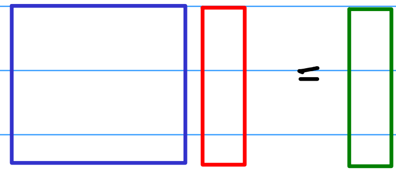
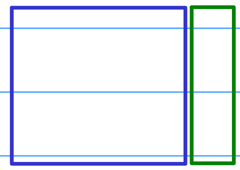
Eigenvalues, Eigenvectors (H.1)

20150723

Augmented Matrix

Zill & Wright





augmented

연립방정식 $x, y, z,$ u, v, w

$$
\begin{aligned}
2x + 6y + z &= 7 \\
x + 2y - z &= -1 \\
5x + 7y - 4z &= 9
\end{aligned}
\qquad
\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & -4 \end{bmatrix}
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
=
\begin{bmatrix} 7 \\ -1 \\ 9 \end{bmatrix}
$$

$$
\begin{aligned}
2u + 6v + w &= 7 \\
+ 2v - w &= -1 \\
5u + 7v - 4w &= 9
\end{aligned}
\qquad
\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & -4 \end{bmatrix}
\begin{bmatrix} u \\ v \\ w \end{bmatrix}
=
\begin{bmatrix} 7 \\ -1 \\ 9 \end{bmatrix}
$$

$$
\left[\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right]
$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 5 & 2 \\ 4 & 7 & -1 & 8 \end{array}\right)$$

$$\begin{pmatrix} 1 & -3 & 5 \\ 4 & 7 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$1 \cdot x_1 - 3 \cdot x_2 + 5 \cdot x_3 = 2$$

$$4x_1 + 7x_2 - 1 \cdot x_3 = 8$$

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$x_1 - 5x_3 = -1$

$2x_1 + 8x_2 = 7$

$x_2 + 9x_3 = 1$

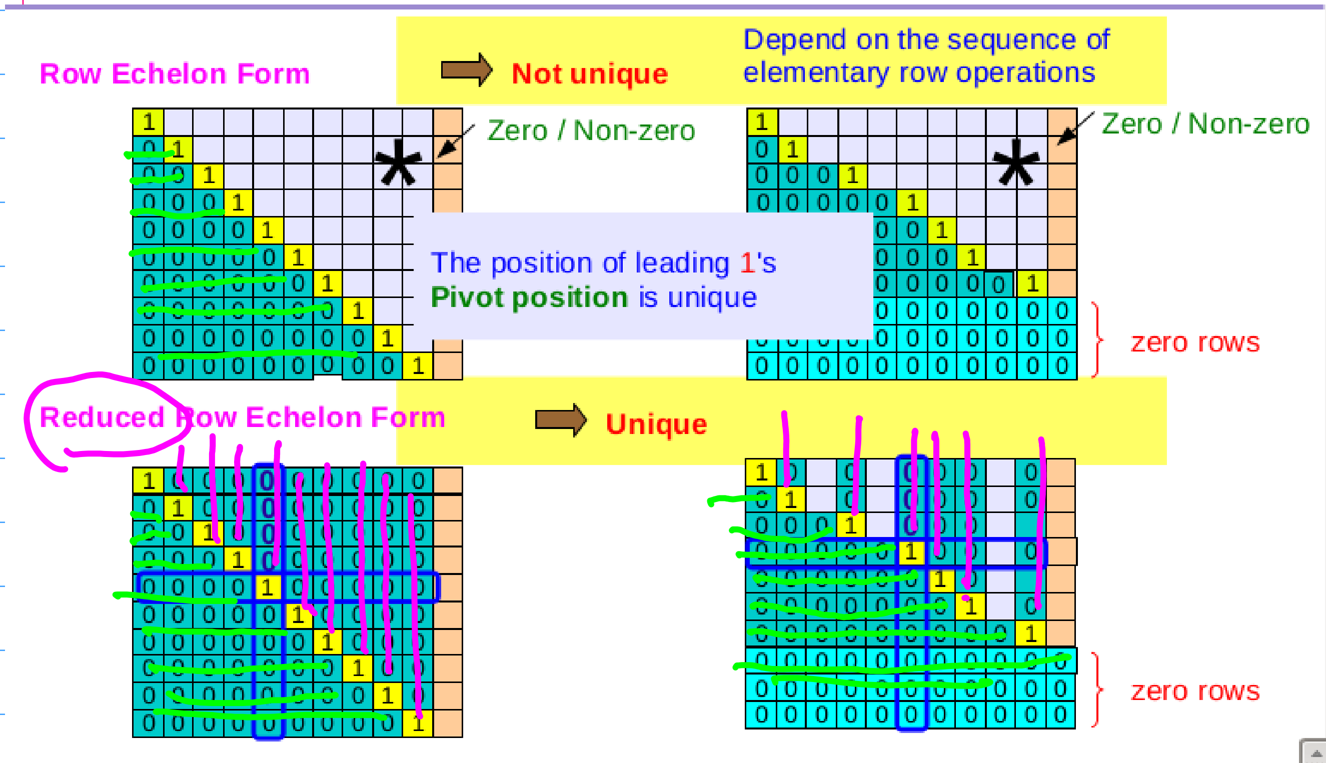
$1\cdot x_1 + 0\cdot x_2 - 5\cdot x_3 = -1$

