

# Determinant (3A)

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# Determinant

## Determinant of order 2

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

## Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

# Determinant

## Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ & b_1 & b_3 \\ & c_1 & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ & b_1 & b_2 \\ & c_1 & c_2 \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

# Rule of Sarrus (1)

Determinant of order 3 only

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Copy and concatenate

Rule of Sarrus

$$\begin{array}{ccc} + & + & + \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} & - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} & \end{array}$$

# Determinant – Rule of Sarrus (2)

## Determinant of order 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32})$$

$$- a_{12}(a_{21}a_{33} - a_{23}a_{31})$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

## Recursive Method

## Determinant of order 3 only

$$\begin{bmatrix} + & & - \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$+ a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$

$$\begin{bmatrix} & + & & - \\ a_{11} & a_{12} & a_{13} & a_{11} \\ a_{21} & a_{22} & a_{23} & a_{21} \\ a_{31} & a_{32} & a_{33} & a_{31} \end{bmatrix} \begin{matrix} a_{12} & a_{13} & a_{11} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$+ a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}$$

## Rule of Sarrus

$$\begin{bmatrix} & & + & & - \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix} \begin{matrix} a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

# Minor

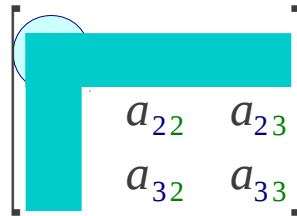
The **minor** of entry  $a_{ij}$   $M_{ij}$

The determinant of the submatrix  
that remains after **deleting**  $i$ -th row and  $j$ -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$


$$\begin{bmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

# Cofactor

The **cofactor** of entry  $a_{ij}$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The **minor** of entry  $a_{ij}$

$$M_{ij}$$

The determinant of the submatrix that remains after **deleting**  $i$ -th row and  $j$ -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

$$\begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} = \begin{bmatrix} (-1)^2 & (-1)^3 & (-1)^4 \\ (-1)^3 & (-1)^4 & (-1)^5 \\ (-1)^4 & (-1)^5 & (-1)^6 \end{bmatrix}$$



# Minor Example (1)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} & & & \\ & a_{22} & a_{23} & \\ & a_{32} & a_{33} & \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = +M_{11}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} & & & \\ a_{21} & & a_{23} & \\ a_{31} & & a_{33} & \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = -M_{12}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & & \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = +M_{13}$$

# Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} & a_{12} & a_{13} \\ & & \\ a_{32} & & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = -M_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & & a_{13} \\ & & \\ a_{31} & & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = +M_{22}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \\ & & \\ a_{31} & a_{32} & \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{23} = -M_{23}$$

# Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & & \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Minor / Cofactor

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = +M_{31}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & & a_{13} \\ a_{21} & & a_{23} \\ & & & \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = -M_{32}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & & \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = +M_{33}$$

# Determinant

The **determinant** of an  $n \times n$  matrix  $\mathbf{A}$   $\det(\mathbf{A})$

Cofactor expansion along the  $i$ -th row  
(elements of the  $i$ -th row)  $\cdot$  (cofactors at the  $i$ -th row)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ \det(\mathbf{A}) &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ \det(\mathbf{A}) &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \end{aligned}$$

Cofactor expansion along the  $j$ -th column  
(elements of the  $j$ -th column)  $\cdot$  (cofactors at the  $j$ -th column)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ \det(\mathbf{A}) &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ \det(\mathbf{A}) &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$

# Adjoint

The **cofactor** of entry  $a_{ij}$   $C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry  $a_{ij}$   $M_{ij}$

The determinant of the submatrix  
that remains after deleting  $i$ -th row and  $j$ -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$a_{11} \Leftrightarrow C_{11}$$

$$a_{12} \Leftrightarrow C_{12}$$

$$a_{13} \Leftrightarrow C_{13}$$

$$a_{21} \Leftrightarrow C_{21}$$

$$a_{22} \Leftrightarrow C_{22}$$

$$a_{23} \Leftrightarrow C_{23}$$

$$a_{31} \Leftrightarrow C_{31}$$

$$a_{32} \Leftrightarrow C_{32}$$

$$a_{33} \Leftrightarrow C_{33}$$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose



Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

# Inverse Matrix

The **cofactor** of entry  $a_{ij}$   $C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry  $a_{ij}$   $M_{ij}$

The determinant of the submatrix that remains after deleting  $i$ -th row and  $j$ -th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose



Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \mathit{adj}(\mathbf{A}) = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

