Complex Series (3C)

Young Won Lim
3/3/14

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Taylor Series

A **power series** in powers of $(z - z_0)$

non-negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + a_4(z-z_0)^4 + \cdots$$

The **Taylor series** of a function $f(z)$

non-negative powers

$$f^{(1)}(z) = a_1 + 2a_2(z-z_0)^1 + 3a_3(z-z_0)^2 + 4a_4(z-z_0)^3 + \cdots$$

$$f^{(2)}(z) = + 2!a_2 + 3\cdot 2 a_3(z-z_0)^1 + 4\cdot 3 a_4(z-z_0)^2 + \cdots$$

$$f^{(3)}(z) = + 3!a_3 + 4\cdot 3\cdot 2 a_4(z-z_0)^1 + \cdots$$

$$f^{(n)}(z_0) = n!\, a_n$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\,(z - z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

converges for all z in the open disk with center z_0 and radius generally equal to the distance from z_0 to the nearest singularity of $f(z)$

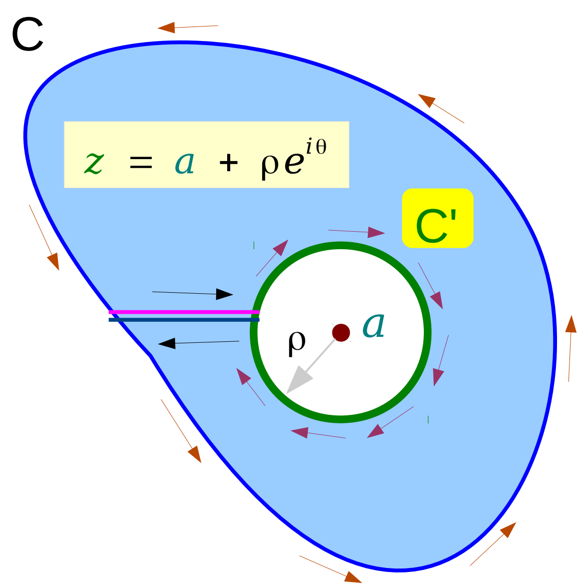
Cauchy's Integral Formula

$f(z)$: **analytic on** and **inside** simple close curve C

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$

the value of $f(z)$ at a point $z = a$ inside C

$$\oint_{ccw\ C} \frac{f(z)}{z-a}\,dz = \oint_{ccw\ C'} \frac{f(z)}{z-a}\,dz$$



$z = a + \rho e^{i\theta}$ ➡ $dz = i\rho e^{i\theta} d\theta$

➡ $\dfrac{dz}{z-a} = \dfrac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$

$$\oint_{ccw\ C} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} f(z)\, i\, d\theta = 2\pi i\ f(a)$$

$$\frac{d}{dz} f(z) = \frac{d}{dz}\left\{\frac{1}{2\pi i}\oint \frac{f(w)}{w-z}\,dw\right\}$$

$$f'(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{(w-z)^2}\,dw$$

Taylor Series Coefficients

A **power series** in powers of $(z-z_0)$

non-negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + a_4(z-z_0)^4 + \cdots$$

The **Taylor series** of a function $f(z)$

non-negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\,(z - z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

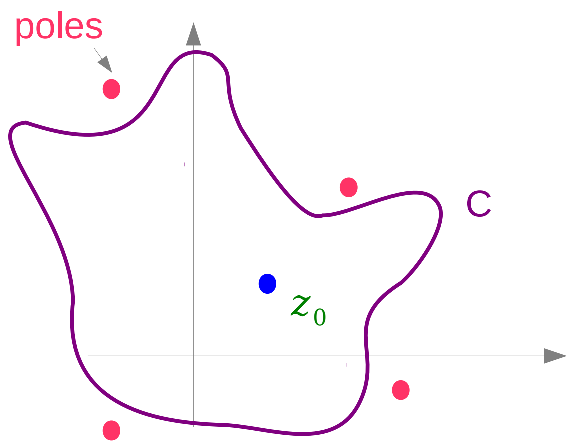
$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

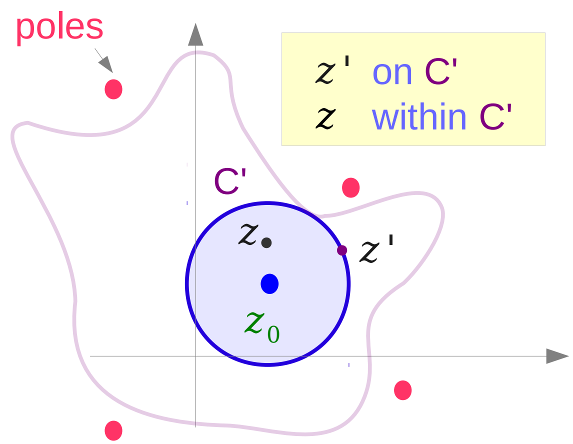
$$f(z) = \sum_{n=0}^{\infty} \left[\left(\frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}} dw \right) (z - z_0)^n \right]$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}} dw$$

Taylor Series From Cauchy's Integral Formula

$$f(z) = \frac{1}{2\pi i}\oint_{C'} \frac{f(w)}{w-z}\,dw \;\Rightarrow\; f(z) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\oint_{C'} \frac{f(z')}{(z'-z_0)} \cdot \sum_{n=0}^{+\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n dz'$$





Integration along the arbitrary contour C

Deformation Theorem

Integration along the contour C'

$$\frac{1}{z'-z} = \frac{1}{(z'-z_0)+(z_0-z)}$$

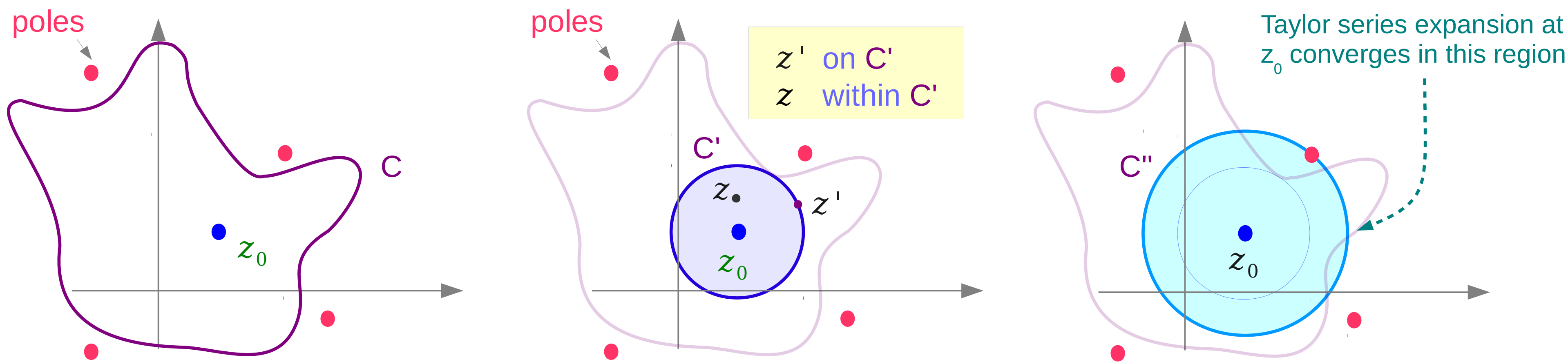
$$= \frac{1}{(z'-z_0)\left(1+\frac{(z_0-z)}{(z'-z_0)}\right)}$$

$$= \frac{1}{(z'-z_0)} \cdot \frac{1}{1-\left(\frac{z-z_0}{z'-z_0}\right)}$$

$$= \frac{1}{(z'-z_0)} \cdot \sum_{n=0}^{+\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$$

$$\frac{(z-z_0)}{(z'-z_0)} < 1 \;\Rightarrow\; \frac{1}{1-\left(\frac{z-z_0}{z'-z_0}\right)} = \sum_{n=0}^{+\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n \;\Rightarrow$$

