

Initial Value Problems (3B)

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Complex Exponentials

$$c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t}$$

$$(c_1 + c_2) = c_3$$

$$i(c_1 - c_2) = c_4$$

$$\begin{aligned} &= e^{-\sigma t} (c_1 e^{+i\omega t} + c_2 e^{-i\omega t}) \\ &= e^{-\sigma t} [c_1 (\cos(\omega t) + i \sin(\omega t)) + c_2 (\cos(\omega t) - i \sin(\omega t))] \\ &= e^{-\sigma t} [(c_1 + c_2) \cos(\omega t) + i(c_1 - c_2) \sin(\omega t)] \\ &= c_3 e^{-\sigma t} \cos(\omega t) + c_4 e^{-\sigma t} \sin(\omega t) \end{aligned}$$

$$c_3 e^{-\sigma t} \cos(\omega t) + c_4 e^{-\sigma t} \sin(\omega t)$$

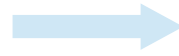
$$\frac{(c_3 - c_4 i)}{2} = c_1$$

$$\frac{(c_3 + c_4 i)}{2} = c_2$$

$$\begin{aligned} &= c_3 e^{-\sigma t} (e^{+i\omega t} + e^{-i\omega t})/2 + c_4 e^{-\sigma t} (e^{+i\omega t} - e^{-i\omega t})/2i \\ &= c_3 e^{-\sigma t} (e^{+i\omega t} + e^{-i\omega t})/2 + c_4 e^{-\sigma t} (-ie^{+i\omega t} + ie^{-i\omega t})/2 \\ &= \frac{(c_3 - c_4 i)}{2} e^{-\sigma t} e^{+i\omega t} + \frac{(c_3 + c_4 i)}{2} e^{-\sigma t} e^{-i\omega t} \\ &= c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t} \end{aligned}$$

Complex Exponentials

$$c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t}$$

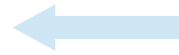


$$c_3 e^{-\sigma t} \cos(\omega t) + c_4 e^{-\sigma t} \sin(\omega t)$$

$$(c_1 + c_2) = c_3$$

$$i(c_1 - c_2) = c_4$$

$$c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t}$$



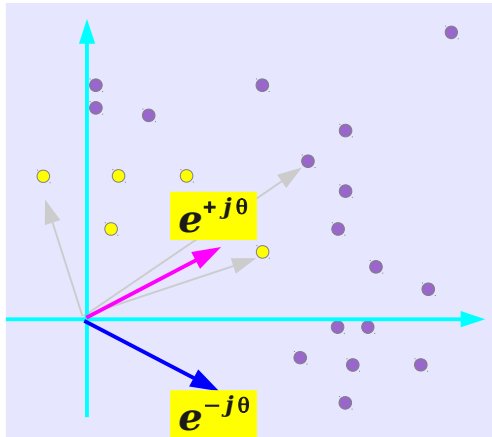
$$c_3 e^{-\sigma t} \cos(\omega t) + c_4 e^{-\sigma t} \sin(\omega t)$$

$$c_1 = \frac{(c_3 - c_4 i)}{2}$$

$$c_2 = \frac{(c_3 + c_4 i)}{2}$$

Basis of the Complex Plane

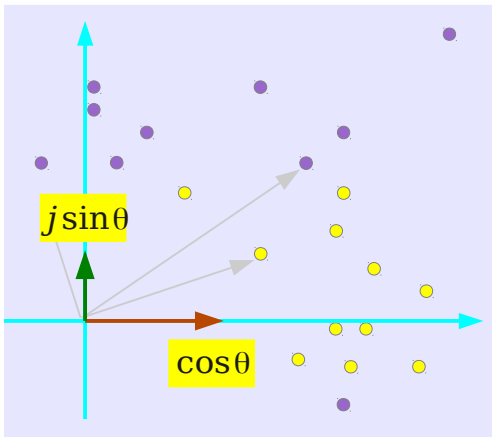
Basis : a set of linear independent spanning vectors



every complex number can be represented by

$$\boxed{k_1} e^{+j\theta} + \boxed{k_2} e^{+j\theta}$$

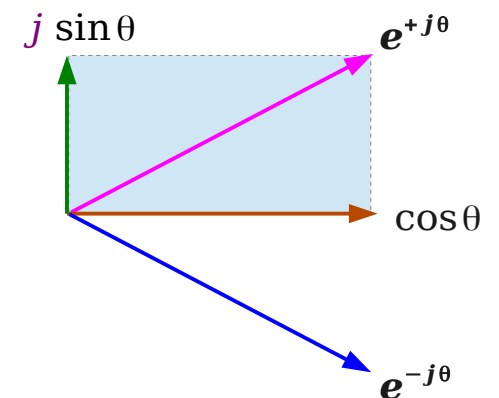
linear combination of $e^{+j\theta}$ and $e^{-j\theta}$ which are one set of linear independent two vectors



