$x_1(t) = c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t + c_3\cos 2\sqrt{3}t + c_4\sin 2\sqrt{3}t$

$x_2(t) = c_5\cos\sqrt{2}t + c_6\sin\sqrt{2}t + c_7\cos 2\sqrt{3}t + c_8\sin 2\sqrt{3}t$

Chap 3. 3.12 Solving Systems of Linear Equations

Example 4.



$\boxed{D = \frac{d}{dt}}$ $\boxed{D^2 = \frac{d^2}{dt^2}}$

$$(D^2 + 10)x_1 - 4x_2 = 0 \quad \cdots ①$$
$$-4x_1 + (D^2 + 4)x_2 = 0 \quad \cdots ②$$

$$(D^2 + 10)x_1 - 4x_2 = 0$$
$$\left(\frac{d^2}{dt^2} + 10\right)x_1 - 4x_2 = 0$$
$$\frac{d^2}{dt^2}x_1(t) + 10 \cdot x_1(t) - 4 \cdot x_2(t) = 0$$
$$x_1'' + 10x_1 - 4x_2 = 0$$

$$-4x_1 + (D^2 + 4)x_2 = 0$$
$$-4x_1 + \left(\frac{d^2}{dt^2} + 4\right)x_2 = 0$$
$$-4x_1 + \frac{d^2}{dt^2}x_2 + 4x_2 = 0$$
$$-4x_1(t) + \frac{d^2}{dt^2}x_2(t) + 4x_2(t) = 0$$
$$-4x_1 + x_2'' + 4x_2 = 0$$

$$x_1(t) = c_1 \cos\sqrt{2}t + c_2 \cos\sqrt{2}t + c_3 \cos 2\sqrt{3}t + c_4 \cos 2\sqrt{3}t$$
$$x_2(t) = c_5 \cos\sqrt{2}t + c_6 \cos\sqrt{2}t + c_7 \cos 2\sqrt{3}t + c_8 \cos 2\sqrt{3}t$$

substitute

$$\boxed{\begin{aligned} x_1'' + 10x_1 \quad -4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned}}$$

$$x_1(t) = c_1 \cos\sqrt{2}t + c_2 \cos\sqrt{2}t + c_3 \cos\sqrt{3}t + c_4 \cos\sqrt{3}t$$
$$x_2(t) = 2c_1 \cos\sqrt{2}t + 2c_2 \cos\sqrt{2}t - \tfrac{1}{2}c_3 \cos\sqrt{3}t - \tfrac{1}{2}c_4 \cos\sqrt{3}t$$

$$\boxed{\begin{aligned} x_1(0) &= 0 & x_2(0) &= 0 \\ x_1'(0) &= 1 & x_2'(0) &= -1 \end{aligned}}$$

$c_1 = 0 \quad c_2 = -\sqrt{2}/10, \quad c_3 = 0, \quad c_4 = \sqrt{3}/5$

$$x_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t$$

$$x_2(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t$$

Zill & Write Example 1 in sec 4.6

$$\boxed{\begin{aligned} x_1'' + 10x_1 \quad - 4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} x_1(0) &= 0 & x_2(0) &= 0 \\ x_1'(0) &= 1 & x_2'(0) &= -1 \end{aligned}}$$

$$\left(s^2 X_1(s) - s\, x_1(0) - x_1'(0)\right) + 10X_1(s) - 4X_2(s) = 0$$

$$-4X_1(s) + \left(s^2 X_1(s) - s\, x_2(0) - x_2'(0)\right) + 4X_2(s) = 0$$

$$(s^2+10)X_1(s) - 4X_2(s) = 1 \quad \cdots\cdots ①$$

$$-4X_1(s) + (s^2+4)X_2(s) = -1 \quad \cdots\cdots ②$$

Find $X_1(s)$, $X_2(s)$

② $\rightarrow$ $X_1(s) = \frac{1}{4}(s^2+4)X_2(s)$

$\rightarrow$ ①

$$(s^2+10)\frac{1}{4}(s^2+4)X_2(s) - 4X_2(s) = 1$$

$$(s^2+10)(s^2+4)X_2(s) - 16X_2(s) = 1$$

$$(s^4 + 14s^2 + 40 - 16)X_2(s) = 1$$

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{\frac{1}{5}}{s^2+2} + \frac{\frac{6}{5}}{s^2+12}$$

$$X_2(s) = \frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{\frac{2}{5}}{s^2+2} - \frac{\frac{3}{5}}{s^2+12}$$