Taylor Series Convergence



Integration along the arbitrary contour C

Deformation Theorem

Integration along the contour C'

$$f(z) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z')}{(z'-z_0)} \cdot \sum_{n=0}^{+\infty} \left(\frac{z-z_0}{z'-z_0}\right)^n dz'$$

$$f(z) = \sum_{n=0}^{+\infty} \left[\frac{1}{2\pi i}\oint_{C'} \frac{f(z')}{(z'-z_0)^{n+1}} dz'\right] (z-z_0)^n$$

$$f(z) = \sum_{n=0}^{+\infty} a_n (z-z_0)^n \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

z must be within the largest circle centered on z_0 that can be inscribed within in C

$$\Rightarrow \quad \frac{(z-z_0)}{(z'-z_0)} < 1$$

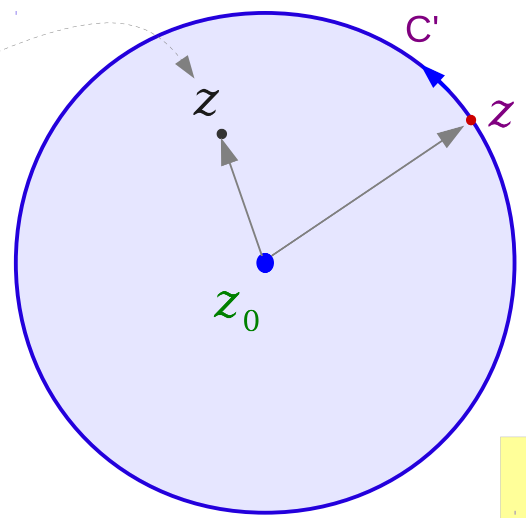
For Taylor Series

Any C must enclose z_0

Any C must not enclose any poles

Taylor Series ROC

$f(z)$: the function value at z



contour integration along C'

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$$

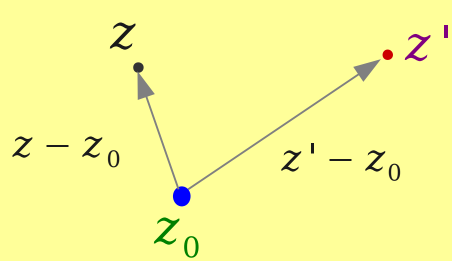
$$f(z_0) = \frac{1}{2\pi i} \oint_{C'} \frac{f(z')}{(z' - z_0)^{n+1}} dz'$$

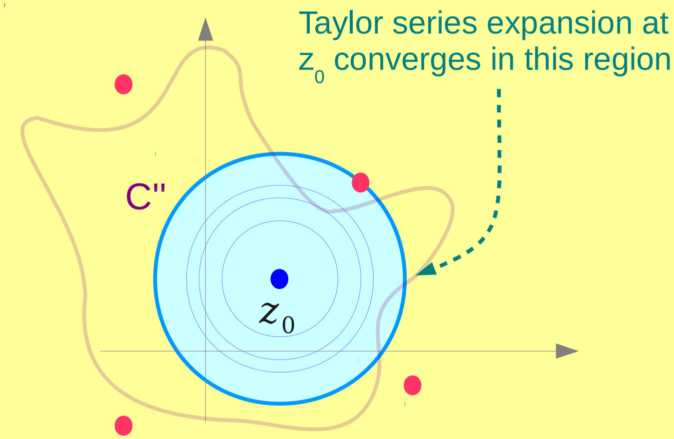
$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

$$f(z) = \sum_{n=0}^{+\infty} \left[\frac{1}{2\pi i} \oint_{C'} \frac{f(z')}{(z' - z_0)^{n+1}} dz' \right] (z - z_0)^n$$

Converges because

$$\frac{(z - z_0)}{(z' - z_0)} < 1$$





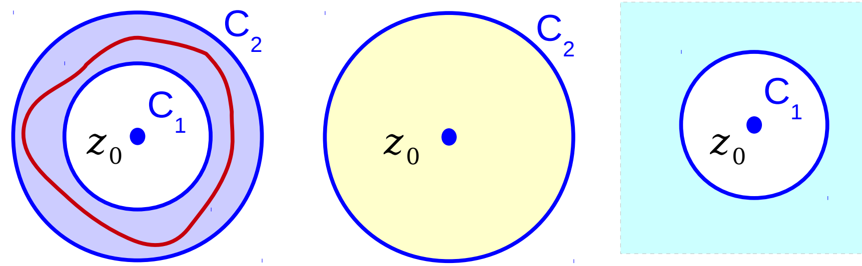
For Taylor Series

Any C must enclose z_0

Any C must not enclose any poles

Laurent's Theorem and Coefficients

$f(z)$: **analytic** in the annular domain D
between concentric circles C_1 and C_2
centered at z_0 $\quad r < |z - z_0| < R$



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$$

convergent in the domain D
any simple closed path C in D

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k$$

$$= \sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz\right)(z - z_0)^k\right]$$