$2\cdot x_1 + 8\, x_2 + 0\cdot x_3 = 7$

$0\cdot x_1 + 1\, x_2 + 9\, x_3 = 1$

$$\begin{bmatrix} 1 & +0\cdot & -5 \\ 2\cdot & +8 & +0 \\ 0\cdot & +1 & +9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & +0\cdot & -5 & -1 \\ 2\cdot & +8 & +0 & 7 \\ 0\cdot & +1 & +9 & 1 \end{array}\right]$$


Row Echelon Form
Not unique
Depend on the sequence of elementary row operations
Zero / Non-zero
Zero / Non-zero
The position of leading 1's
Pivot position is unique
zero rows
Reduced Row Echelon Form
Unique
zero rows

$$\left(\begin{array}{ccc|c} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

row echelon form

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -6 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

reduced row echelon form

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -6 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

reduced row echelon form

eigen values (A^{-1}) ?

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$$A = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix}$$

$$\det(\lambda I - A) = 0 \qquad \begin{vmatrix} \lambda-4 & 0 \\ -2 & \lambda-3 \end{vmatrix} = (\lambda-4)(\lambda-3) = 0$$

$$\lambda = 3, 4$$

$$(\lambda I - A)p = 0$$

$\lambda = 3$

$$(3I - A)p_1 = 0$$

$$\begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-a = 0$

$-2a = 0$

$a = 0$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 13.4 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

all these are possible eigenvectors

$\lambda = 4$

$$(4I - A)p_2 = 0$$

$$\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2c + d = 0$

$d = 2c$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3.5 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} \frac{1}{99} \\ \frac{2}{99} \end{bmatrix}$$

all these are possible eigenvectors

$$A = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix} \qquad A^{-1} = \frac{1}{\Delta}\begin{pmatrix} 3 & 0 \\ -2 & 4 \end{pmatrix}$$

$$\Delta = 12 - 0$$

$$A^{-1} = \frac{1}{12}\begin{pmatrix} 3 & 0 \\ -2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$\begin{vmatrix} \lambda - \frac{1}{4} & 0 \\ +\frac{1}{6} & \lambda - \frac{1}{3} \end{vmatrix} = 0 \qquad (\lambda - \tfrac{1}{4})(\lambda - \tfrac{1}{3}) = 0$$

$$\lambda = \frac{1}{3}, \frac{1}{4}$$

$\lambda = \frac{1}{3}$

$$\begin{bmatrix} \frac{1}{3} - \frac{1}{4} & 0 \\ \frac{1}{6} & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$a = 0$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\lambda = \frac{1}{4}$

$$\begin{bmatrix} 0 & 0 \\ \frac{1}{6} & -\frac{1}{12} \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{c}{6} = \frac{d}{12} \qquad 2c = d$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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A non-singular

λ : eigenvalue

p : eigenvector

A^{-1}

$1/\lambda$: eigenvalue

p : eigenvector.

$$Ap = \lambda p$$

$$A^{-1}Ap = A^{-1}\lambda p = \lambda A^{-1} p$$

$$p = \lambda A^{-1} p$$

$$\frac{1}{\lambda} p = A^{-1} p$$

$$\boxed{A^{-1}}\, p = \boxed{\frac{1}{\lambda}}\, p$$

eigenvalue

Zill & Wright

$$A = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix}$$

$$\det(\lambda I - A) \qquad \begin{vmatrix} \lambda-9 & -1 & -1 \\ -1 & \lambda-9 & -1 \\ -1 & -1 & \lambda-9 \end{vmatrix} = 0$$