# Inverse Matrix

Given matrix

$$\begin{bmatrix} +2 & +2 & 0 \\ -2 & +1 & +1 \\ +3 & 0 & +1 \end{bmatrix}$$

$$\left( \begin{array}{l} + \begin{vmatrix} +1 & +1 \\ 0 & +1 \end{vmatrix} - \begin{vmatrix} -2 & +1 \\ +3 & +1 \end{vmatrix} + \begin{vmatrix} -2 & +1 \\ +3 & 0 \end{vmatrix} \\ - \begin{vmatrix} +2 & 0 \\ 0 & +1 \end{vmatrix} + \begin{vmatrix} +2 & 0 \\ +3 & +1 \end{vmatrix} - \begin{vmatrix} +2 & +2 \\ +3 & 0 \end{vmatrix} \\ + \begin{vmatrix} +2 & 0 \\ +1 & +1 \end{vmatrix} - \begin{vmatrix} +2 & 0 \\ -2 & +1 \end{vmatrix} + \begin{vmatrix} +2 & +2 \\ -2 & +1 \end{vmatrix} \end{array} \right)$$

$$+2 \cdot \begin{vmatrix} +1 & +1 \\ 0 & +1 \end{vmatrix} + 2 \cdot \begin{vmatrix} -2 & +1 \\ +3 & +1 \end{vmatrix} + 0 \cdot \begin{vmatrix} -2 & +1 \\ +3 & 0 \end{vmatrix} = 12$$

$$\mathbf{A}^{-1} = \frac{1}{\mathit{det}(\mathbf{A})} \mathit{adj}(\mathbf{A})$$

a number                  a matrix

$$\mathbf{A}^{-1} = \frac{1}{12} \begin{bmatrix} +1 & -2 & +2 \\ +5 & +2 & -2 \\ -3 & +6 & +6 \end{bmatrix}$$

Matrix of Cofactors

$$\begin{bmatrix} +1 & +5 & -3 \\ -2 & +2 & +6 \\ +2 & -2 & +6 \end{bmatrix}$$

Adjoint  $\mathit{adj}(\mathbf{A})$

$$\begin{bmatrix} +1 & -2 & +2 \\ +5 & +2 & -2 \\ -3 & +6 & +6 \end{bmatrix}$$

# Matrix Transpose

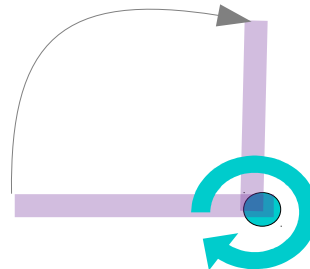
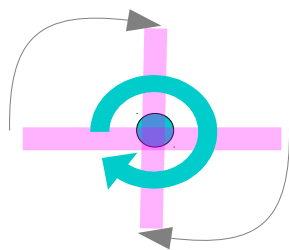
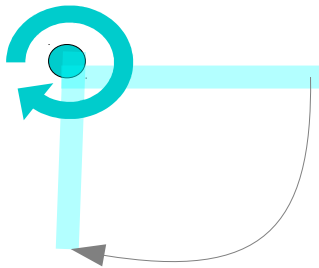
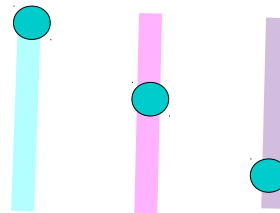
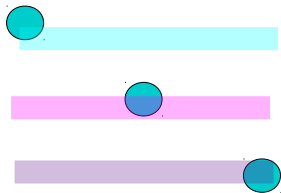
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\det(A) = \det(A^T)$$

$[a_{ij}]$

$[a_{ji}]$





# Cofactor Expansion and Determinant

$A$   $n \times n$

zero row

zero col

has

$$\begin{bmatrix} \text{ } \\ \text{ } \\ \text{00 0} \\ \text{ } \end{bmatrix} \text{ or } \begin{bmatrix} \text{0} \\ \text{0} \\ \text{0} \\ \text{ } \end{bmatrix}$$



$$\det(A) = 0$$

$$\begin{aligned} \det(A) &= a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} \\ &= a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} \\ &= 0 \end{aligned}$$

i-th row cofactor expansion

j-th column cofactor expansion

$A$   $n \times n$

$A^T$   $n \times n$

$$\begin{bmatrix} \text{ } \\ \text{* * *} \\ \text{ } \end{bmatrix} \text{ i-th row}$$

$$\begin{bmatrix} \text{*} \\ \text{*} \\ \text{*} \\ \text{ } \end{bmatrix} \text{ i-th col}$$



$$\det(A^T) = \det(A)$$

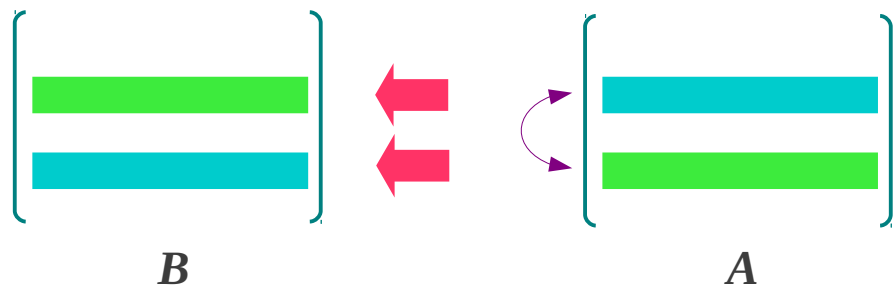
$$\begin{aligned} \det(A) &= a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} \\ &= a_{1i}C_{1i} + a_{2i}C_{2i} + a_{3i}C_{3i} \end{aligned}$$