$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

Contour Integration and Coefficients

$$f(z) = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots$$

$$\int_C f(z)\,dz = \int_C a_0 z^0\,dz + \int_C a_1 z^1\,dz + \int_C a_2 z^2\,dz + \cdots + \int_C b_1 z^{-1}\,dz + \int_C b_2 z^{-2}\,dz + \cdots$$

$$\int_C \frac{f(z)}{z}\,dz = \int_C \frac{a_0}{z}\,dz + \int_C a_1 z^0\,dz + \int_C a_2 z^1\,dz + \cdots \qquad \int_C \frac{f(z)}{z}\,dz = a_0 \cdot 2\pi i \qquad \frac{1}{z^{0+1}}$$

$$\int_C \frac{f(z)}{z^2}\,dz = \int_C a_0 z^{-2}\,dz + \int_C \frac{a_1}{z}\,dz + \int_C a_2 z^0\,dz + \cdots \qquad \int_C \frac{f(z)}{z^2}\,dz = a_1 \cdot 2\pi i \qquad \frac{1}{z^{1+1}}$$

$$\int_C \frac{f(z)}{z^3}\,dz = \int_C a_0 z^{-3}\,dz + \int_C a_1 z^{-2}\,dz + \int_C \frac{a_2}{z}\,dz + \cdots \qquad \int_C \frac{f(z)}{z^3}\,dz = a_2 \cdot 2\pi i \qquad \frac{1}{z^{2+1}}$$

$$\int_C f(z)\,dz = \cdots + \int_C \frac{b_1}{z}\,dz + \int_C \frac{b_2}{z^2}\,dz + \int_C \frac{b_3}{z^3}\,dz + \cdots \qquad \int_C f(z)\,dz = b_1 \cdot 2\pi i \qquad \frac{1}{z^{-1+1}}$$

$$b_1 = a_{-1}$$

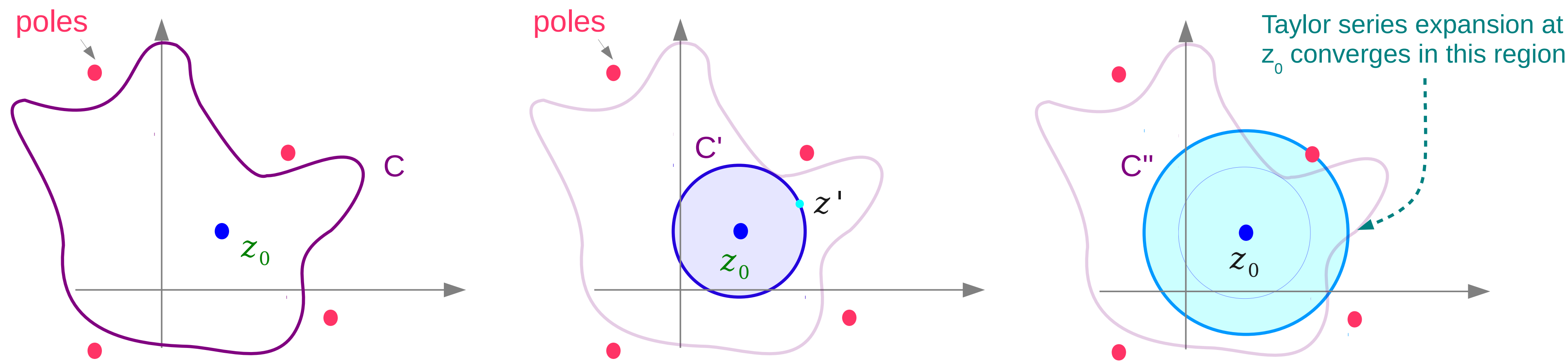
$$\int_C f(z)\cdot z\,dz = \cdots + \int_C b_1\,dz + \int_C \frac{b_2}{z}\,dz + \int_C \frac{b_3}{z^2}\,dz + \cdots \qquad \int_C f(z)\cdot z\,dz = b_2 \cdot 2\pi i \qquad \frac{1}{z^{-2+1}}$$

$$b_2 = a_{-2}$$

$$\int_C f(z)\cdot z^2\,dz = \cdots + \int_C b_1 z\,dz + \int_C b_2\,dz + \int_C \frac{b_3}{z}\,dz + \cdots \qquad \int_C f(z)\cdot z^2\,dz = b_3 \cdot 2\pi i \qquad \frac{1}{z^{-3+1}}$$

$$b_3 = a_{-3}$$

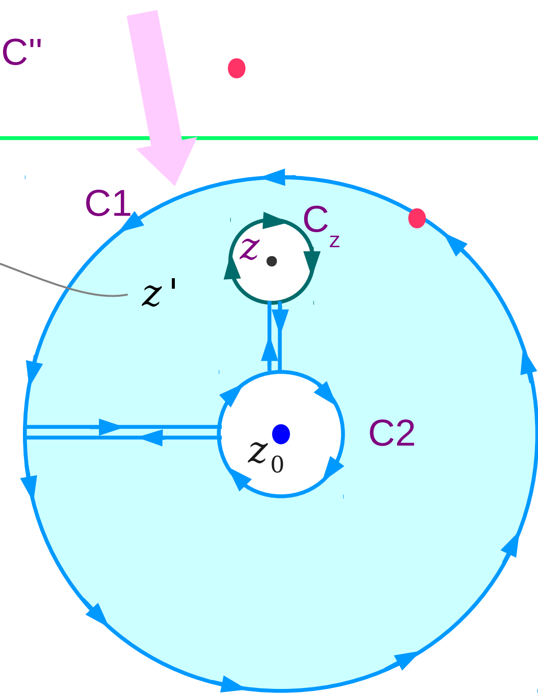
Contours : C1, C2, Cz



Integration along the arbitrary contour C

Deformation Theorem

Integration along the contour C'



z_0 the center

z the evaluation point

z' the contour point

$f(z')$	**: analytic**	for all z'	in this region
$\dfrac{f(z')}{(z'-z)}$	**: analytic**	for all z'	in this region
			z is excluded

$$\frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}dz' = \frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}dz' + \frac{1}{2\pi i}\oint_{Cz}\frac{f(z')}{z'-z}dz'$$

$$\frac{1}{2\pi i}\oint_{Cz}\frac{f(z')}{z'-z}dz' \Rightarrow f(z)$$

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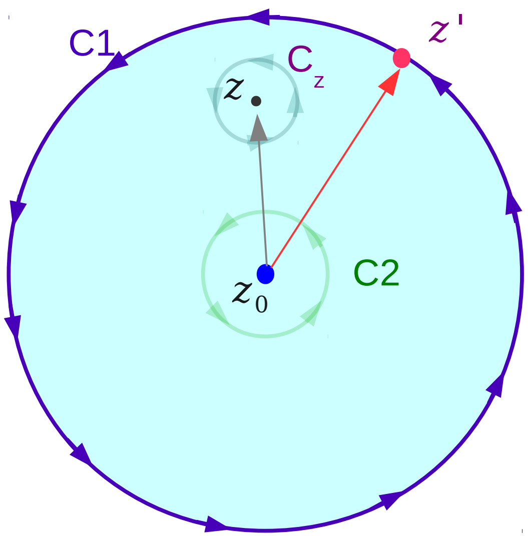
Two Converging Regions