$$\overset{+}{\lambda-9}\begin{vmatrix} \lambda-9 & -1 \\ -1 & \lambda-9 \end{vmatrix} \qquad \overset{-}{-1}\begin{vmatrix} -1 & -1 \\ -1 & \lambda-9 \end{vmatrix} \qquad \overset{+}{-1}\begin{vmatrix} -1 & \lambda-9 \\ -1 & -1 \end{vmatrix}$$

$$(\lambda-9)\begin{vmatrix} \lambda-9 & -1 \\ -1 & \lambda-9 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & \lambda-9 \end{vmatrix} + (-1)\begin{vmatrix} -1 & \lambda-9 \\ -1 & -1 \end{vmatrix}$$

$$= (\lambda-9)\left((\lambda-9)^2-1\right) + (-\lambda+9-1) + (-1-\lambda+9)$$

$$= (\lambda-9)\left\{\lambda^2-18\lambda+81-1\right\} - 2\lambda+16$$

$$= -(\lambda-11)(\lambda-8)^2 = 0$$

11, 8, 8

$$\lambda = 11, \quad \lambda = 8$$

$\lambda = 11$

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$a - c = 0$

$b - c = 0$

$a = c$

$b = c$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 8$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$d + e + f = 0$

$$\begin{array}{ccc} -1 & 1 & 0 \\ -1 & 0 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 11$

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & -\frac{3}{2} & +\frac{3}{2} & 0 \end{array}\right)$$

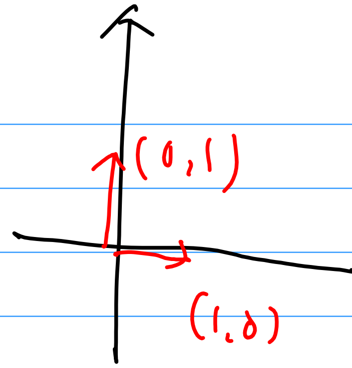
$$\begin{array}{cccc} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ -(& 1 & -2 & 1\ ) \\ \hline & 0 & +\frac{3}{2} & -\frac{3}{2} \end{array} \qquad \begin{array}{cccc} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ -(& 1 & 1 & -2\ ) \\ \hline & 0 & -\frac{3}{2} & +\frac{3}{2} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{cccc} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2}(& 0 & 1 & -1\ ) \\ \hline & 1 & 0 & -1 \end{array}$$

$d + e + f = 0$

$$\begin{array}{ccc} -2 & 1 & 1 \\ -4 & 1 & 3 \end{array}$$



$$c_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

non-singular

$$Ax = b$$

① A^{-1} $\qquad x = A^{-1}b$

② Cramer's rule

③ Gauss-Jordan Elimination

Geometric Interpretation of eigenvectors

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \qquad \begin{vmatrix} \lambda-5 & +2 \\ +2 & \lambda-2 \end{vmatrix} = (\lambda-5)(\lambda-2) - 4$$

$$= \lambda^2 - 7\lambda + 6$$

$$= (\lambda-1)(\lambda-6) = 0$$

$$X = Ax = \lambda x$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 5x - 2y = \lambda x \\ -2x + 2y = \lambda y \end{cases}$$