every complex number can also be represented by

$$\boxed{l_1} \cos\theta + \boxed{l_2} j \sin\theta$$

$$\boxed{l_1} \cos\theta + \boxed{l_2} j \sin\theta$$



Real Coefficients C_1 & C_2

$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$$

real number

real number

$$c_1 = (c_3 - c_4 i) / 2$$

$$c_2 = (c_3 + c_4 i) / 2$$

$$1 \cdot e^{+i\omega t} + 0 \cdot e^{-i\omega t}$$

$$\frac{1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega t}$$

$$0 \cdot e^{+i\omega t} + 1 \cdot e^{-i\omega t}$$

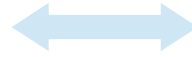
$$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega t}$$

$$-1 \cdot e^{+i\omega t} + 0 \cdot e^{-i\omega t}$$

$$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega t}$$

$$0 \cdot e^{+i\omega t} - 1 \cdot e^{-i\omega t}$$

$$\frac{1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega t}$$



$$c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

$$c_3 = (c_1 + c_2)$$

$$c_4 = i(c_1 - c_2)$$

real number

imaginary number

$$1 \cdot \cos(\omega t) + 1i \cdot \sin(\omega t)$$

$$\sqrt{2} \cdot \cos(\omega t) + 0i \cdot \sin(\omega t)$$

$$1 \cdot \cos(\omega t) - 1i \cdot e^{-\sigma t} \sin(\omega t)$$

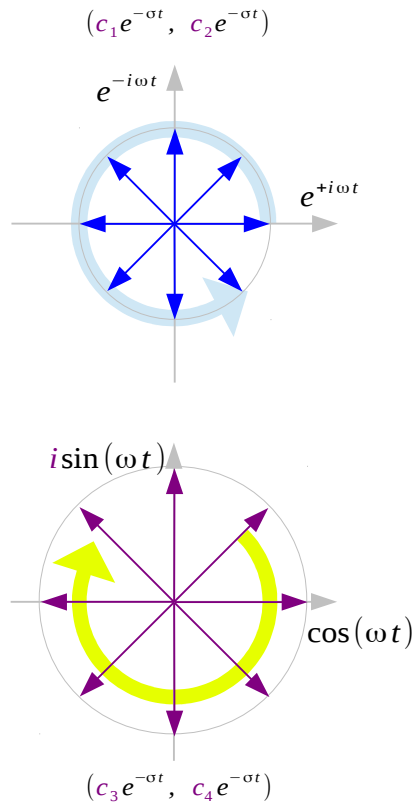
$$0 \cdot \cos(\omega t) - \sqrt{2}i \cdot \sin(\omega t)$$

$$-1 \cdot \cos(\omega t) - 1i \cdot \sin(\omega t)$$

$$-\sqrt{2} \cdot \cos(\omega t) - 0i \cdot \sin(\omega t)$$

$$-1 \cdot \cos(\omega t) + 1i \cdot \sin(\omega t)$$

$$0 \cdot \cos(\omega t) + \sqrt{2}i \cdot \sin(\omega t)$$



Complex Plane Basis

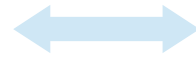
$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$$