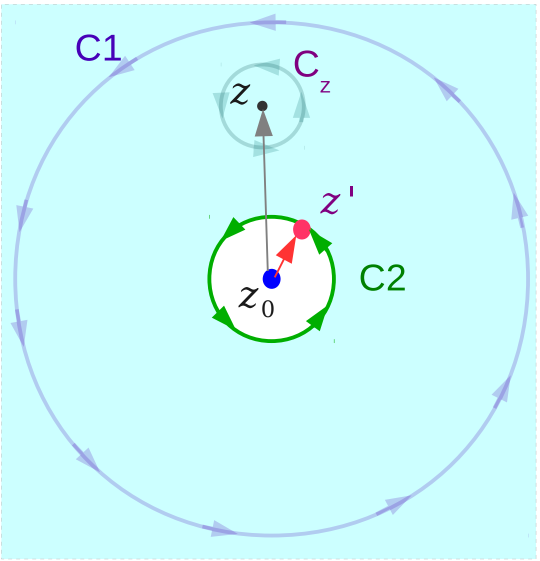
$$\frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}dz' = \frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}dz' + \frac{1}{2\pi i}\oint_{Cz}\frac{f(z')}{z'-z}dz'$$

$$f(z) = \frac{1}{2\pi i}\oint_{Cz}\frac{f(z')}{z'-z}dz' = \frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}dz' - \frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}dz'$$



$$\frac{(z-z_0)}{(z'-z_0)} < 1$$

$$\frac{1}{2\pi i}\oint_{C1}\frac{f(z')}{z'-z}dz'$$



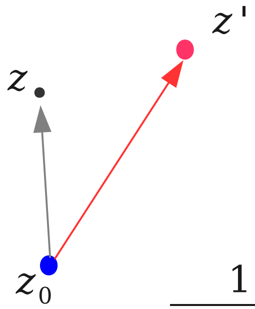
$$\frac{(z-z_0)}{(z'-z_0)} > 1$$

$$\frac{(z'-z_0)}{(z-z_0)} < 1$$

$$-\frac{1}{2\pi i}\oint_{C2}\frac{f(z')}{z'-z}dz'$$

Two Representations of 1 / (z' – z)

$$f(z) = \frac{1}{2\pi i}\oint_{Cz} \frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\oint_{C1} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\oint_{C2} \frac{f(z')}{z'-z}\,dz'$$



$$\frac{(z - z_0)}{(z' - z_0)} < 1$$

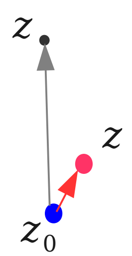
$$\frac{1}{(z'-z)} = \frac{1}{(z'-z_0)} \cdot \frac{1}{1 - \left(\frac{z-z_0}{z'-z_0}\right)}$$

$$= \frac{1}{(z'-z_0)} \cdot \sum_{n=0}^{+\infty} \left(\frac{z-z_0}{z'-z_0}\right)^n$$

$$= \sum_{n=0}^{+\infty} \frac{(z-z_0)^n}{(z'-z_0)^{n+1}}$$

$$= \sum_{k=0}^{+\infty} \frac{(z-z_0)^k}{(z'-z_0)^{k+1}}$$

$$\frac{1}{(z'-z)} = \sum_{k=0}^{+\infty} \frac{(z-z_0)^k}{(z'-z_0)^{k+1}}$$



$$\frac{(z' - z_0)}{(z - z_0)} < 1$$

$$-\frac{1}{(z'-z)} = \frac{1}{(z-z_0)} \cdot \frac{1}{1 - \left(\frac{z'-z_0}{z-z_0}\right)}$$

$$= \frac{1}{(z-z_0)} \cdot \sum_{n=0}^{+\infty} \left(\frac{z'-z_0}{z-z_0}\right)^n$$

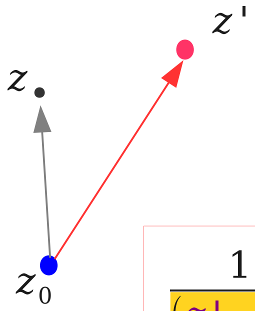
$m = n + 1$ ➡ $$= \sum_{n=0}^{+\infty} \frac{(z'-z_0)^n}{(z-z_0)^{n+1}} = \sum_{m=1}^{+\infty} \frac{(z'-z_0)^{m-1}}{(z-z_0)^m}$$

$k = -m$ ➡ $$= \sum_{k=-\infty}^{-1} \frac{(z-z_0)^k}{(z'-z_0)^{k+1}}$$

$$\frac{-1}{(z'-z)} = \sum_{k=-\infty}^{-1} \frac{(z-z_0)^k}{(z'-z_0)^{k+1}}$$

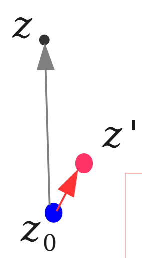
Two Contour Integrations

$$f(z) = \frac{1}{2\pi i}\oint_{Cz} \frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\oint_{C1} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\oint_{C2} \frac{f(z')}{z'-z}\,dz'$$



$$\frac{(z-z_0)}{(z'-z_0)} < 1$$

$$\frac{1}{(z'-z)} = \sum_{k=0}^{+\infty} \frac{(z-z_0)^k}{(z'-z_0)^{k+1}}$$



$$\frac{(z'-z_0)}{(z-z_0)} < 1$$

$$\frac{-1}{(z'-z)} = \sum_{k=-\infty}^{-1} \frac{(z-z_0)^k}{(z'-z_0)^{k+1}}$$

$$\frac{1}{2\pi i}\oint_{C1} \frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\oint_{C1} f(z')\left[\sum_{k=0}^{+\infty} \frac{(z-z_0)^k}{(z'-z_0)^{k+1}}\right]dz'$$

$$-\frac{1}{2\pi i}\oint_{C2} \frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\oint_{C2} f(z')\left[\sum_{k=-\infty}^{-1} \frac{(z-z_0)^k}{(z'-z_0)^{k+1}}\right]dz'$$

$$f(z) = \frac{1}{2\pi i}\oint_{Cz} \frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\oint_{C} f(z')\left[\sum_{k=-\infty}^{+\infty} \frac{(z-z_0)^k}{(z'-z_0)^{k+1}}\right]dz' = \sum_{n=-\infty}^{+\infty}\left[\frac{1}{2\pi i}\oint_{C} \frac{f(z')}{(z'-z_0)^{k+1}}\,dz'\right](z-z_0)^k$$

z-Transform

Unilateral z-Transform

$$X(z) = \sum_{n=0}^{\infty} x_k z^{-k}$$

Inverse z-Transform

$$x_k = \frac{1}{2\pi i} \oint_C X(z)\, z^{k-1}\, dz$$

Bilateral z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x_k z^{-k}$$

Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k$$

Laurent Series coefficients

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{k+1}}\, dz$$

$$a_k = \frac{1}{2\pi i} \oint_C f(z)(z-z_0)^{-k-1}\, dz$$

Transform vs. Series Expansion

Bilateral z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x_k z^{-k}$$

x_k — transform → $X(z)$?

$$X(z) = \cdots + x_{-2} z^{+2} + x_{-1} z^{+1} + x_0 z^{0} + x_1 z^{-1} + x_2 z^{-2} + \cdots$$

Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k$$

a_k ? ← expansion — $f(z)$

$$f(z) = \cdots + a_{-1}(z-z_0)^{-1} + a_0(z-z_0)^{0} + a_1(z-z_0)^{+1} + \cdots$$

