Strum-Liouville (H.1)
Background

20160102

b

HW #

Paul's Online Math Note : Differential Equations

Boundary Value Problem & Fourier Series

Fourier Cosine

Fourier Sine

Fourier Series

.

Zill & Wright   Sec 3.3 & 3.6

Advanced Engineering Mathematics in plain view
Wikiversity

Second Order
    Linear Equation
    Cauchy-Euler Equation

Hyperbolic Cos   cosh
Hyperbolic Sin   sinh
Calculus in plain view

Linear Homogeneous Eq

$$\begin{cases} a\,\boxed{y''} + b\,\boxed{y'} + c\,\boxed{y} = 0 & \text{constant coefficients} \\ a(t)\,\boxed{y''} + b(t)\,\boxed{y'} + c(t)\,\boxed{y} = 0 & \text{variable coefficients} \end{cases}$$

$$a\,y'' + b\,y' + c\,y = 0$$

Linear Eq with constant coefficient

$$a m^2 + b m + c = 0 \quad \text{aux}$$

$$m = m_1, \quad m_2$$

$$c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$a(t)\,y'' + b(t)\,y' + c(t)\,y = 0$$

$y' + \alpha y = 0$ $\qquad$ $y(x) = Ce^{-\alpha x}$

$y'' + \alpha^2 y = 0 \quad (\alpha > 0)$ $\qquad$ $y(x) = C_1 e^{-i\alpha x} + C_2 e^{+i\alpha x}$

$m^2 + \alpha^2 = 0$

$m^2 = -\alpha^2$ $\qquad$ $m = \pm i\alpha$ $\qquad$ $= C_3 \cos(\alpha x) + C_4 \sin(\alpha x)$

$y'' - \alpha^2 y = 0 \quad (\alpha > 0)$ $\qquad$ $y(x) = C_1 e^{-\alpha x} + C_2 e^{+\alpha x}$

$m^2 - \alpha^2 = 0$ $\qquad$ $= C_3 \cosh(\alpha x) + C_4 \sinh(\alpha x)$

$m^2 = \alpha^2$ $\qquad$ $m = \pm\alpha$

☆ Cauchy-Euler Equation

$x^2 y'' + x y' - \alpha^2 y = 0 \quad \alpha \geq 0$

$y(x) = C_1 x^{+\alpha} + C_2 x^{-\alpha} \quad \alpha > 0$

$m^2 - \alpha^2 = 0$ $\qquad$ $y(x) = C_1 x^0 + C_2 x^0 \cdot \ln x \quad \alpha = 0$

$m^2 = \alpha^2 \quad m = \pm\alpha$ $\qquad$ $= C_1 + C_2 \ln x$

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

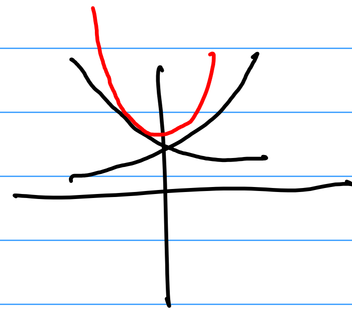
$$e^{x} = \cosh x + i\sinh x$$
$$e^{-x} = \cosh x - i\sinh x$$

$$\frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta$$

$$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \sin\theta$$

$$\frac{e^{x}+e^{-x}}{2} = \underline{\cosh(x)}$$

$$\frac{e^{x}-e^{-x}}{2} = \underline{\sinh(x)}$$



Bessel's Equation

$$x^2 \boxed{y''} + x \boxed{y'} + (x^2 - \nu^2) \boxed{y} = 0$$

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

Parametric Bessel's Equation

$$x^2 y'' + x y' + (\alpha^2 x^2 - \nu^2) y = 0$$

$$y = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$$

Parametric Bessel's Equation Order $\nu = 0$

integer

$$x^2 y'' + x y' + \alpha^2 x^2 y = 0$$

$$x y'' + y' + \alpha^2 x y = 0$$

$$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$$

Legendre's Equation

Zill & Wright 5.3.2

$$(1-x^2)\boxed{y''}-2x\boxed{y'}+n(n+1)\boxed{y}=0$$

order n

Legendre Polynomial

Order

	$(1-x^2)y''-2xy'+n(n+1)y=0$	
$n=0$	$(1-x^2)y''-2xy'+0\,y=0$	$y=P_0(x)=1$
$n=1$	$(1-x^2)y''-2xy'+2\,y=0$	$y=P_1(x)=x$
$n=2$	$(1-x^2)y''-2xy'+6\,y=0$	$y=P_2(x)=\frac{1}{2}(3x^2-1)$
$n=3$	$(1-x^2)y''-2xy'+12\,y=0$	$y=P_3(x)=\frac{1}{2}(5x^3-3x)$

☆ Linear Ordinary Differential Equations (ODE)

$y' + \alpha y = 0$ $\quad$ $y(x) = Ce^{-\alpha x}$

$y'' + \alpha^2 y = 0 \quad (\alpha > 0)$ $\quad$ $y(x) = c_1 e^{-i\alpha x} + c_2 e^{+i\alpha x}$

