Conformal Mapping (6A)

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12/24/13

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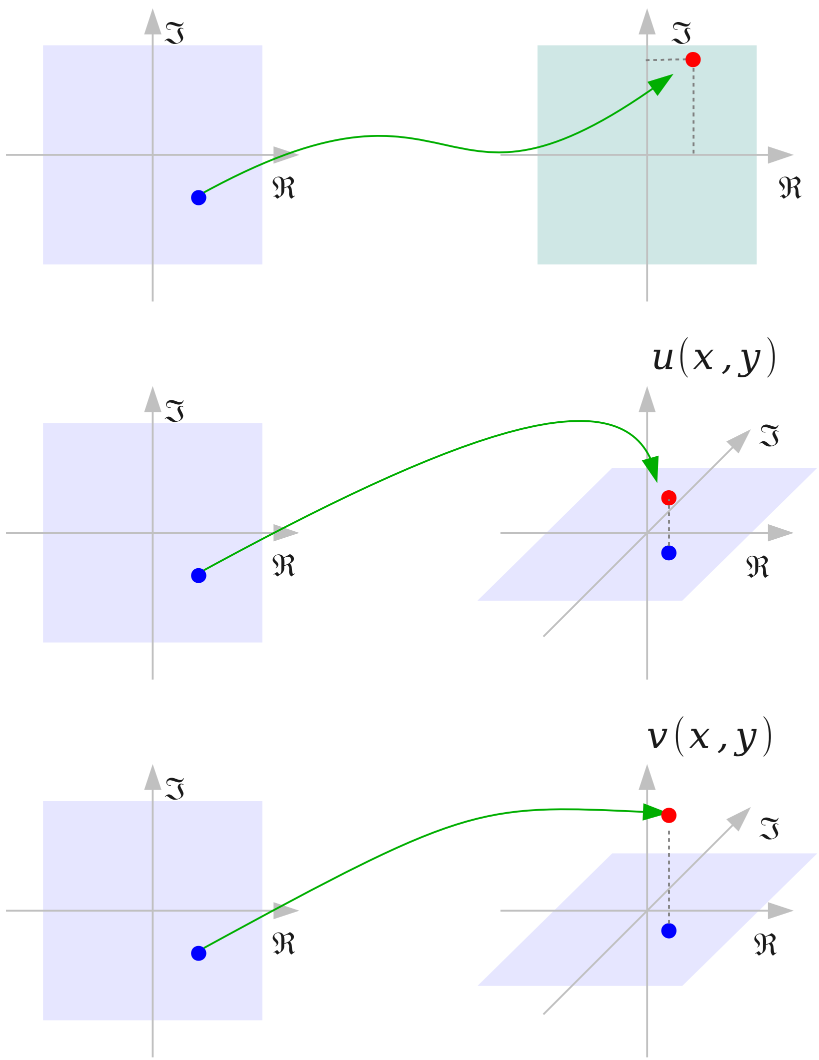
Visualizing Functions of a complex variable

$z = x + i\,y$ $\qquad$ $f(z) = u(x, y) + i\,v(u, y)$



Isocontour

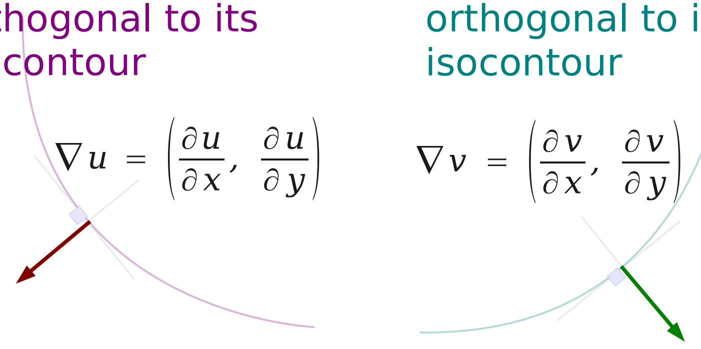
$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \qquad \nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

$$\nabla u \cdot \nabla v = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)$$

the gradient of $u(x,y)$: orthogonal to its isocontour

the gradient of $v(x,y)$: orthogonal to its isocontour



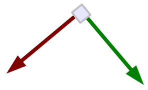
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