$z_0 = 0$

$$f(z) = \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 z^{0} + a_1 z^{+1} + a_2 z^{+2} + \cdots$$

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Types of Complex Series

The **Taylor series** of a function $f(z)$

non-negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\,(z - z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

The **MacLaurin series** of a function $f(z)$

non-negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, z^n$$

$$a_n = \frac{1}{n!} f^{(n)}(0)$$

The **Laurent series** of a function $f(z)$

$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{k+1}}\, dz$$

The **z-transform** of a series $\{a_k\}$

$$f(z) = \sum_{n=-\infty}^{\infty} a_k z^{-k}$$

$$a_k = \frac{1}{2\pi i} \oint_C f(z)\, z^{k-1}\, dz$$

Regions in Laurent Series and Taylor Series

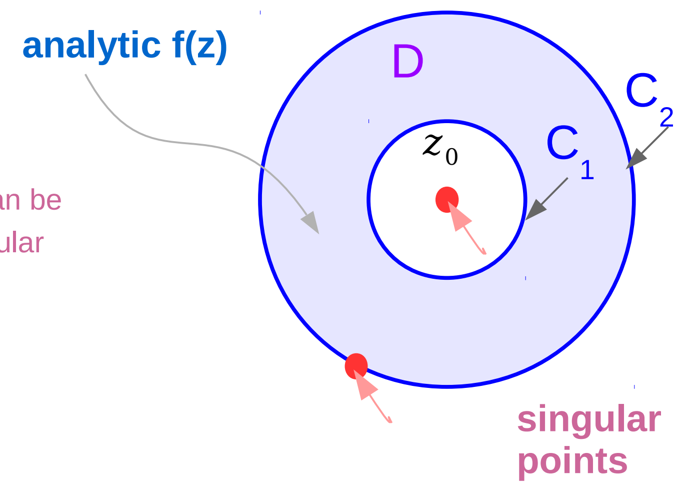
The **Laurent series** of a function $f(z)$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n \neq \frac{1}{n!} f^{(n)}(z_0)$$

z_0 can be singular

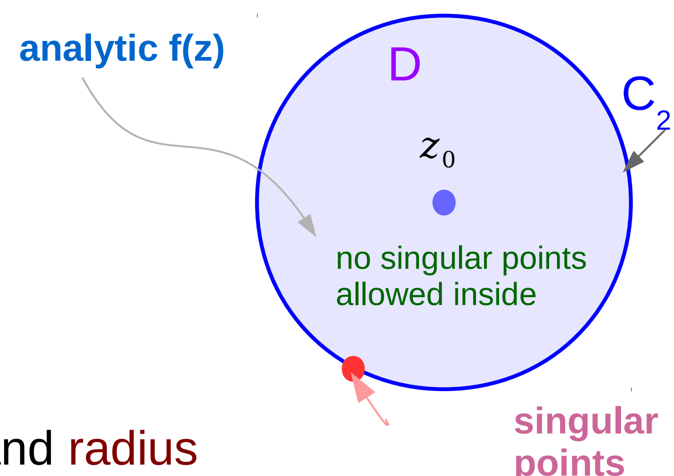
converges in the region D between circles C_1 and C_2 centered at z_0 where f(z) is **analytic**



The **Taylor series** of a function $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$



converges for all z in the open disk with center z_0 and radius generally equal to the distance from z_0 to the nearest singularity of $f(z)$

Coefficients of Laurent Series and Taylor Series

Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{k+1}} \, dz$$

$$= \sum_{z_m} Res\left(\frac{f(z)}{(z-z_0)^{k+1}}, z_m\right)$$

for general f(z)

the contour **C** can enclose poles

Taylor series

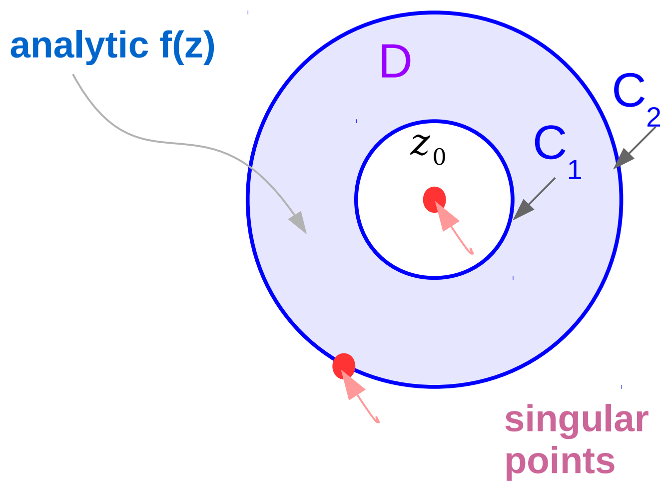
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

only for analytic f(z)

the contour **C** must not enclose any pole

For a given region



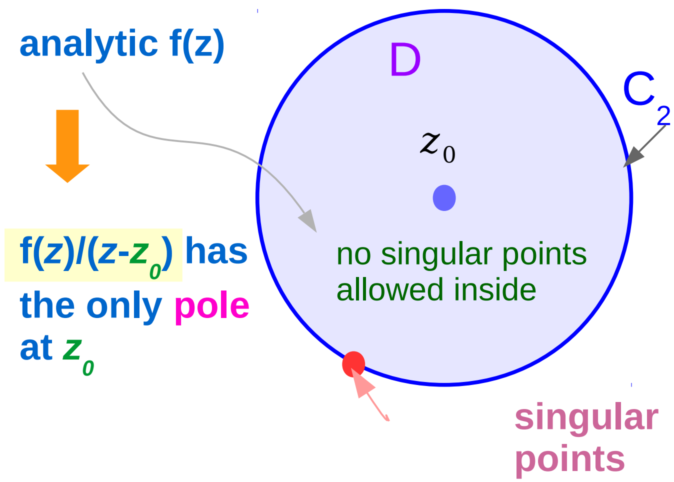
Laurent series expansion only

$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{k+1}} \, dz$$

non-singular z_0: $a_k = \frac{1}{k!} f^{(k)}(z_0)$

singular z_0: $a_k \neq \frac{1}{k!} f^{(k)}(z_0)$



Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

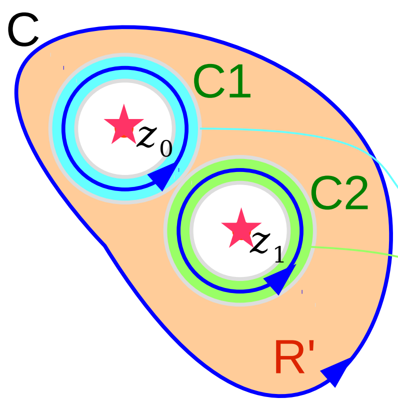
$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Laurent series expansion