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{\frac{1}{5}}{s^2+2} + \frac{\frac{6}{5}}{s^2+12}$$

$$x_1(t) = -\frac{1}{5\sqrt{2}} \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} + \frac{6}{5\sqrt{12}} \mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\}$$

$$= -\frac{\sqrt{2}}{10} \sin\sqrt{2}\, t + \frac{\sqrt{3}}{5} \sin 2\sqrt{3}\, t$$

$$X_2(s) = \frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{\frac{2}{5}}{s^2+2} - \frac{\frac{3}{5}}{s^2+12}$$

$$x_1(t) = -\frac{2}{5\sqrt{2}} \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} - \frac{3}{5\sqrt{12}} \mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\}$$

$$= -\frac{\sqrt{2}}{5} \sin\sqrt{2}\, t - \frac{\sqrt{3}}{10} \sin 2\sqrt{3}\, t$$

(Diagonal Matrix) ^k

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1^2 & 0 & 0 \\ 0 & b_2^2 & 0 \\ 0 & 0 & c_3^2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}^2$$

$$\begin{bmatrix} a_1^k & 0 & 0 \\ 0 & b_2^k & 0 \\ 0 & 0 & c_3^k \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}^k$$

(A)^k & Diagonalization

$$AP = P\Lambda$$

$$P^{-1}AP = \Lambda$$

$$A = P\Lambda P^{-1}$$

$$A^2 = AA = (P\Lambda \cancel{P^{-1}})(\cancel{P}\Lambda P^{-1}) = P\Lambda^2 P^{-1}$$

$$A^k = P\Lambda^k P^{-1}$$

$$= P \begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \lambda_n^k \end{bmatrix} P^{-1}$$

Taylor Series

$$f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{\lambda t} = 1 + \frac{\lambda t}{1!} + \frac{\lambda^2 t^2}{2!} + \frac{\lambda^3 t^3}{3!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k$$

$$e^A = ?$$

$$f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$f(A) = e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

Summation inside a Matrix

$$e^{\lambda t} = \boxed{\sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_1^k & & 0 \\ & \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_2^k & \\ 0 & & \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_n^k \end{bmatrix}$$

Summation over the simlar matrices

$k=0$

$$\begin{bmatrix} \frac{t^0}{0!}\lambda_1^0 & & \\ & \frac{t^0}{0!}\lambda_2^0 & \\ & & \ddots & \\ & & & \frac{t^0}{0!}\lambda_n^0 \end{bmatrix} = \frac{t^0}{0!}\Lambda^0$$

$k=1$

$$\begin{bmatrix} \frac{t^1}{1!}\lambda_1^1 & & \\ & \frac{t^1}{1!}\lambda_2^1 & \\ & & \ddots & \\ & & & \frac{t^1}{1!}\lambda_n^1 \end{bmatrix} = \frac{t^1}{1!}\Lambda^1$$

$k=2$

$$\begin{bmatrix} \frac{t^2}{2!}\lambda_1^2 & & \\ & \frac{t^2}{2!}\lambda_2^2 & \\ & & \ddots & \\ & & & \frac{t^2}{2!}\lambda_n^2 \end{bmatrix} = \frac{t^2}{2!}\Lambda^2$$

$\vdots$

$k=n$

$$\begin{bmatrix} \frac{t^n}{n!}\lambda_1^n & & \\ & \frac{t^n}{n!}\lambda_2^n & \\ & & \ddots & \\ & & & \frac{t^n}{n!}\lambda_n^n \end{bmatrix} = \frac{t^n}{n!}\Lambda^n$$