⇒ orthogonal isocontours

the isocontours of $u(x,y)$ and $v(x,y)$ are orthogonal to each other whenever they cross with each other

$$\nabla u \cdot \nabla v = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right) = 0$$

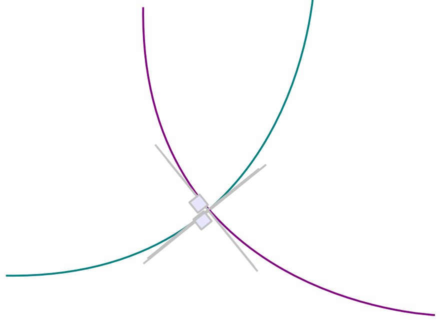
$$\nabla u \cdot \nabla v = 0 \qquad \nabla u \perp \nabla v$$



$\nabla u \perp \nabla v$

isocontour of $u(x,y)$

isocontour of $v(x,y)$



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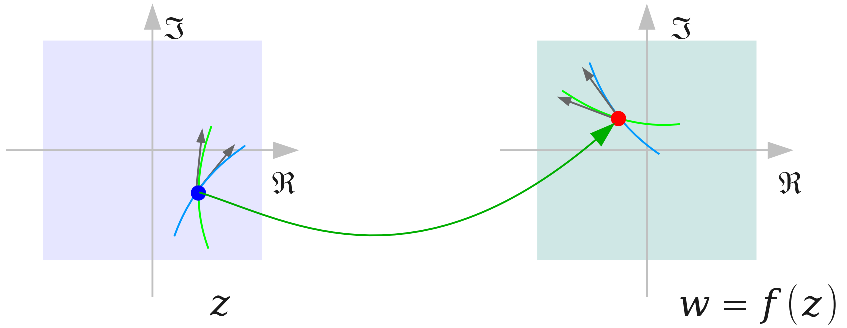
Angle Preserving Mappings

$z = x + iy$ $\qquad f(z) = u(x, y) + i v(u, y)$

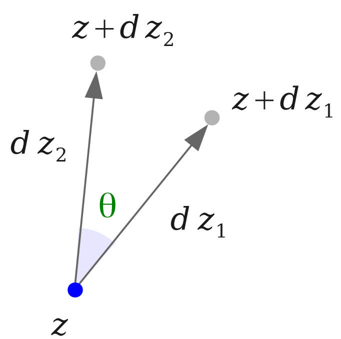


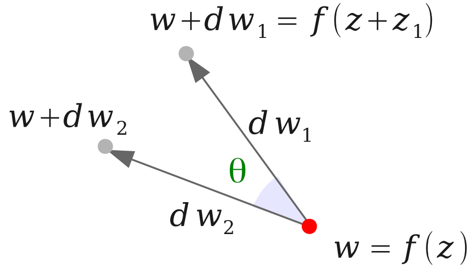
all angles are preserved
by the mapping w = f(z)
except where f'(z) = 0 and
$dw_1 = dw_2 = 0$ at first order.

$$f(z+\Delta z) - f(z) \approx f'(z)\Delta z$$

dw_1, dw_2 at first order

$$f(z+\Delta z) - f(z) = f'(z)\Delta z$$





$$f(z+z_1) - f(z) = w + dw_1 - w$$

$$f(z+z_2) - f(z) = w + dw_2 - w$$

$$dz_1 = |dz_1| e^{j\theta_1}$$

$$dz_2 = |dz_2| e^{j\theta_2}$$

$$\frac{dz_2}{dz_1} = \frac{|dz_2|}{|dz_1|} e^{j(\theta_2 - \theta_1)}$$

dw_1, dw_2 at first order

$$f'(z)\, dz_1 = dw_1$$

$$f'(z)\, dz_2 = dw_2$$

$$arg\left(\frac{dw_2}{dw_1}\right) = arg\left(\frac{dz_2}{dz_1}\right) = \theta$$

f'(z) = 0 and
$dw_1 = dw_2 = 0$ at first order.