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

$$= Res\left(\frac{f(z)}{(z - z_0)^{n+1}}, z_0 \right)$$



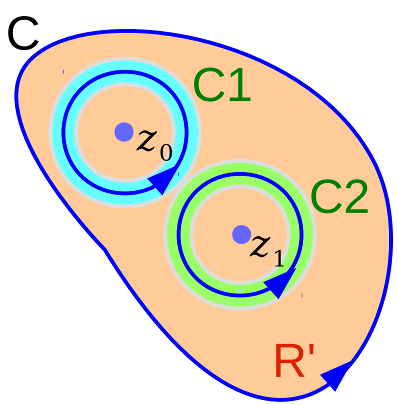
z_0, z_1 : isolated singularities

$$\oint_C f(z)\,dz = 2\pi i\,\{Res(f(z), z_0) + Res(f(z), z_1)\}$$

Laurent series expansion around z_0 $\quad f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$

Laurent series expansion around z_1 $\quad f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-z_1)^n$

$$\oint_C f(z)\,dz = \oint_{C1} f(z)\,dz + \oint_{C2} f(z)\,dz = 2\pi i \cdot a_{-1} + 2\pi i \cdot c_{-1}$$



z_0, z_1 : regular points

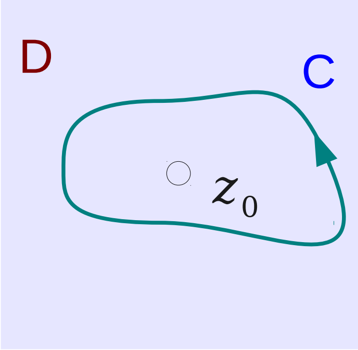
$$\oint_C f(z)\,dz = \oint_{C1} f(z)\,dz = \oint_{C2} f(z)\,dz = 0$$

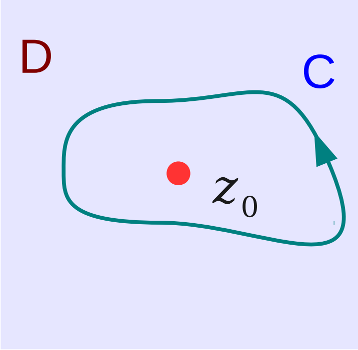
- $f(z)$ is analytic within and on **C** ⬅ **Cauchy Integral Theorem**
- non-constant $f(z) = F'(z)$ on **C** ⬅ **Fundamental Theorem**

$$\oint_C f(z)\,dz = 0$$

- $f(z)$ is analytic within and on **C**
- non-constant $f(z) = F'(z)$ on **C**

$$\oint_C f(z)\,dz = 2\pi i \sum_k Res(f(z), z_k) = 2\pi i \cdot a_{-1}$$





$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k \Rightarrow \frac{f(z)}{(z-z_0)^{k+1}} = \sum_{k=-\infty}^{+\infty} \frac{a_k}{(z-z_0)}$$

for general f(z)

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz = \sum_{z_m} Res\left(\frac{f(z)}{(z-z_0)^{k+1}}, z_m\right)$$

Applying residue theorem recursively

$$\oint_C f(z)\,dz = 2\pi i \sum_k Res\big(f(z), z_k\big) = 2\pi i \cdot a_{-1}$$

$f(z)$

for general f(z)

$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz = \sum_{z_m} Res\left(\frac{f(z)}{(z-z_0)^{k+1}}, z_m\right)$$

$$\frac{f(z)}{(z-z_0)^{k+1}}$$

only for analytic f(z)

$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz = \sum_{z_m} Res\left(\frac{f(z)}{(z-z_0)^{k+1}}, z_m\right) = \frac{1}{k!} f^{(k)}(z_0)$$

$$\frac{f(z)}{(z-z_0)^{k+1}}$$

analytic f(z) ⇒ **$f(z)/(z-z_0)$ has the only pole at z_0**

Laurent Expansion Example

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$z = +2$ Not an isolated singular point

$$\frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{(z-2)\left(1+\frac{1}{z-2}\right)}$$

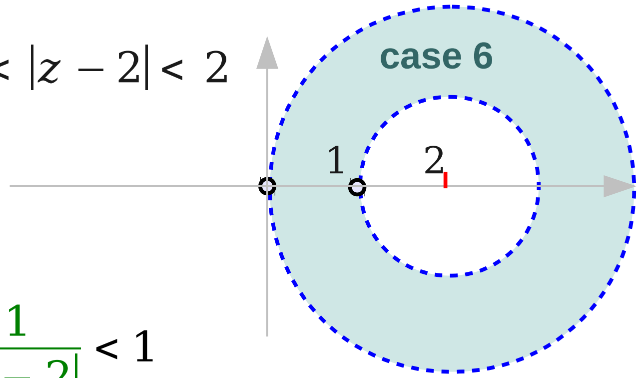
$$= \frac{1}{z-2}\left[1 - \frac{1}{(z-2)} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} \cdots\right]$$

$$-\frac{1}{z} = -\frac{1}{2+z-2} = -\frac{1}{2\left(1+\frac{z-2}{2}\right)}$$

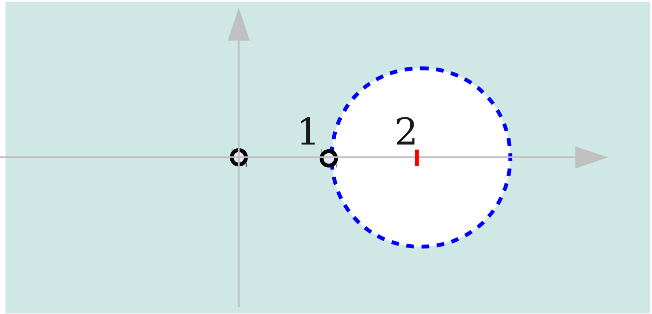
$$= -\frac{1}{2}\left[1 - \frac{(z-2)}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} \cdots\right]$$