$\lambda = 1$

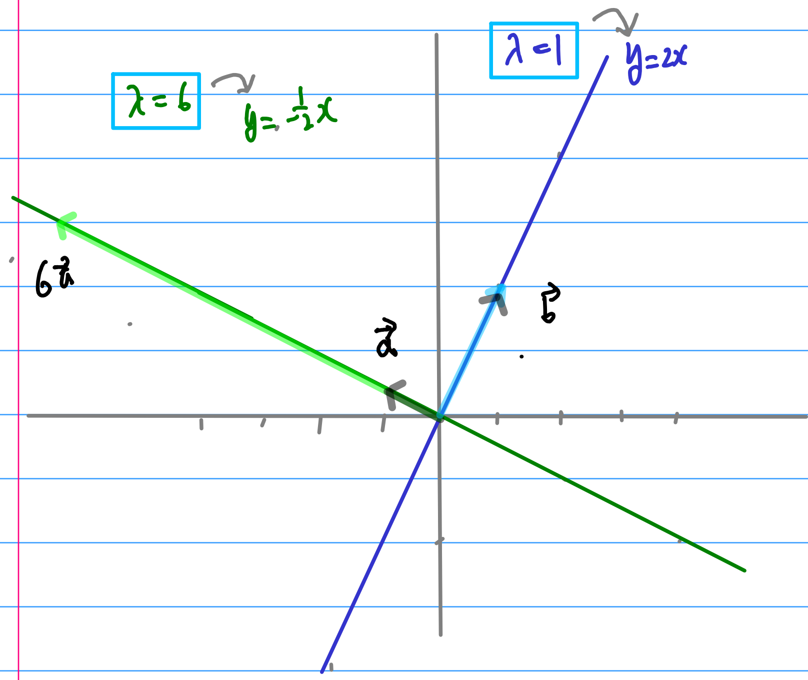
$4x - 2y = 0$

$-2x + y = 0$

$\lambda = 6$

$x + 2y = 0$

$-2x - 4y = 0$



$$X = Aa = 6a$$

$$X = Ab = 1b$$

$$a \xrightarrow[A]{\text{transform}} 6a$$

any vector in $y = -\frac{1}{2}x$

$$b \xrightarrow[A]{\text{transform}} 1b$$

any vector in $y = 2x$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \qquad \lambda = 1,\ 6$$

$$\left.\begin{aligned} \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= 6 \begin{pmatrix} a \\ b \end{pmatrix} \\ \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} &= 1 \begin{pmatrix} c \\ d \end{pmatrix} \end{aligned}\right\}$$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c & a \\ d & b \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

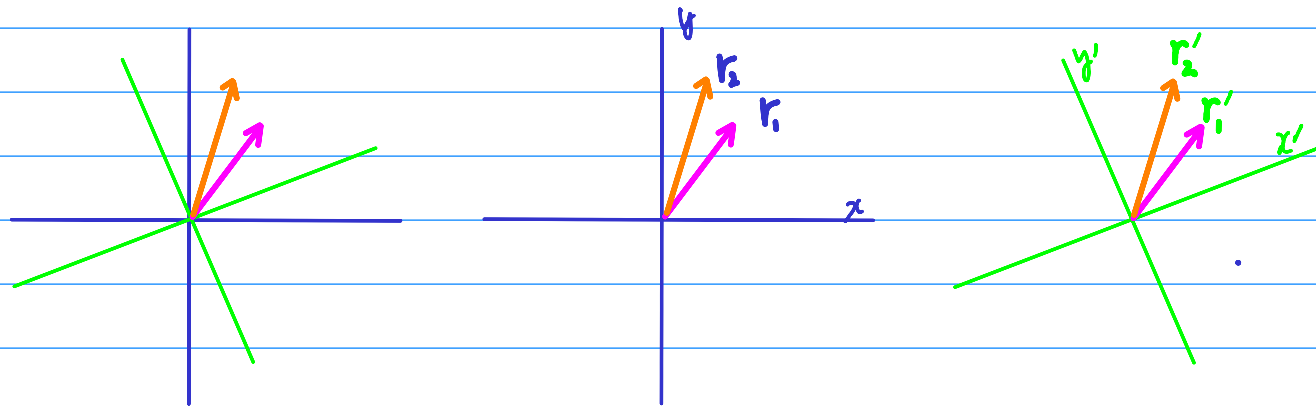
$$A \cdot P = P \cdot \Lambda$$

pick unit vectors $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$P = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

Boas



$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{P} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

* (perpendicular eigenvectors)

$$r_1 = P\, r_1' \qquad r_1' = P^{-1} \cdot r_1$$

$$r_2 = P\, r_2' \qquad r_2' = P^{-1} \cdot r_2$$

$$r_2 = A\, r_1$$ deformation in the (x, y) system

$$P\, r_2' = A\, P\, r_1'$$

$$r_2' = P^{-1} A\, P\, r_1' \qquad P^{-1} A\, P = B$$

$$r_2' = B\, r_1'$$ deformation in the (x', y') system

$r_2 = A \, r_1$ deformation in the (x, y) system

$r_2' = B \, r_1'$ deformation in the (x', y') system

$$P^{-1} A P = B$$

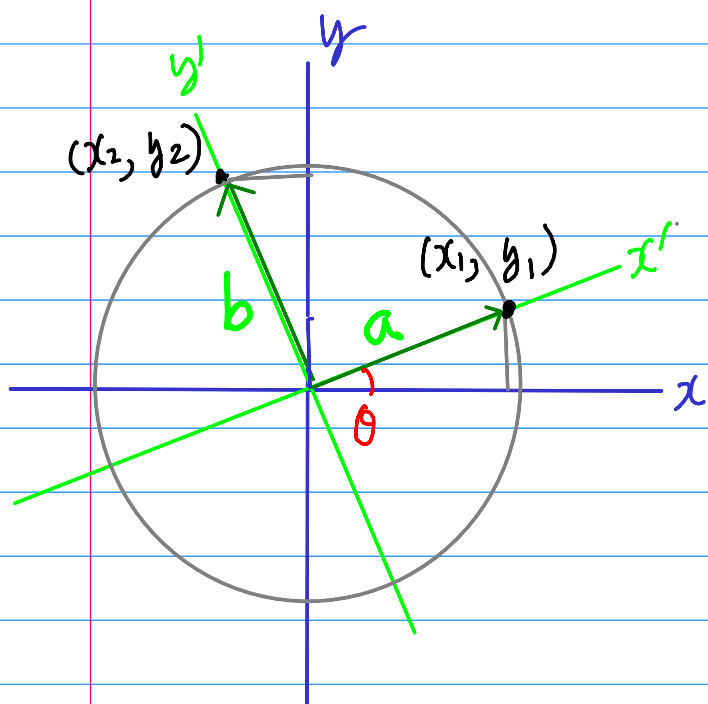
if P is chosen to make $P^{-1} A P = B \Rightarrow \Lambda$ a diagonal matrix