$\vdots$ $\qquad\qquad$ $\vdots$

$+)$

$$e^{\Lambda t} = \qquad \sum^{\infty} \frac{t^k}{k!}\Lambda^k$$

$$e^{\Lambda t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Lambda^k$$

$$= \boxed{\begin{matrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_1^k & & 0 \\ & \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_2^k & \\ 0 & & \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_n^k \end{matrix}}$$

$$P e^{\Lambda t} P^{-1} = P\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \Lambda^k\right) P^{-1}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \boxed{P \Lambda^k P^{-1}}$$

$$\overbrace{\underbrace{\boxed{P\Lambda P^{-1}}}_{A}\;\underbrace{\boxed{P\Lambda P^{-1}}}_{A}\;\cdots\;\underbrace{\boxed{P\Lambda P^{-1}}}_{A}}^{k} = A^k$$

$$P e^{\Lambda t} P^{-1} = P\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \Lambda^k\right) P^{-1}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \boxed{P \Lambda^k P^{-1}}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

$$e^{at} = \sum_{k=0}^{\infty} \frac{t^k}{k!} a^k \qquad e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

$$\boxed{\begin{aligned} e^{a(t_1+t_2)} &= e^{at_1} \cdot e^{at_2} \\ e^{at} \cdot e^{-at} &= 1 \\ (e^{at})^{-1} &= e^{-at} \\ \frac{d}{dt}(e^{at}) &= a e^{at} \end{aligned}} \qquad \boxed{\begin{aligned} e^{A(t_1+t_2)} &= e^{At_1} \cdot e^{At_2} \\ e^{At} \cdot e^{-At} &= I \\ (e^{At})^{-1} &= e^{-At} \\ \frac{d}{dt}(e^{At}) &= A e^{At} \end{aligned}}$$

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\left(I + \frac{A}{1!} t + \frac{A^2}{2!} t^2 + \frac{A^3}{3!} t^3 + \cdots \right)$$

$$= \left(0 + \frac{A}{1!} + \frac{A^2}{1!} t + \frac{A^3}{2!} t^2 + \right)$$

$$= A \left(I + \frac{A}{1!} t + \frac{A^2}{2!} t^2 + \frac{A^3}{3!} t^3 + \cdots \right)$$

$$= A e^{At}$$

$$X' = AX \qquad \text{Solution} \quad X = e^{At} C$$

$$X' = \frac{d}{dt} e^{At} = A e^{At}$$

$$X' = \frac{d}{dt} e^{At} C = A e^{At} C$$

$$\begin{array}{cc} & 1 \\ n & \begin{bmatrix} \\ X \\ \\ \end{bmatrix} \end{array} = \begin{array}{cc} & n \\ n & \begin{bmatrix} & & \\ & e^{At} & \\ & & \end{bmatrix} \end{array} \begin{array}{c} 1 \\ \begin{bmatrix} \\ C \\ \\ \end{bmatrix} \end{array}$$

Fundamenta Matrix

$$X = \boxed{\phi(t)}\boxed{C}$$

① non-singular (invertible)

② $\phi'(t) = A\phi(t)$

Wronskian Sol. Vectors

$$\det(\phi(t)) = W(\underbrace{x_1, x_2, \cdots, x_n})$$

$\det \neq 0$ $\Leftarrow$ linear independent

$\phi^{-1}(t)$ exists

$$\Psi(t) = e^{At}$$

$$\Psi'(t) = A\,\Psi(t)$$

$$\Psi(0) = e^{A0} = I$$

$$\det(\Psi(0)) \neq 0$$

sufficient condition

$\Psi(t)$ is a fundamental matrix of the system

$$X' = AX$$

Non-homogeneous System

$$X = X_c + X_p =$$

$$e^{At} C + e^{At} \int_{t_0}^{t} e^{-At} F(\tau)\, d\tau$$

Computing e^{At}

① Using Laplace Transform

$x = e^{At}$ is the solution of an IVP

$$\boxed{x' = Ax \qquad x(0) = I}$$

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$x' \quad = \quad A \quad x$$

$$\begin{pmatrix} sX_1(s) - x_1(0) \\ sX_2(s) - x_2(0) \\ \vdots \\ sX_n(s) - x_n(0) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{pmatrix}$$