~~essential singularity~~

$1 < |z-2| < 2$

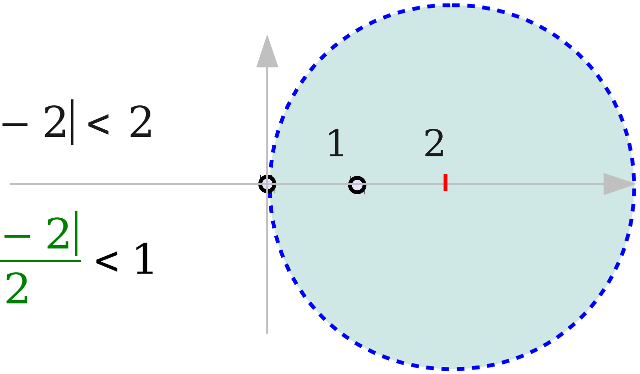


$\frac{1}{|z-2|} < 1$



$|z-2| < 2$

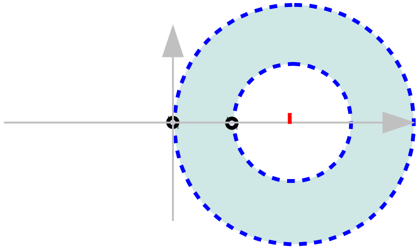
$\frac{|z-2|}{2} < 1$



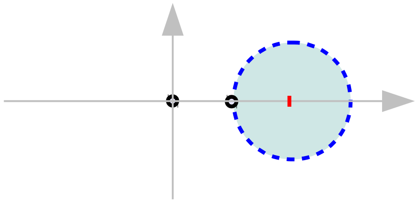
Laurent Expansion – at the three ROC's

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

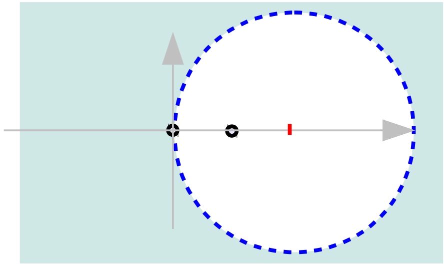
$$f(z) = \left[\cdots + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)} - \frac{1}{2} + \frac{(z-2)}{2^2} - \frac{(z-2)^2}{2^3} + \cdots \right]$$



$$f(z) = \left[+ \frac{1}{2} - \frac{3}{2^2}(z-2) + \frac{7}{2^3}(z-2)^2 + \cdots \right]$$

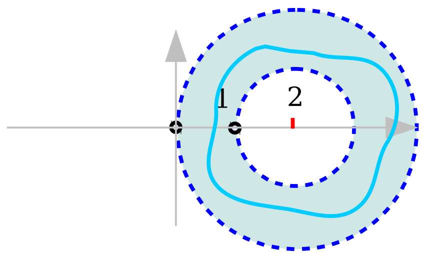


$$f(z) = \left[+ \frac{1}{(z-2)^2} - \frac{3}{(z-2)^3} + \frac{7}{(z-2)^4} + \cdots \right]$$



Laurent Expansion Coefficients at the ROC 1

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$



$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz = \sum_{z_m} Res\left(\frac{f(z)}{(z-z_0)^{k+1}}, z_m\right)$$

$$a_2 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^3}\,dz = \sum_{z_m=1,2} Res\left(\frac{1}{z(z-1)(z-2)^3}, z_m\right) = \frac{1}{1(1-2)^3} + \left(-\frac{1}{2^3} + \frac{1}{(2-1)^3}\right) = -\frac{1}{2^3}$$

$$a_1 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^2}\,dz = \sum_{z_m=1,2} Res\left(\frac{1}{z(z-1)(z-2)^2}, z_m\right) = \frac{1}{1(1-2)^2} + \left(\frac{1}{2^2} - \frac{1}{(2-1)^2}\right) = +\frac{1}{2^2}$$

$$a_0 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^1}\,dz = \sum_{z_m=1,2} Res\left(\frac{1}{z(z-1)(z-2)^1}, z_m\right) = \frac{1}{1(1-2)^1} + \left(-\frac{1}{2} + \frac{1}{(2-1)}\right) = -\frac{1}{2}$$

$$a_{-1} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^0}\,dz = \sum_{z_m=1} Res\left(\frac{1}{z(z-1)}, z_m\right) = 1$$

$$a_{-2} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^{-1}}\,dz = \sum_{z_m=1} Res\left(\frac{(z-2)}{z(z-1)}, z_m\right) = -1$$

z_0 : the n-th order pole

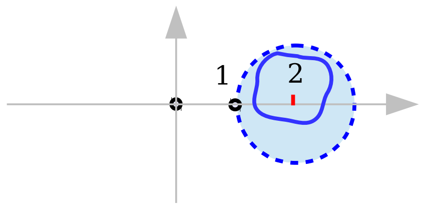
$$Res(f(z), z_0) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)$$

$$\frac{1}{1!}\frac{d}{dz}\left\{\frac{1}{z(z-1)}\right\} = \frac{1}{1!}\frac{d}{dz}\left\{-z^{-1}+(z-1)^{-1}\right\} = z^{-2}-(z-1)^{-2}$$

$$\frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{1}{z(z-1)}\right\} = \frac{1}{2!}\frac{d}{dz}\left\{z^{-2}-(z-1)^{-2}\right\} = -z^{-3}+(z-1)^{-3}$$

Laurent Expansion Coefficients at the ROC 2

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$



$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz = \sum_{z_m} Res\left(\frac{f(z)}{(z-z_0)^{k+1}}, z_m\right)$$

$$a_2 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^3}\,dz = \sum_{z_m=2} Res\left(\frac{1}{z(z-1)(z-2)^3}, z_m\right) = \left(-\frac{1}{2^3}+\frac{1}{(2-1)^3}\right) = +\frac{7}{2^3}$$

$$a_1 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^2}\,dz = \sum_{z_m=2} Res\left(\frac{1}{z(z-1)(z-2)^2}, z_m\right) = \left(\frac{1}{2^2}-\frac{1}{(2-1)^2}\right) = -\frac{3}{2^2}$$

$$a_0 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^1}\,dz = \sum_{z_m=2} Res\left(\frac{1}{z(z-1)(z-2)^1}, z_m\right) = \left(-\frac{1}{2}+\frac{1}{(2-1)}\right) = +\frac{1}{2}$$

$$a_{-1} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^0}\,dz = \sum_{z_m=\{\}} Res\left(\frac{1}{z(z-1)}, z_m\right) = 0$$

$$a_{-2} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^{-1}}\,dz = \sum_{z_m=\{\}} Res\left(\frac{(z-2)}{z(z-1)}, z_m\right) = 0$$

z_0 : the n-th order pole

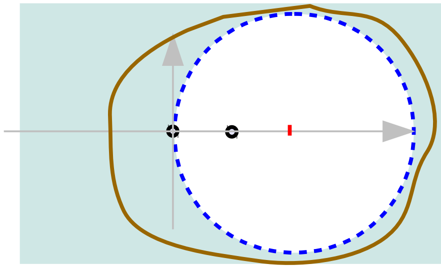
$$Res(f(z), z_0) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)$$

$$\frac{1}{1!}\frac{d}{dz}\left\{\frac{1}{z(z-1)}\right\} = \frac{1}{1!}\frac{d}{dz}\left\{-z^{-1}+(z-1)^{-1}\right\} = z^{-2}-(z-1)^{-2}$$

$$\frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{1}{z(z-1)}\right\} = \frac{1}{2!}\frac{d}{dz}\left\{z^{-2}-(z-1)^{-2}\right\} = -z^{-3}+(z-1)^{-3}$$