$$c_1 = (c_3' + c_4')/2$$

$$c_2 = (c_3' - c_4')/2$$

real number

real number



$$c_3' \cos(\omega t) + c_4' i \sin(\omega t)$$

$$c_3' = (c_1 + c_2)$$

$$c_4' = (c_1 - c_2)$$

real number

real number

C¹

$$1 \cdot e^{+i\omega t} + 0 \cdot e^{-i\omega t}$$

$$\frac{1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega t}$$

$$0 \cdot e^{+i\omega t} + 1 \cdot e^{-i\omega t}$$

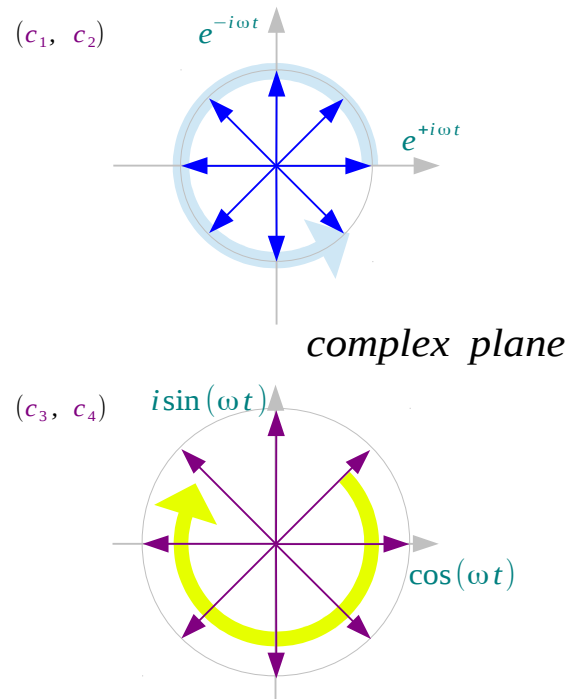
$$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega t}$$

$$-1 \cdot e^{+i\omega t} + 0 \cdot e^{-i\omega t}$$

$$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega t}$$

$$0 \cdot e^{+i\omega t} - 1 \cdot e^{-i\omega t}$$

$$\frac{1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega t}$$



$$1 \cdot \cos(\omega t) + 1i \cdot \sin(\omega t)$$

$$\sqrt{2} \cdot \cos(\omega t) + 0i \cdot \sin(\omega t)$$

$$1 \cdot \cos(\omega t) - 1i \cdot \sin(\omega t)$$

$$0 \cdot \cos(\omega t) - \sqrt{2}i \cdot \sin(\omega t)$$

$$-1 \cdot \cos(\omega t) - 1i \cdot \sin(\omega t)$$

$$-\sqrt{2} \cdot \cos(\omega t) - 0i \cdot \sin(\omega t)$$

$$-1 \cdot \cos(\omega t) + 1i \cdot \sin(\omega t)$$

$$0 \cdot \cos(\omega t) + \sqrt{2}i \cdot \sin(\omega t)$$

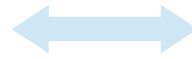
Real Coefficients C_3 & C_4

$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$$

$$c_1 = (c_3 - c_4 i) / 2$$

$$c_2 = (c_3 + c_4 i) / 2$$

conjugate
complex number



$$c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

$$c_3 = (c_1 + c_2)$$

$$c_4 = i(c_1 - c_2)$$

real number
real number

\mathbb{R}^2 ^{+2*real part}
_{-2*imag part}

$$\frac{(+1-0i)}{2} \cdot e^{+i\omega t} + \frac{(+1+0i)}{2} \cdot e^{-i\omega t}$$

$$\frac{(+1-i)}{2\sqrt{2}} \cdot e^{+i\omega t} + \frac{(+1+i)}{2\sqrt{2}} \cdot e^{-i\omega t}$$

$$\frac{(0-i)}{2} \cdot e^{+i\omega t} + \frac{(0+i)}{2} \cdot e^{-i\omega t}$$

$$\frac{(-1-i)}{2\sqrt{2}} \cdot e^{+i\omega t} + \frac{(-1+i)}{2\sqrt{2}} \cdot e^{-i\omega t}$$

$$\frac{(-1-0i)}{2} \cdot e^{+i\omega t} + \frac{(-1+0i)}{2} \cdot e^{-i\omega t}$$

$$\frac{(-1+i)}{2\sqrt{2}} \cdot e^{+i\omega t} + \frac{(-1-i)}{2\sqrt{2}} \cdot e^{-i\omega t}$$

$$\frac{(0+i)}{2} \cdot e^{+i\omega t} + \frac{(0-i)}{2} \cdot e^{-i\omega t}$$

$$\frac{(+1+i)}{2\sqrt{2}} \cdot e^{+i\omega t} + \frac{(+1-i)}{2\sqrt{2}} \cdot e^{-i\omega t}$$

$$1 \cdot \cos(\omega t) + 0 \cdot \sin(\omega t)$$

$$\frac{1}{\sqrt{2}} \cdot \cos(\omega t) + \frac{1}{\sqrt{2}} \cdot \sin(\omega t)$$

$$0 \cdot \cos(\omega t) + 1 \cdot \sin(\omega t)$$

$$\frac{-1}{\sqrt{2}} \cdot \cos(\omega t) + \frac{1}{\sqrt{2}} \cdot \sin(\omega t)$$

