Inverse Laplace Transform (2A)

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5/15/15

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Forward and Inverse Laplace Transform

Forward Laplace Transform

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

Integration with a real variable t

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt$$

Inverse Laplace Transform

$$f(t) \xleftarrow{\mathcal{L}^{-1}} F(s)$$

Integration with a complex variable s

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{+st}\, ds$$

Some Laplace Transform Pairs

1	$\Longleftrightarrow$	$\dfrac{1}{s}$	e^{-at}	$\Longleftrightarrow$	$\dfrac{1}{s+a}$
t	$\Longleftrightarrow$	$\dfrac{1}{s^2}$	$\sin(\omega t)$	$\Longleftrightarrow$	$\dfrac{\omega}{s^2+\omega^2}$
t^2	$\Longleftrightarrow$	$\dfrac{2}{s^3}$	$\cos(\omega t)$	$\Longleftrightarrow$	$\dfrac{s}{s^2+\omega^2}$
t^3	$\Longleftrightarrow$	$\dfrac{6}{s^4}$	$\sinh(\omega t)$	$\Longleftrightarrow$	$\dfrac{\omega}{s^2-\omega^2}$
t^n	$\Longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$	$\cosh(\omega t)$	$\Longleftrightarrow$	$\dfrac{s}{s^2-\omega^2}$

Partial Fraction Methods

$$\frac{1}{\cdots(ax+b)\cdots} \Rightarrow \cdots + \frac{A}{(ax+b)} + \cdots$$

$$\frac{1}{\cdots(ax+b)^k\cdots} \Rightarrow \cdots + \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k} + \cdots$$

$$\frac{1}{\cdots(ax^2+bx+c)\cdots} \Rightarrow \cdots + \frac{Ax+b}{(ax^2+bx+c)} + \cdots$$

$$\frac{1}{\cdots(ax^2+bx+c)^k\cdots} \Rightarrow \cdots + \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k} + \cdots$$

Repeated Pole Case

$$X(s) = \frac{P(s)}{(s+p)(s+r)^k} = \frac{K}{(s+p)} + \frac{A_0}{(s+r)^k} + \frac{A_1}{(s+r)^{k-1}} + \cdots + \frac{A_{k-1}}{(s+r)^1}$$

$$A_0 = \left([X(s)(s+r)^k] |_{s=-r} \right)$$

$$A_1 = \frac{d}{ds} \left([X(s)(s+r)^k] |_{s=-r} \right)$$

$$A_2 = \frac{1}{2!} \frac{d^2}{ds^2} \left([X(s)(s+r)^k] |_{s=-r} \right)$$

$$A_m = \frac{1}{m!} \frac{d^m}{ds^m} \left([X(s)(s+r)^k] |_{s=-r} \right)$$

$$A_{k-1} = \frac{1}{k-1!} \frac{d^{k-1}}{ds^{k-1}} \left([X(s)(s+r)^k] |_{s=-r} \right)$$

Cover-up Method (1)

$$\frac{2s-1}{(s+3)(s-2)} = \frac{A}{(s+3)} + \frac{B}{(s-2)}$$

$$\frac{2s-1}{(s+3)(s-2)}(s+3) \Rightarrow A = \frac{2s-1}{(s-2)}\bigg|_{s=-3} = \frac{-6-1}{-3-2} = +\frac{7}{5}$$

$$\frac{2s-1}{(s+3)(s-2)}(s-2) \Rightarrow B = \frac{2s-1}{(s+3)}\bigg|_{s=2} = \frac{4-1}{2+3} = +\frac{3}{5}$$

Cover-up Method (2)

$$\frac{2s-1}{(s+3)(s-2)} = \frac{A}{(s+3)} + \frac{B}{(s-2)}$$

$$\left.\frac{2s-1}{\cancel{(s+3)}(s-2)}\cancel{(s+3)}\right|_{s=-3} = \left.\frac{A}{\cancel{(s+3)}}\cancel{(s+3)} + \frac{B}{(s-2)}(s+3)\right|_{s=-3} = A$$

$$\left.\frac{2s-1}{(s+3)\cancel{(s-2)}}\cancel{(s-2)}\right|_{s=2} = \left.\frac{A}{(s+3)}(s-2) + \frac{B}{\cancel{(s-2)}}\cancel{(s-2)}\right|_{s=2} = B$$