$$sX(s) - x(0) \quad = \quad A \quad X(s)$$

$$sX(s) - x(0) = AX(s)$$

$$(sI - A)X(s) = x(0) = I$$

$$X(s) = (sI - A)^{-1} = \mathcal{L}\{x(t)\} = \mathcal{L}\{e^{At}\}$$

$$\boxed{e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}}$$

Computing e^{At}

① Using Caley-Hamilton Theorem

Characteristic Eq distinct eigenvalues

$$\boxed{\begin{aligned} &(-1)\,\lambda^{n} + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda^{1} + c_0 = 0 \\ &(-1)\,A^{n} + c_{n-1}A^{n-1} + \cdots + c_1A^{1} + c_0 I = 0 \end{aligned}}$$

recursive application

$m = 0, \pm 1, \pm 2, \cdots$

$$\boxed{\begin{aligned} &A^{m} = k_{m0} I + k_{m1}A^{1} + k_{m2}A^{2} + \cdots + k_{m\,n-1}A^{n-1} \\ &\lambda^{m} = k_{m0} + k_{m1}\lambda^{1} + k_{m2}\lambda^{2} + \cdots + k_{m\,n-1}\lambda^{n-1} \end{aligned}}$$

linear combiton of $A^0, A^1, A^2, \cdots, A^{n-1}$

linear combiton of $\lambda^0, \lambda^1, \lambda^2, \cdots, \lambda^{n-1}$

rewrite eq's

$$A^{k} = c_0 I + c_1 A^{1} + c_2 A^{2} + \cdots + c_{k-1} A^{k-1}$$

$$\lambda^{k} = c_0 + c_1\lambda^{1} + c_2\lambda^{2} + \cdots + c_{k-1}\lambda^{k-1}$$

$$e^{\lambda t} = 1 + \frac{\lambda t}{1!} + \frac{\lambda^2 t^2}{2!} + \frac{\lambda^3 t^3}{3!} + \cdots$$

$$e^{At} = I + \frac{At}{1!} + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$$

$$e^{\lambda t} = 1 + \lambda \frac{t}{1!} + \lambda^2 \frac{t^2}{2!} + \lambda^3 \frac{t^3}{3!} + \cdots$$

$$e^{At} = 1 + A \frac{t}{1!} + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \cdots$$

for any k

$A^k \Longrightarrow$ linear combition of $A^0, A^1, A^2, \cdots, A^{n-1}$

$\lambda^k \Longrightarrow$ linear combition of $\lambda^0, \lambda^1, \lambda^2, \cdots, \lambda^{n-1}$

$e^{\lambda t} \Longrightarrow$ linear combition of $A^0, A^1, A^2, \cdots, A^{n-1}$

$e^{At} \Longrightarrow$ linear combition of $\lambda^0, \lambda^1, \lambda^2, \cdots, \lambda^{n-1}$

$$e^{\lambda t} = 1 + \lambda \frac{t}{1!} + \lambda^2 \frac{t^2}{2!} + \lambda^3 \frac{t^3}{3!} + \cdots$$

$$e^{At} = 1 + A \frac{t}{1!} + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \cdots$$

for any k

$e^{\lambda t} \Longrightarrow$ linear combition of $A^0, A^1, A^2, \cdots, A^{n-1}$

$e^{At} \Longrightarrow$ linear combition of $\lambda^0, \lambda^1, \lambda^2, \cdots, \lambda^{n-1}$

$$e^{\lambda t} = \sum_{j=0}^{n-1} b_j(t)\, \lambda^j \qquad b_j(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} c_{kj}$$

$$e^{At} = \sum_{j=0}^{n-1} b_j(t)\, A^j \qquad b_j(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} c_{kj}$$

$$\begin{array}{cccccc}
\boxed{k=0} & \boxed{k=1} & \boxed{k=2} & \boxed{k=3} & & \boxed{k=\infty}
\end{array}$$