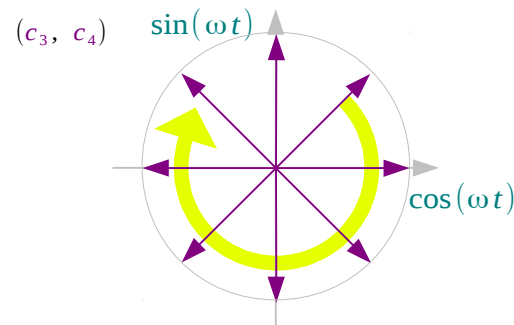
$$-1 \cdot \cos(\omega t) + 0 \cdot \sin(\omega t)$$

$$\frac{-1}{\sqrt{2}} \cdot \cos(\omega t) + \frac{-1}{\sqrt{2}} \cdot \sin(\omega t)$$

$$0 \cdot \cos(\omega t) - 1 \cdot \sin(\omega t)$$

$$\frac{1}{\sqrt{2}} \cdot \cos(\omega t) - \frac{1}{\sqrt{2}} \cdot \sin(\omega t)$$

real plane

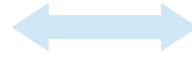


Real Coefficients C_3 & C_4

$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$$

$$\begin{aligned} c_1 &= (c_3 - c_4 i) / 2 \\ c_2 &= (c_3 + c_4 i) / 2 \end{aligned}$$

*conjugate
complex number*



$$c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

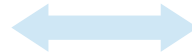
\mathbb{R}^2 +2*real part
-2*imag part

$$\begin{aligned} c_3 &= (c_1 + c_2) \\ c_4 &= i(c_1 - c_2) \end{aligned}$$

real number

real number

$$A \cos(\omega t - \varphi)$$



$$c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

$$\begin{aligned} \sqrt{c_3^2 + c_4^2} &= A \\ \frac{c_3}{\sqrt{c_3^2 + c_4^2}} &= \cos(\varphi) \\ \frac{c_4}{\sqrt{c_3^2 + c_4^2}} &= \sin(\varphi) \end{aligned}$$

C^1 and R^2 Spaces

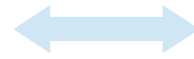
$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$$

$$c_1 = (c_3 + c_4)/2$$

$$c_2 = (c_3 - c_4)/2$$

real number

real number



$$c_3 \cos(\omega t) + c_4 i \sin(\omega t)$$

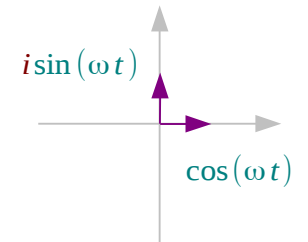
$$c_3 = (c_1 + c_2)$$

$$c_4 = (c_1 - c_2)$$

real number

real number

C^1



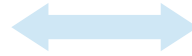
$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$$

$$c_1 = (c_3 - c_4 i)/2$$

$$c_2 = (c_3 + c_4 i)/2$$

conjugate

complex number



$$c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

+2*real part

-2*imag part

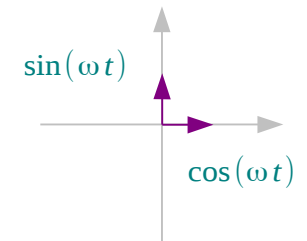
$$c_3 = (c_1 + c_2)$$

$$c_4 = i(c_1 - c_2)$$

real number

real number

R^2



Signal Spaces and Phasors

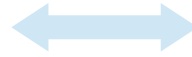
$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$$