Laurent Expansion Coefficients at the ROC 3

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$



$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz = \sum_{z_m} Res\left(\frac{f(z)}{(z-z_0)^{k+1}}, z_m\right)$$

$$a_2 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^3}\,dz = \sum_{z_m=0,1,2} Res\left(\frac{1}{z(z-1)(z-2)^3}, z_m\right) = \frac{1}{(-1)(-2)^3} + \frac{1}{1(1-2)^3} + \left(-\frac{1}{2^3} + \frac{1}{(2-1)^3}\right) = 0$$

$$a_1 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^2}\,dz = \sum_{z_m=0,1,2} Res\left(\frac{1}{z(z-1)(z-2)^2}, z_m\right) = \frac{1}{(-1)(-2)^2} + \frac{1}{1(1-2)^2} + \left(\frac{1}{2^2} - \frac{1}{(2-1)^2}\right) = 0$$

$$a_0 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^1}\,dz = \sum_{z_m=1,2} Res\left(\frac{1}{z(z-1)(z-2)^1}, z_m\right) = \frac{1}{(-1)(-2)} + \frac{1}{1(1-2)^1} + \left(-\frac{1}{2} + \frac{1}{(2-1)}\right) = 0$$

$$a_{-1} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^0}\,dz = \sum_{z_m=0,1} Res\left(\frac{1}{z(z-1)}, z_m\right) = \frac{1}{(-1)} + \frac{1}{1} = 0$$

$$a_{-2} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^{-1}}\,dz = \sum_{z_m=0,1} Res\left(\frac{(z-2)}{z(z-1)}, z_m\right) = \frac{(-2)}{(-1)} + \frac{(1-2)}{1} = +1$$

$$a_{-3} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^{-2}}\,dz = \sum_{z_m=0,1} Res\left(\frac{(z-2)^2}{z(z-1)}, z_m\right) = \frac{(-2)^2}{(-1)} + \frac{(1-2)^2}{1} = -3$$

$$a_{-4} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)^{-3}}\,dz = \sum_{z_m=0,1} Res\left(\frac{(z-2)^3}{z(z-1)}, z_m\right) = \frac{(-2)^3}{(-1)} + \frac{(1-2)^3}{1} = +7$$

Young Won Lim
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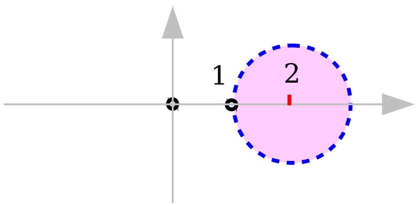
Taylor Expansion at the ROC 2

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$



$$a_k = \frac{1}{k!} f^{(k)}(z_0)$$

$$a_2 = \frac{1}{2!} f^{(2)}(2) \quad = -z^{-3} + (z-1)^{-3}\Big|_{z_0=2} \quad = \left(-\frac{1}{2^3} + \frac{1}{(2-1)^3}\right) \quad = +\frac{7}{2^3}$$

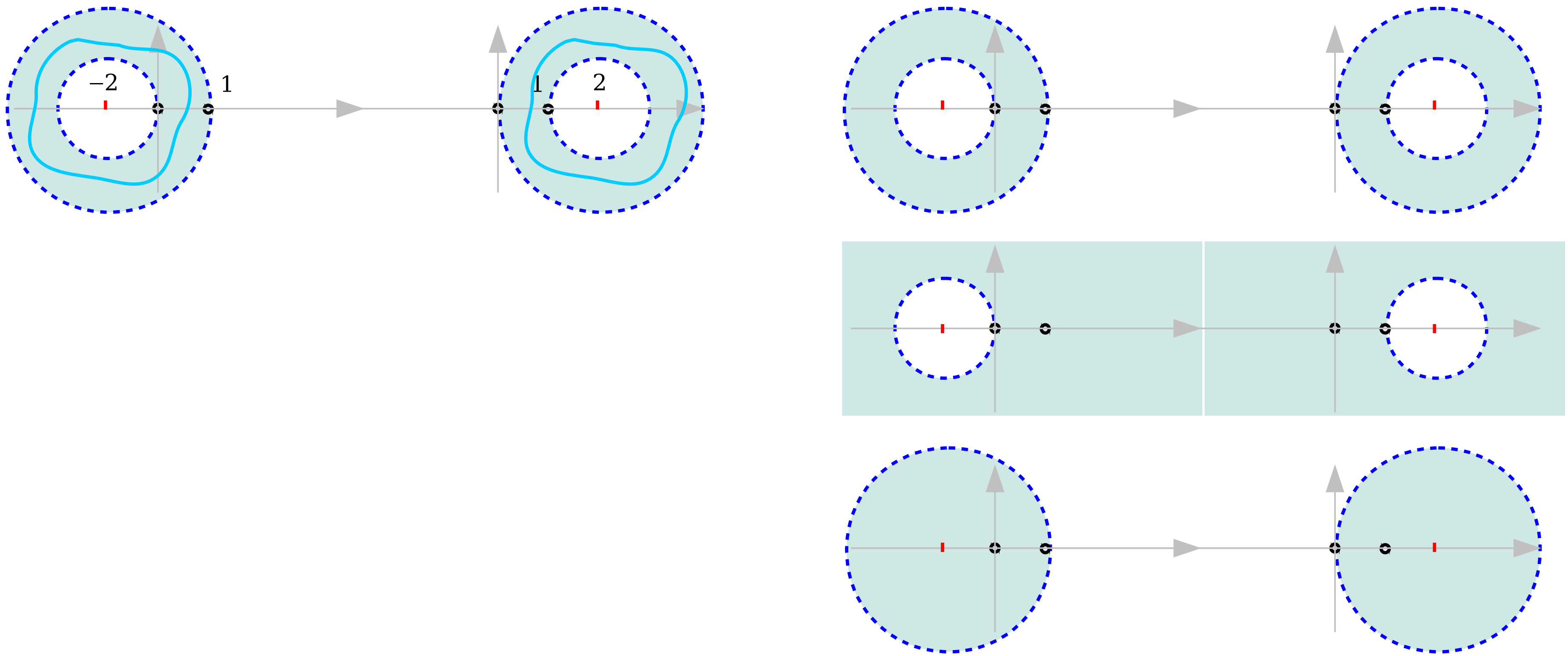
$$a_1 = \frac{1}{1!} f^{(1)}(2) \quad = z^{-2} - (z-1)^{-2}\Big|_{z_0=2} \quad = \left(\frac{1}{2^2} - \frac{1}{(2-1)^2}\right) \quad = -\frac{3}{2^2}$$

$$a_0 = \frac{1}{0!} f^{(0)}(2) \quad = -z^{-1} + (z-1)^{-1}\Big|_{z_0=2} \quad = \left(-\frac{1}{2} + \frac{1}{(2-1)}\right) \quad = +\frac{1}{2}$$

$$\frac{1}{1!}\frac{d}{dz}\left\{\frac{1}{z(z-1)}\right\} = \frac{1}{1!}\frac{d}{dz}\left\{-z^{-1} + (z-1)^{-1}\right\} = z^{-2} - (z-1)^{-2}$$

$$\frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{1}{z(z-1)}\right\} = \frac{1}{2!}\frac{d}{dz}\left\{z^{-2} - (z-1)^{-2}\right\} = -z^{-3} + (z-1)^{-3}$$

Laurent Expansion Example (5)



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