$= c_3 \cos(\alpha x) + c_4 \sin(\alpha x)$

$y'' - \alpha^2 y = 0 \quad (\alpha > 0)$ $\quad$ $y(x) = c_1 e^{-\alpha x} + c_2 e^{+\alpha x}$

$= c_3 \cosh(\alpha x) + c_4 \sinh(\alpha x)$

※ coefficient : α ····· constant

↑ constant coefficient

$x^2 y'' + x y' - \alpha^2 y = 0$

~~constant~~ coefficient

$$\boxed{y' + \alpha y = 0}$$

$m + \alpha = 0$ $\qquad$ $\underline{m = -\alpha}$

$y(x) = c\, e^{-\alpha x}$

$$\boxed{y'' + \alpha^2 y = 0 \quad (\alpha > 0)}$$

$\underline{m^2 + \alpha^2} = 0$ $\qquad$ $m = \pm i\alpha$ $\qquad$ $\left(D = 0^2 - 4\alpha^2 < 0\right)$

2 complex conjugate

$y(x) = c_1 e^{0 \cdot x} e^{+i\alpha x} + c_2 e^{0x} e^{-i\alpha x}$

$= c_3 \cos(\alpha x) + c_4 \sin(\beta x)$

$$\boxed{y'' - \alpha^2 y = 0 \quad (\alpha > 0)}$$

$m^2 - \alpha^2 = 0$ $\qquad$ $m = \pm\alpha$ $\qquad$ $\left(D = 0^2 + 4\alpha^2 > 0\right)$

2 distinct real.

$y(x) = c_1 e^{+\alpha x} + c_2 e^{-\alpha x}$

$= c_3 \cosh(\alpha x) + c_4 \sinh(\beta x)$

Linear Eq

$$ay'' + by' + cy = 0$$

$$am^2 + bm + c = 0$$

$$D < 0 \qquad m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$$

$$\boxed{c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t}}$$

$$W\left(e^{(\alpha + i\beta)t}, e^{(\alpha - i\beta)t}\right) \neq 0$$

linearly independent → Fundamental set of solution

$c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$ 일 때 solution이 된다

$$\frac{1}{2}\left(e^{(\alpha + i\beta)t} + e^{(\alpha - i\beta)t}\right) = e^{\alpha t}\cos(\beta t)$$

$c_1 = \frac{1}{2i}$, $c_2 = \frac{-1}{2i}$ 일 때도 solution이 된다

$$\frac{1}{2i}\left(e^{(\alpha + i\beta)t} - e^{(\alpha - i\beta)t}\right) = e^{\alpha t}\sin(\beta t)$$

$$\boxed{C_1 e^{(\alpha+i\beta)t} + C_2 e^{(\alpha-i\beta)t}}$$ General Solution

$C_1 = \frac{1}{2}$, $C_2 = \frac{1}{2}$ 일 때 solution이 된다

$$\frac{1}{2}\left(e^{(\alpha+i\beta)t} + e^{(\alpha-i\beta)t}\right) = e^{\alpha t}\cos(\beta t)$$

$C_1 = \frac{1}{2i}$, $C_2 = \frac{-1}{2i}$ 일 때도 solution이 된다

$$\frac{1}{2i}\left(e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t}\right) = e^{\alpha t}\sin(\beta t)$$

$e^{\alpha t}\cos(\beta t)$ 은 solution이고

$e^{\alpha t}\sin(\beta t)$ 도 solution이다

$$\boxed{C_3 e^{\alpha t}\cos(\beta t) + C_4 e^{\alpha t}\sin(\beta t)}$$ 도 solution이다.

$$W\left(\underline{e^{\alpha t}\cos(\beta t)}, \underline{e^{\alpha t}\sin(\beta t)}\right) \neq 0$$

linearly independent. → Fundamental set of solution

$$\boxed{C_3 e^{\alpha t}\cos(\beta t) + C_4 e^{\alpha t}\sin(\beta t)}$$ General Solution

$$e^{(2+\sqrt{5}i)t} \qquad e^{(2-\sqrt{5}i)t}$$

$$c_1 \underline{e^{2t} \cdot e^{\sqrt{5}it}} + c_2 \underline{e^{2t} e^{-\sqrt{5}it}}$$ sol

$$\frac{1}{2} e^{2t} \cdot e^{\sqrt{5}it} + \frac{1}{2} e^{2t} e^{-\sqrt{5}it}$$ → 에정 solution

$$c_3 e^{2t} \cos\sqrt{5}t + c_4 e^{2t} \sin\sqrt{5}t$$

$$y'' + 4y = 0$$

$$m^2 + 4 = 0 \qquad m = +2i, \ -2i$$

$$y = c_1 e^{2it} + c_2 e^{-2it}$$

$$= c_3 \cos(2t) + c_4 \sin(2t)$$

$$a y'' + b y' + c y = 0$$

if y_1 is a solution $\quad a y_1'' + b y_1' + c y_1 = 0$

if y_2 is a solution $\quad a y_2'' + b y_2' + c y_2 = 0$

$$a(y_1+y_2)'' + b(y_1+y_2)' + c(y_1+y_2) = 0$$

y_1+y_2 is also a solution

$c_1 y_1 + c_2 y_2$ is also a solution

If $c_1 e^{2t} \cdot e^{\sqrt{5}it} + c_2 e^{2t} e^{-\sqrt{5}it}$ is a general solution