$$0 \cdot dz_1 = dw_1 = 0$$

$$0 \cdot dz_2 = dw_2 = 0$$

⇨ $arg\left(\frac{dw_2}{dw_1}\right)$ *undefined*

regardless of $\theta = arg\left(\frac{dz_2}{dz_1}\right)$

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Angle Preserving Mappings

f'(z) = 0
f''(z) ≠ 0
dw_1 and dw_2 at second order

at such points,
angles are doubled

$$\frac{f''(z)}{2} dz_1^2 = dw_1$$

$$\frac{f''(z)}{2} dz_2^2 = dw_2$$

$$\frac{dz_2}{dz_1} = \left(\frac{dz_2}{dz_1}\right)^2 = \left(\frac{|z_2|}{|z_1|}\right)^2 e^{j2(\theta_2-\theta_1)} = 2\theta$$

$$f(z) = z^2$$

$$f'(z) = 2z$$

$$f''(z) = 2$$

$$z = re^{i\theta}$$

$$z^2 = r^2 e^{i2\theta}$$

f'(z) = 0
f''(z) = 0
$f^{(3)}(z) \neq 0$
dw_1 and dw_2 at third order

at such points,
angles are trippled

$$\frac{f^{(3)}(z)}{3!} dz_1^3 = dw_1$$

$$\frac{f^{(3)}(z)}{3!} dz_2^2 = dw_2$$

$$\frac{dz_2}{dz_1} = \left(\frac{dz_2}{dz_1}\right)^3 = \left(\frac{|z_2|}{|z_1|}\right)^3 e^{j3(\theta_2-\theta_1)} = 3\theta$$

$$f(z) = z^3$$

$$f'(z) = 3z^2$$

$$f''(z) = 6z$$

$$f^{(3)}(z) = 6$$

$$z = re^{i\theta}$$

$$z^3 = r^3 e^{i3\theta}$$

all angles are preserved by the mapping w = f(z) except where f'(z) = 0 and $dw_1 = dw_2 = 0$ at first order.

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A type of a critical point

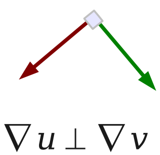
f'(z) = 0 and
$dw_1 = dw_2 = 0$ at first order.

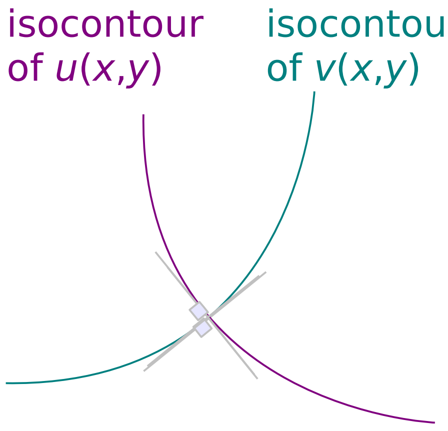
$$0 \cdot dz_1 = dw_1 = 0$$
$$0 \cdot dz_2 = dw_2 = 0$$

$$\Rightarrow \quad arg\left(\frac{dw_2}{dw_1}\right) \text{ undefined}$$

$$\text{regardless of } \theta = arg\left(\frac{dz_2}{dz_1}\right)$$

Except a critical point





all angles are preserved
by the mapping w = f(z)

At a critical point,
∇u = 0, then ∇v = ∇⊥ u = 0
the vectors do **not define tangent directions**

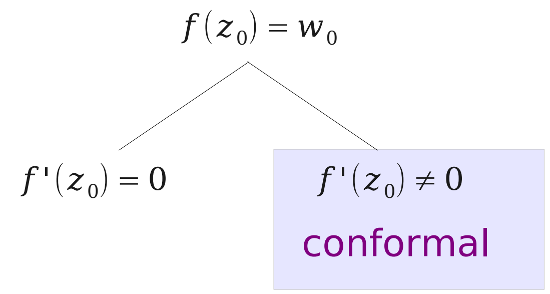
the orthogonality of the level curves
does not necessarily hold at critical points.