$$c_1 = (c_3 - c_4 i) / 2$$

$$c_2 = (c_3 + c_4 i) / 2$$

conjugate

complex number



$$c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

+2*real part

-2*imag part

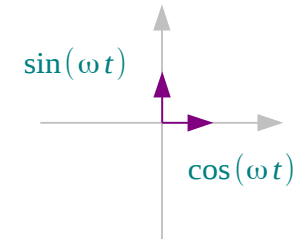
$$c_3 = (c_1 + c_2)$$

$$c_4 = i(c_1 - c_2)$$

real number

real number

\mathbb{R}^2



Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

auxiliary equation

try a solution $y = e^{mx}$



$$(am^2 + bm + c) = 0$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$



(A) $b^2 - 4ac > 0$ Real, distinct m_1, m_2

$$y_1 = e^{m_1 x} = y_2 = e^{m_2 x}$$



(B) $b^2 - 4ac = 0$ Real, equal m_1, m_2

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$$



(C) $b^2 - 4ac < 0$ Conjugate complex m_1, m_2

Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

$$\frac{d^2 y}{dx^2} + \frac{b}{a} \frac{dy}{dx} + \frac{c}{a} y = 0$$

$$\frac{d^2 y}{dx^2} + 2\lambda \frac{dy}{dx} + \omega^2 y = 0$$

$$\lambda = \frac{b}{2a} \quad \omega^2 = \frac{c}{a}$$

$$D = \left(\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)$$

$$D = (2\lambda)^2 - 4\omega^2$$

$$D = (b^2 - 4ac)$$

$$D = \lambda^2 - \omega^2$$

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

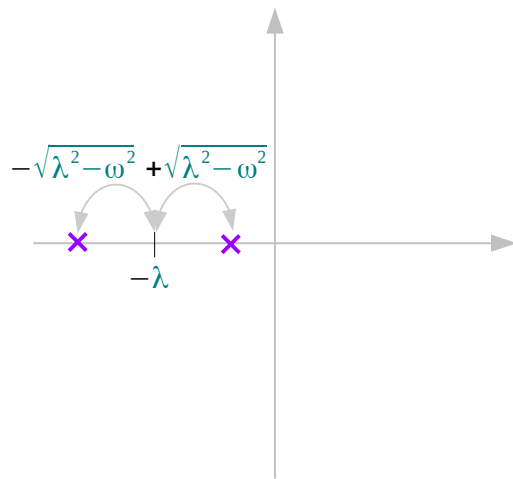
$$m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$$

Homogeneous Second Order DEs with Constant Coefficients

$$\frac{d^2 y}{dx^2} + \frac{b}{a} \frac{dy}{dx} + \frac{c}{a} y = 0$$

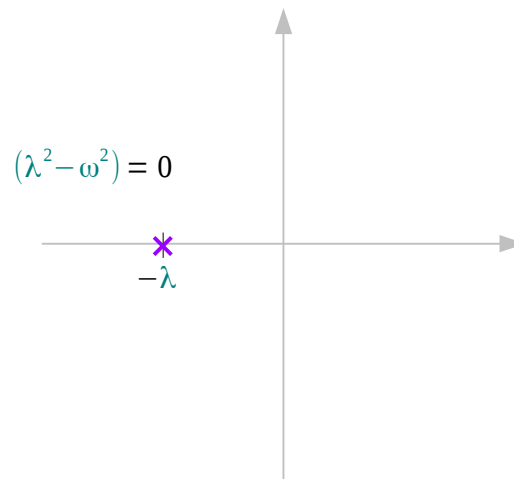
$$\frac{d^2 y}{dx^2} + 2\lambda \frac{dy}{dx} + \omega^2 y = 0$$

$$\lambda = \frac{b}{2a} \quad \omega^2 = \frac{c}{a}$$



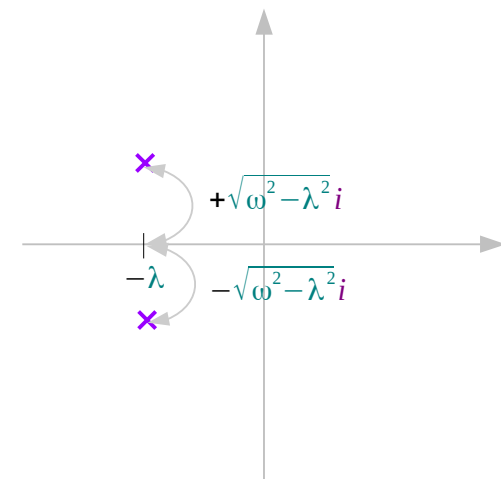
overdamped $y(t)$

$$e^{-\lambda t} (c_1 e^{+\sqrt{\lambda^2 - \omega^2} t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2} t})$$



critically damped $y(t)$

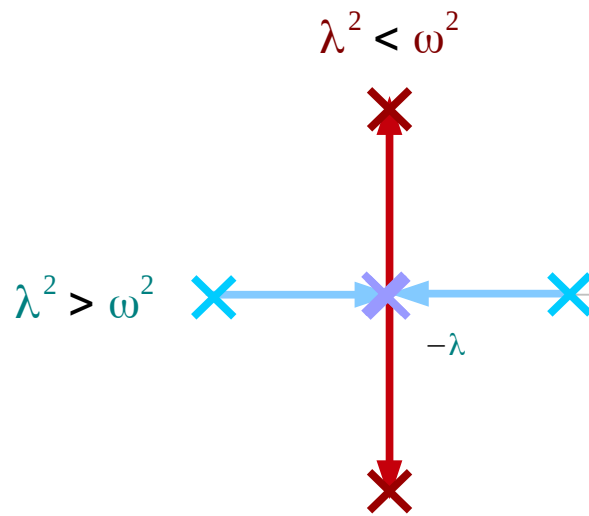
$$e^{-\lambda t} (c_1 + c_2 t)$$



underdamped $y(t)$

$$e^{-\lambda t} (c_1 \cos \sqrt{\lambda^2 - \omega^2} t + c_2 \sin \sqrt{\lambda^2 - \omega^2} t)$$