$$\frac{1}{2} e^{2t} \cdot e^{\sqrt{5}it} + \frac{1}{2} e^{2t} e^{-\sqrt{5}it}$$

$$e^{2t}\left(\frac{e^{\sqrt{5}it} + e^{-\sqrt{5}it}}{2}\right)$$

$$e^{2t} \cos(\sqrt{5}t)$$

c_1 $\quad c_2$

$$\frac{1}{2i} e^{2t} \cdot e^{\sqrt{5}it} - \frac{1}{2i} e^{2t} e^{-\sqrt{5}it}$$

$$e^{2t}\left(\frac{e^{\sqrt{5}it} - e^{-\sqrt{5}it}}{2i}\right)$$

$$e^{2t} \sin(\sqrt{5}t)$$

then $e^{2t} \cos(\sqrt{5}t)$: a Solution

$e^{2t} \sin(\sqrt{5}t)$: a solution

$C_3 e^{2t} \cos(\sqrt{5}t) + C_4 e^{2t} \sin(\sqrt{5}t)$: a general solu

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$+ \quad e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

$$\left(e^{i\theta} + e^{-i\theta}\right) = 2\cos(\theta)$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$- \quad e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

$$\left(e^{i\theta} - e^{-i\theta}\right) = 2i\cos(\theta)$$

$$\cos(\theta) = \frac{e^{+i\theta} + e^{-i\theta}}{2}$$

$$\sin(\theta) = \frac{e^{+i\theta} - e^{-i\theta}}{2i}$$

$$y(x) = c_1 e^{+\alpha x} + c_2 e^{-\alpha x}$$

$$= c_3 \cosh(\alpha x) + c_4 \sinh(\beta x)$$

$$c_1 e^{+\alpha x} + c_2 e^{-\alpha x}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\frac{1}{2} e^{+\alpha x} + \frac{1}{2} e^{-\alpha x} = \frac{1}{2}\left(e^{+\alpha x} + e^{-\alpha x}\right) = \cosh(\alpha x)$$

$$\downarrow \qquad \qquad \downarrow$$

$$\frac{1}{2} e^{+\alpha x} - \frac{1}{2} e^{-\alpha x} = \frac{1}{2}\left(e^{+\alpha x} - e^{-\alpha x}\right) = \sinh(\alpha x)$$

Cauchy-Euler Equation

$$x^2 y'' + x y' - \alpha^2 y = 0 \qquad \alpha \geq 0$$

$y = x^m$ Suppose

$$x^2 m(m-1)x^{m-2} + x \cdot m x^{m-1} - \alpha^2 x^m = 0$$

$$(m^2 - \cancel{m} + \cancel{m} - \alpha^2)\, x^m = 0$$

$$(m^2 - \alpha^2) = 0 \qquad \cancel{x^m = 0}$$

$$m^2 = \alpha^2 \qquad m = \pm \alpha \qquad m = 0$$

$$y(x) = c_1 x^{+\alpha} + c_2 x^{-\alpha} \qquad \alpha > 0$$

$$y(x) = c_1 x^0 + c_2 x^0 \cdot \ln x \qquad \alpha = 0$$

$$= c_1 + c_2 \ln x$$

Bessel's Equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$\boxed{y = \sum_{n=0}^{\infty} c_n x^{n+r}}$$

$$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$$

Bessel Functions of the 1st kind

$(r = \nu)$
$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$(r = -\nu)$
$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

Bessel Functions of the 2nd kind

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

3.6 cauchy-euler
3.9 Boundary Value Problem

5.3 special function (Bessel, Legendre)

12.1
12.2
12.3
12.4
12.5

$$a y'' + b y' + c y = 0 \quad \leftarrow y_1, \ \leftarrow y_2$$

$$a y_1'' + b y_1' + c y_1 \Rightarrow 0$$

$$a y_2'' + b y_2' + c y_2 \Rightarrow 0$$

$$a(y_1'' + y_2'') + b(y_1' + y_2') + c(y_1 + y_2) \Rightarrow 0$$

$$a(y_1 + y_2)'' + b(y_1 + y_2)' + c(y_1 + y_2) \Rightarrow 0$$

$$\left.\begin{array}{l} y_1 \text{ sol.} \\ y_2 \text{ sol} \end{array}\right\} \quad c_1 y_1 + c_2 y_2 \text{ is sol}$$

$$W(y_1, y_2) \neq 0$$

lin. indep

$$c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t}.$$

$y_3 = e^{\alpha t} \cos\beta t \to$ sol

$y_4 = e^{\alpha t} \sin\beta t \to$ sol

$c_3 y_3 + c_4 y_4 \Rightarrow$ sol.

$\vdots$