the critical points of u
= the critical points of v
= the critical points of f (z)
= points where its complex derivative vanishes: **f ′(z) = 0**.

$$f'(z) = 0$$

Conformal Condition

For every point z where f is **holomorphic** and **f'(z) ≠ 0**,
the mapping z → w = f (z) is **conformal**,
i.e., it preserves angles.



The phrase "holomorphic at a point z0" means
not just differentiable at z0,
but differentiable everywhere
within some neighborhood of z0.

The existence of a complex derivative
in a neighborhood is a very strong condition,
for it implies that any holomorphic function is
actually **infinitely differentiable** and
equal to its own **Taylor series**.
.

tangent vectors *dz* to each curve at z_0
are transformed into vectors *dw* at $w_0 = f(z_0)$ which are

- **magnified** by factor $|f'(z_0)|$
- **rotated** through angle ψ0 = $\arg\{f'(z_0)\}$

⇒ angles between curves remain the same
(conformal mapping)

Critical Points

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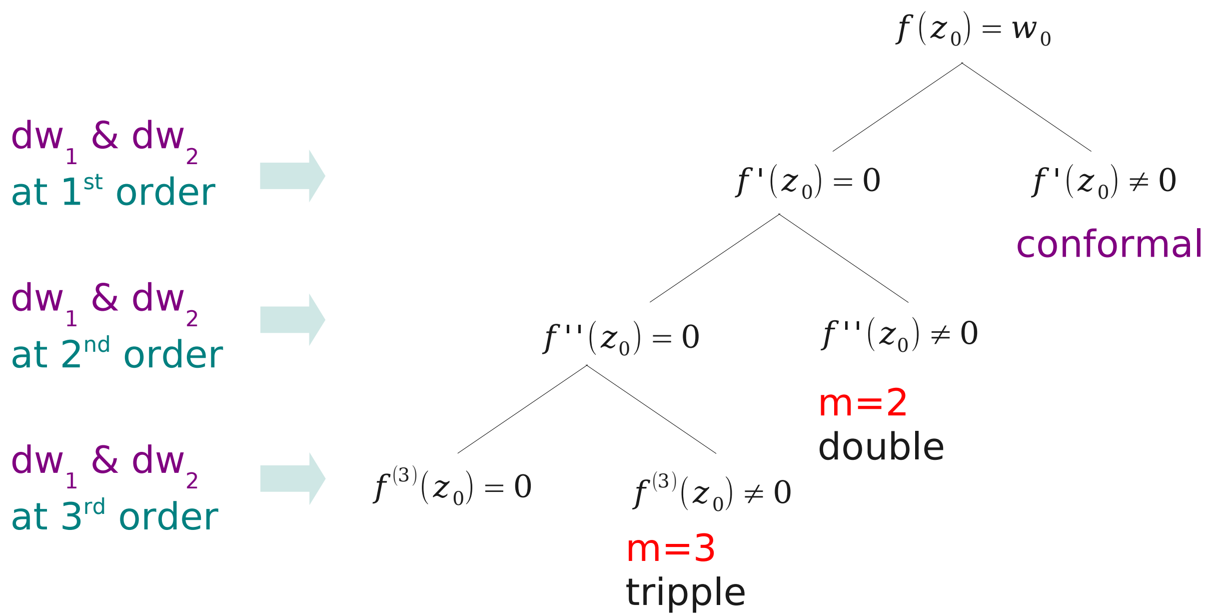
(conformal mapping)



Conformal Mapping

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Riemann Surface

A one-dimensional complex manifold.
can be thought of as "deformed versions" of the complex plane: locally near every point they look like patches of the complex plane, but the global topology can be quite different.
For example, they can look like a sphere or a torus or a couple of sheets glued together.

References

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