for a given λ
increasing ω



underdamped $y(t)$ $\lambda^2 < \omega^2$

$$e^{-\lambda t} (c_1 \cos \sqrt{\lambda^2 - \omega^2} t + c_2 \sin \sqrt{\lambda^2 - \omega^2} t)$$

complex m_1, m_2

overdamped $y(t)$ $\lambda^2 > \omega^2$

$$e^{-\lambda t} (c_1 e^{+\sqrt{\lambda^2 - \omega^2} t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2} t})$$

real m_1, m_2

critically damped $y(t)$ $\lambda^2 = \omega^2$

$$e^{-\lambda t} (c_1 + c_2 t)$$

real $m_1 = m_2$

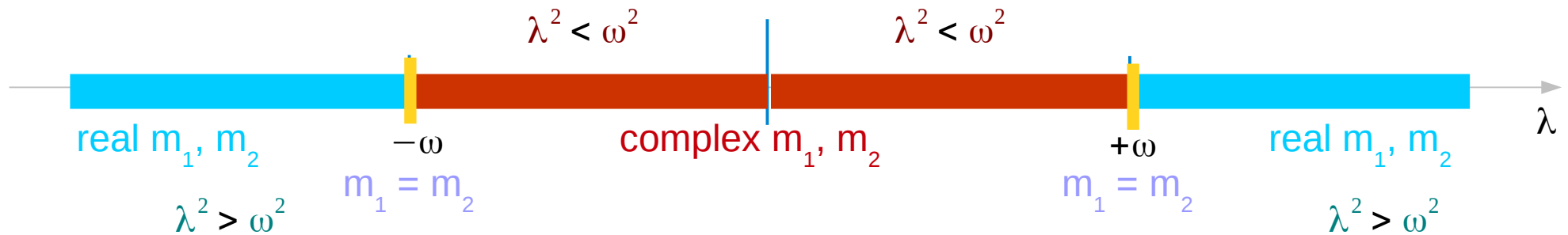
critically damped $y(t)$

$$e^{-\lambda t}(c_1 + c_2 t)$$

$$\lambda^2 = \omega^2$$

underdamped $y(t)$

$$e^{-\lambda t}(c_1 \cos \sqrt{\lambda^2 - \omega^2} t + c_2 \sin \sqrt{\lambda^2 - \omega^2} t)$$



overdamped $y(t)$

$$e^{-\lambda t}(c_1 e^{+\sqrt{\lambda^2 - \omega^2} t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2} t})$$