then the new x' and y' axes are in the direction of eigenvectors of A

$$P = \begin{pmatrix} \boxed{} & \boxed{} \end{pmatrix}$$

eigenvectors

if eigenvectors are perpendicular, then the x' and y' axes are a set of perpendicular axes rotated by angle θ



unit eigenvectors a, b

$x_1 = |a| \cos\theta$ $\qquad$ $x_2 = -|b| \sin\theta$

$y_1 = |a| \sin\theta$ $\qquad$ $y_2 = |b| \cos\theta$

$$P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

a matrix P which diagonalize A is the rotation matrix P when x' and y' axes are along the directions of perpendicular eigenvectors of A

☆ perpendicular eigenvectors of A $\Leftrightarrow$ P : orthogonal

$\Leftrightarrow$ A : symmetric

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$X' = x'$
$Y' = y'$

each point (x', y') has
its x' coordinate unchanged
its y' coordinate multiplied by 6

$\rightarrow$ stretching y' direction only

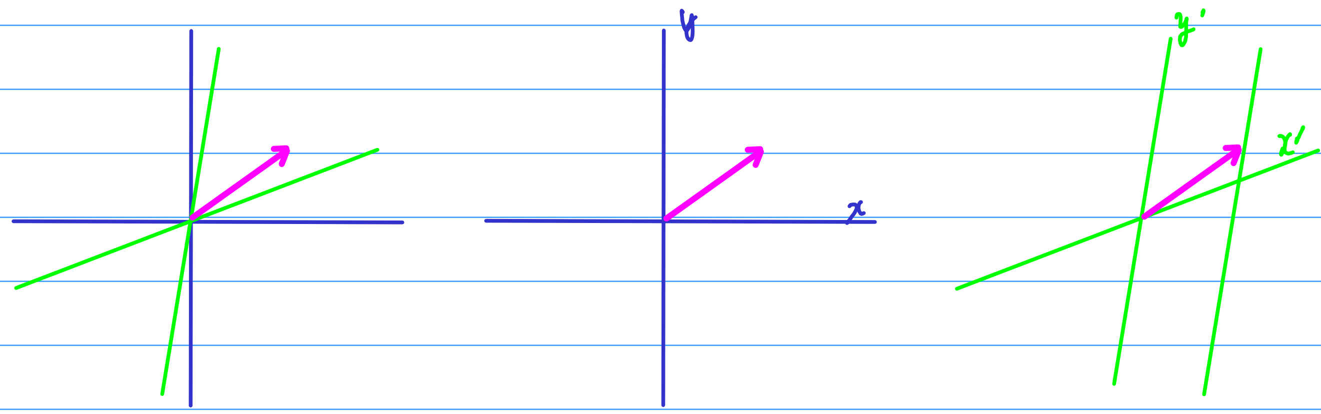
$A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$ symmetric

$P = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$ orthogonal $\qquad P^{-1} = P^{T}$

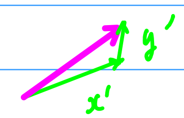
$\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = 1 \qquad \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = 0$

$\begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = 0 \qquad \begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = 1$

if eigenvectors are NOT perpendicular



$$\begin{pmatrix} x \\ y \end{pmatrix} = \cancel{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} x' \\ y' \end{pmatrix}$$



P just any non-singular matrix

$r_2 = A\, r_1$ deformation in the (x, y) system

$r_2' = B\, r_1'$ deformation in the (x', y') system

$$\boxed{P^{-1} A P = B}$$

still holds

if P is chosen
to make $P^{-1} A P = B \Rightarrow \Lambda$ a diagonal matrix

then the new x' and y' axes are
in the direction of eigenvectors of A