$$e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$$

$$\begin{array}{cccccc}
c_{00}A^0 & c_{10}A^0 & c_{20}A^0 & c_{30}A^0 & & b_0(t) \\
c_{01}A^1 & c_{11}A^1 & c_{21}A^1 & c_{31}A^1 & & b_1(t) \\
c_{02}A^2 & c_{12}A^2 & c_{22}A^2 & c_{32}A^2 & & b_2(t) \\
\vdots & \vdots & \vdots & \vdots & & \\
c_{0m}A^{n-1} & c_{1m}A^{n-1} & c_{2m}A^{n-1} & c_{3m}A^{n-1} & & b_{n-1}(t)
\end{array}$$

$$b_j(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} c_{kj}$$

Using time domain technique yh + yp 20150626

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{pmatrix} X$$

$\lambda_1 = -2$ $\lambda_2 = -4$ $\lambda_3 = -3$

$$x_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ -4 \\ 16 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}$$

$$x = c_1 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -4 \\ 16 \end{pmatrix} e^{-4t} + c_3 \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix} e^{-3t}$$

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$$

$$x_p = \begin{pmatrix} a \\ b \\ c \end{pmatrix} e^{-t} \qquad x_p' = \begin{pmatrix} -a \\ -b \\ -c \end{pmatrix} e^{-t}$$

$$\begin{pmatrix} -a \\ -b \\ -c \end{pmatrix} e^{-t} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$$

$$= \begin{pmatrix} b \\ c \\ -24a - 26b - 9c + 1 \end{pmatrix} \qquad \begin{aligned} b &= -a \\ c &= a \\ 6a &= 1 \end{aligned}$$

$$x_p = \frac{1}{6} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-t}$$

$$x(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$x = c_1 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -4 \\ 16 \end{pmatrix} e^{-4t} + c_3 \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix} e^{-3t} + \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{6} \\ \frac{1}{6} \end{pmatrix} e^{-t}$$

$$x(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + c_3 + \frac{1}{6} \\ -2c_1 - 4c_2 - 3c_3 - \frac{1}{6} \\ 4c_1 + 16c_2 + 9c_3 + \frac{1}{6} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & -4 & -3 \\ 4 & 16 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} \\ \frac{1}{6} \\ \frac{11}{6} \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{13}{2} \\ \frac{23}{6} \\ -\frac{19}{2} \end{pmatrix}$$

$$x = \frac{13}{2}\begin{pmatrix}1\\-2\\4\end{pmatrix}e^{-2t} + \frac{23}{6}\begin{pmatrix}1\\-4\\16\end{pmatrix}e^{-4t} - \frac{19}{2}\begin{pmatrix}1\\-3\\9\end{pmatrix}e^{-3t} + \begin{pmatrix}\frac{1}{6}\\-\frac{1}{6}\\\frac{1}{6}\end{pmatrix}e^{-t}$$

$$y = \begin{bmatrix}1 & 1 & 0\end{bmatrix} x$$

$$= -\frac{13}{2}e^{-2t} - \frac{23}{2}e^{-4t} + 19e^{-3t}$$

(2) Using Laplace

20150626

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{pmatrix} X$$

$$(sI-A) = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 24 & 26 & s+9 \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{1}{\Delta} \begin{pmatrix} s^2+9s+26 & s+9 & 1 \\ -24 & s^2+9s & s \\ -24s & -26s-24 & s^2 \end{pmatrix}$$

$$\Delta = s^3 + 9s^2 + 26s + 24$$

$$X(s) = (sI-A)^{-1} \left(x(0) + BU(s) \right)$$

$$= \frac{1}{\Delta} \begin{pmatrix} s^3+10s^2+35s+29 \\ 2s^2-21s-24 \\ s(2s^2-21s-24) \end{pmatrix}$$

$$\Delta = (s+1)(s+2)(s+3)(s+4)$$

$$Y(s) = \frac{s^3+12s^2+16s+5}{(s+1)(s+2)(s+3)(s+4)} = \frac{-\frac{13}{2}}{s+2} + \frac{19}{s+3} - \frac{\frac{23}{2}}{s+4}$$

$$y(t) = -\frac{13}{2} e^{-2t} - \frac{23}{2} e^{-4t} + 19 e^{-3t}$$