for a given ω
increasing λ

Homogeneous Second Order DEs with Constant Coefficients

$$\frac{d^2 y}{dx^2} + \frac{b}{a} \frac{dy}{dx} + \frac{c}{a} y = 0$$

$$\lambda = 1 \quad \omega = 1/2$$

$$\lambda = 1 \quad \omega = 1$$

$$\lambda = 1 \quad \omega = 2$$

$$\frac{d^2 y}{dx^2} + 2\lambda \frac{dy}{dx} + \omega^2 y = 0$$

$$\lambda = \frac{b}{2a} \quad \omega^2 = \frac{c}{a}$$

$$y'' + 2y' + y/4 = 0$$

$$y = c_1 e^{(-1+\sqrt{3}/2)t} + c_2 e^{(-1-\sqrt{3}/2)t}$$

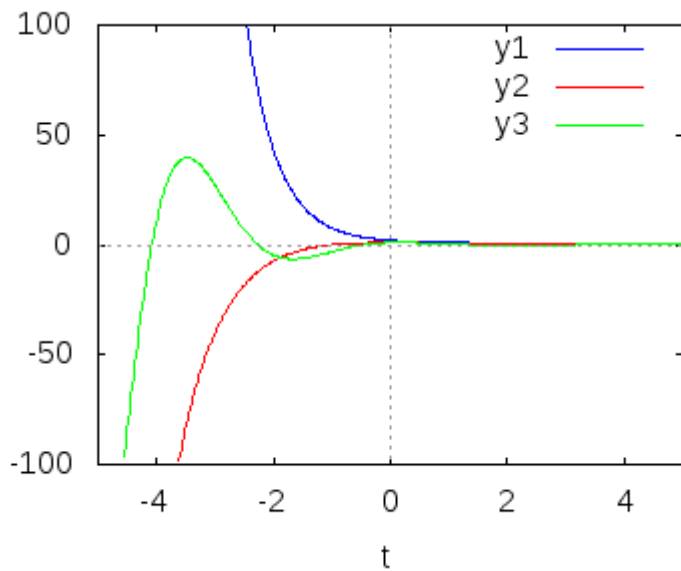
$$y'' + 2y' + y = 0$$

$$y = c_1 e^{-t} + c_2 t e^{-t}$$

$$y'' + 2y' + 4y = 0$$

$$y = c_1 e^{(-1+\sqrt{3}i)t} + c_2 e^{(-1-\sqrt{3}i)t}$$

$$= e^{-t} (c_3 \cos(\sqrt{3}t) + c_4 \sin(\sqrt{3}t))$$



$$\lambda = 1 \quad \omega = 1/2$$

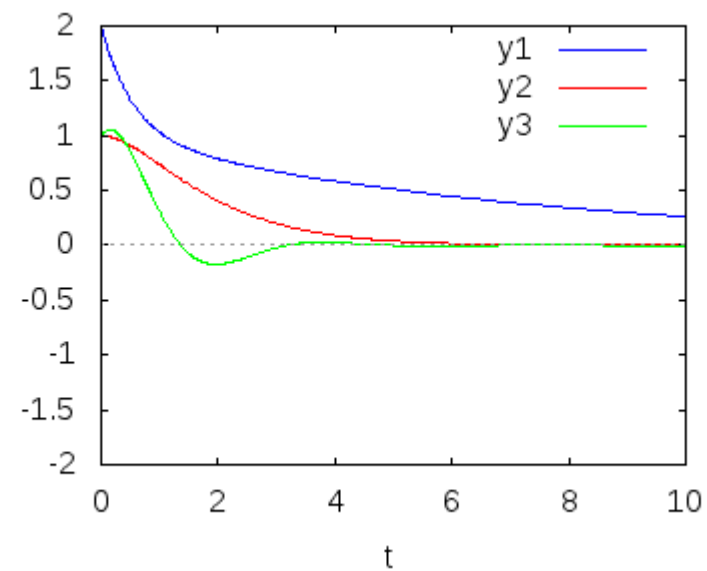
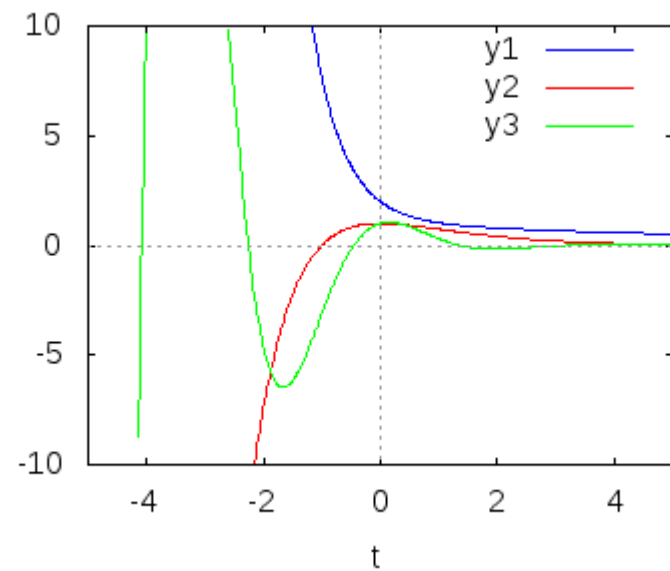
$$y_1(t) = e^{(-1+\sqrt{3}/2)t} + e^{(-1-\sqrt{3}/2)t}$$