Cover-up Method : Repeated Poles (1)

$$\frac{1}{s^2(s+1)} = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{B}{(s+1)}$$

$$\frac{1}{s^2(s+1)} s^2 \quad \Rightarrow \quad A_1 = \frac{1}{(s+1)}\Big|_{s=0} = \frac{1}{1} = +1$$

$$\frac{d}{ds}\left(\frac{1}{s^2(s+1)} s^2\right) \quad \Rightarrow \quad A_2 = \frac{d}{ds}\left(\frac{1}{(s+1)}\right)\Big|_{s=0} = -\frac{1}{(s+1)^2}\Big|_{s=0} = -\frac{1}{1} = -1$$

$$\frac{1}{s^2(s+1)}(s+1) \quad \Rightarrow \quad B = \frac{1}{s^2}\Big|_{s=-1} = \frac{1}{1} = +1$$

Cover-up Method : Repeated Poles (2)

$$\frac{1}{s^2(s+1)} = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{B}{(s+1)}$$

$$\left.\frac{1}{s^2(s+1)}s^2\right|_{s=0} = \left.\frac{A_1}{s^2}s^2 + \frac{A_2}{s}s^2 + \frac{B}{(s+1)}s^2\right|_{s=0} = \left.A_1 + A_2 s + \frac{B}{(s+1)}s^2\right|_{s=0} = A_1$$

$$\left.\frac{d}{ds}\left(\frac{1}{s^2(s+1)}s^2\right)\right|_{s=0} = \left.\frac{d}{ds}\left(A_1 + A_2 s + \frac{B}{(s+1)}s^2\right)\right|_{s=0} = \left.A_2 + \left(\frac{2Bs}{(s+1)} - B\frac{s^2}{(s+1)^2}\right)\right|_{s=0} = A_2$$

$$\left.\frac{1}{s^2(s+1)}(s+1)\right|_{s=-1} = \left.\frac{A_1}{s^2}(s+1) + \frac{A_2}{s}(s+1) + \frac{B}{(s+1)}(s+1)\right|_{s=-1} = B$$

Cover-up Method : Inverse Laplace Transform

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{(s+1)}$$

$$\Longleftrightarrow \quad t - 1 + e^{-t}$$

$$1 \Longleftrightarrow \frac{1}{s}$$

$$t \Longleftrightarrow \frac{1}{s^2}$$

$$e^{-at} \Longleftrightarrow \frac{1}{s+a}$$

Examples

$$\frac{2s+5}{(s-3)^2} = \frac{A_1}{(s-3)} + \frac{A_2}{(s-3)^2}$$

$$\frac{2s+5}{(s-3)^2}(s-3)^2 = \frac{A_1}{(s-3)}(s-3)^2 + \frac{A_2}{(s-3)^2}(s-3)^2 = A_1(s-3) + A_2$$

$$\left.\frac{2s+5}{(s-3)^2}(s-3)^2\right|_{s=3} = \left.(2s+5)\right|_{s=3} = 11 = A_2$$

$$\frac{d}{ds}\left(\frac{2s+5}{(s-3)^2}(s-3)^2\right) = \frac{d}{ds}\left(A_1(s-3) + A_2\right) = A_1$$

$$\left.\frac{d}{ds}\left(\frac{2s+5}{(s-3)^2}(s-3)^2\right)\right|_{s=3} = 2 = A_1$$

Cover-up Method : Inverse Laplace Transform

$$\frac{1}{s^2+2s-3} = \frac{1}{s^2+2s+1-4} = \frac{1}{2}\frac{2}{(s+1)^2-2^2}$$

$$\sinh(2t) \Longleftrightarrow \frac{2}{s^2-2^2}$$

$$\frac{1}{2}e^{-t}\sinh(2t) \Longleftrightarrow \frac{1}{2}\frac{2}{(s+1)^2-2^2}$$

$$\frac{s}{s^2+2s-3} = \frac{s}{s^2+2s+1-4} = \frac{s}{(s+1)^2-2^2}$$

$$\cosh(2t) \Longleftrightarrow \frac{s}{s^2-2^2}$$

$$e^{-t}\cosh(2t) \Longleftrightarrow \frac{s}{(s+1)^2-2^2}$$

Cover-up Method : Inverse Laplace Transform

$$\frac{1}{s^2+2s+5} = \frac{1}{s^2+2s+1+4} = \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

$$\sin(2t) \Longleftrightarrow \frac{2}{s^2+2^2}$$

$$\frac{1}{2}e^{-t}\sin(2t) \Longleftrightarrow \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

$$\frac{s}{s^2+2s+5} = \frac{s}{s^2+2s+1+4} = \frac{s}{(s+1)^2+2^2}$$

$$\cos(2t) \Longleftrightarrow \frac{s}{s^2+2^2}$$

$$e^{-t}\cos(2t) \Longleftrightarrow \frac{s}{(s+1)^2+2^2}$$

Selected Laplace Transform Pairs (2)

References

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