$$\lambda = 1 \quad \omega = 1$$

$$y_2(t) = e^{-t} + te^{-t}$$

$$\lambda = 1 \quad \omega = 2$$

$$y_3(t) = e^{-t}(\cos(\sqrt{3}t) + \sin(\sqrt{3}t))$$



Homogeneous Second Order DEs with Constant Coefficients

$$\frac{d^2 y}{dx^2} + \frac{b}{a} \frac{dy}{dx} + \frac{c}{a} y = 0$$

$$\lambda = 2 \quad \omega = 1$$

$$\lambda = 1 \quad \omega = 1$$

$$\lambda = 1/2 \quad \omega = 1$$

$$\frac{d^2 y}{dx^2} + 2\lambda \frac{dy}{dx} + \omega^2 y = 0$$

$$\lambda = \frac{b}{2a} \quad \omega^2 = \frac{c}{a}$$

$$y'' + 4y' + y = 0$$

$$y = c_1 e^{(-2+\sqrt{3})t} + c_2 e^{(-2-\sqrt{3})t}$$

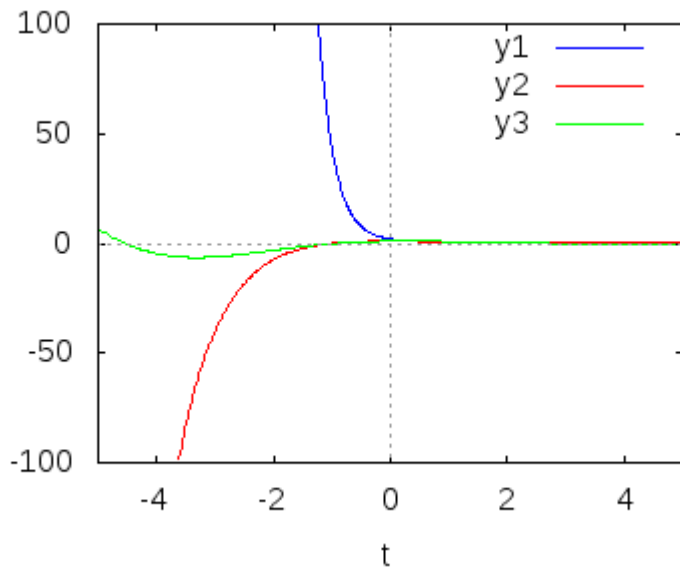
$$y'' + 2y' + y = 0$$

$$y = c_1 e^{-t} + c_2 t e^{-t}$$

$$y'' + y' + y = 0$$

$$y = c_1 e^{(-1+\sqrt{3}i)/2t} + c_2 e^{(-1-\sqrt{3}i)/2t}$$

$$= e^{-t/2} (c_3 \cos(\sqrt{3}/2 t) + c_4 \sin(\sqrt{3}/2 t))$$



$$\lambda = 2 \quad \omega = 1$$

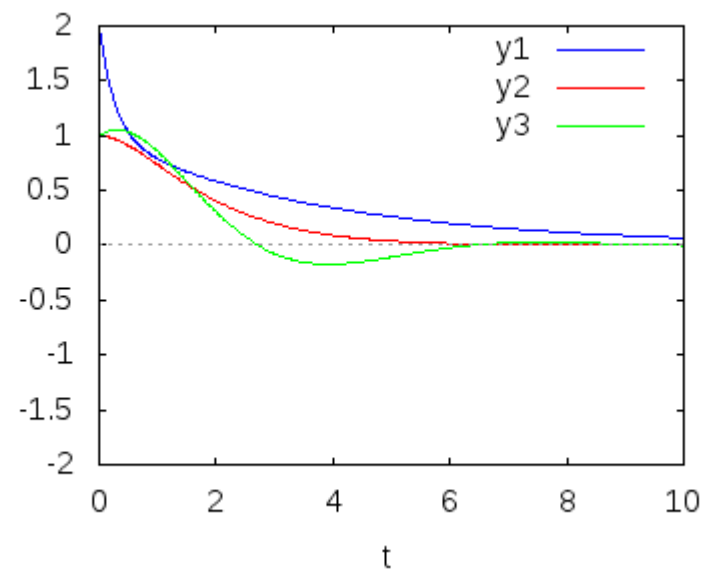
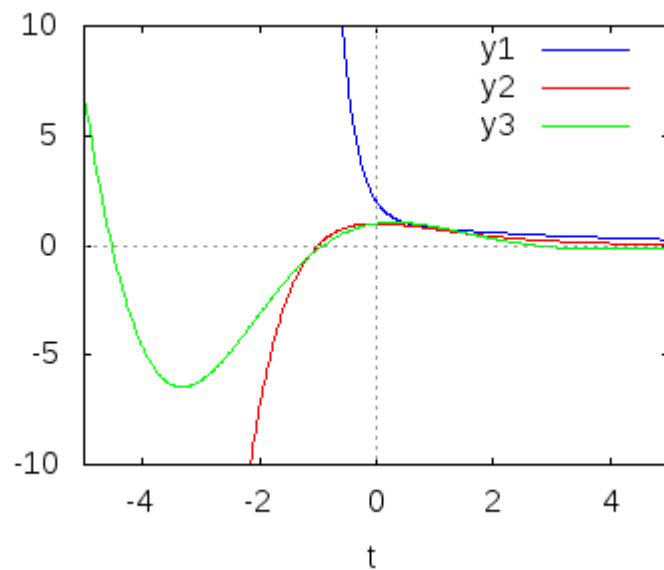
$$y_1(t) = c_1 e^{(-2+\sqrt{3})t} + c_2 e^{(-2-\sqrt{3})t}$$

$$\lambda = 1 \quad \omega = 1$$

$$y_2(t) = c_1 e^{-t} + c_2 t e^{-t}$$

$$\lambda = 1/2 \quad \omega = 1$$

$$y_3(t) = e^{-t/2} (c_3 \cos(\sqrt{3}/2 t) + c_4 \sin(\sqrt{3}/2 t))$$



References

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