i-th row cofactor expansion of  $A$

i-th column cofactor expansion of  $A^T$

# Elementary Matrix and Determinant (1)

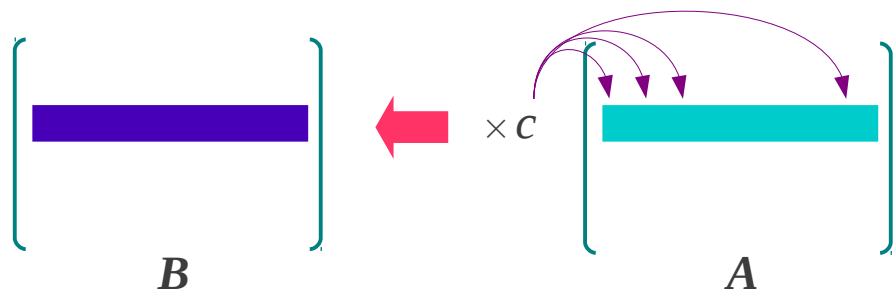
Interchange two rows



$$\det(\mathbf{B}) = -\det(\mathbf{A})$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

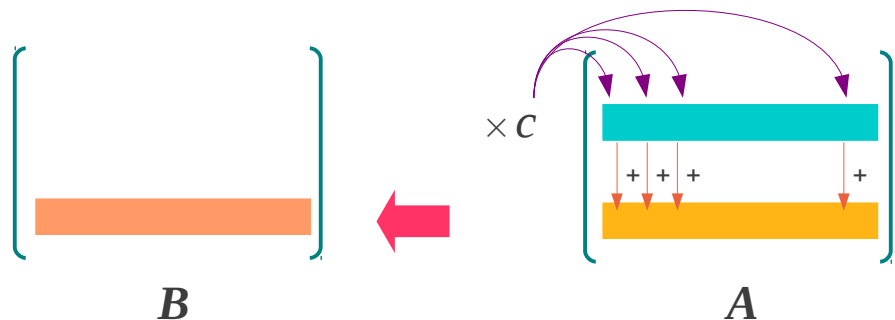
Multiply a row by a nonzero constant



$$\det(\mathbf{B}) = c \det(\mathbf{A})$$

$$\begin{vmatrix} c a_{11} & c a_{12} & c a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Add a multiple of one row to another

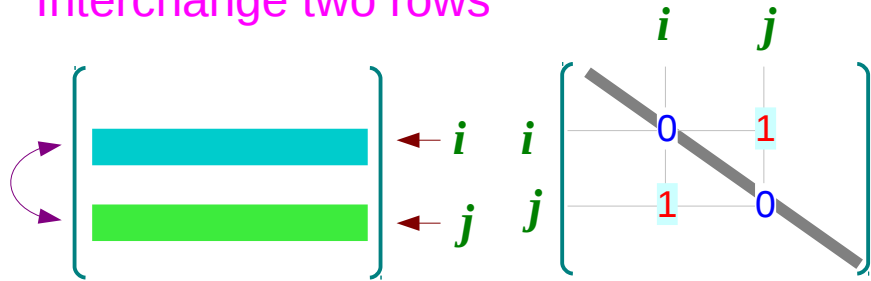


$$\det(\mathbf{B}) = \det(\mathbf{A})$$

$$\begin{vmatrix} a_{11} + c a_{21} & a_{12} + c a_{22} & a_{13} + c a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

# Elementary Matrix and Determinant (2)

Interchange two rows



$\det(\mathbf{B})$

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$\det(\mathbf{A})$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\det(\mathbf{B}) = -\det(\mathbf{A})$$

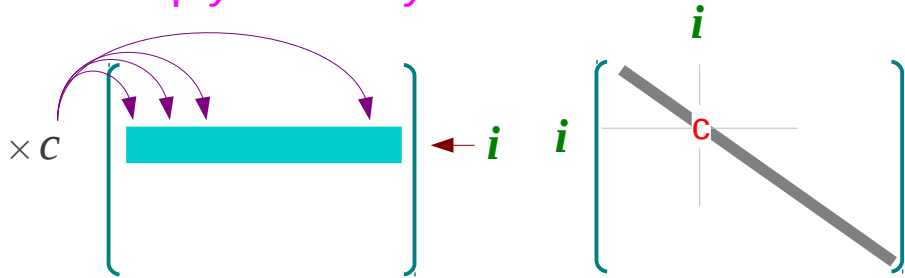
$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} \det(\mathbf{B}) &= b_{21}C_{21} + b_{22}C_{22} + b_{23}C_{23} \\ &= -a_{11}M_{21} + a_{12}M_{22} - a_{13}M_{23} \end{aligned}$$

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \end{aligned}$$

# Elementary Matrix and Determinant (3)

Multiply a row by a nonzero constant



$$\det(\mathbf{B}) = c \det(\mathbf{A})$$

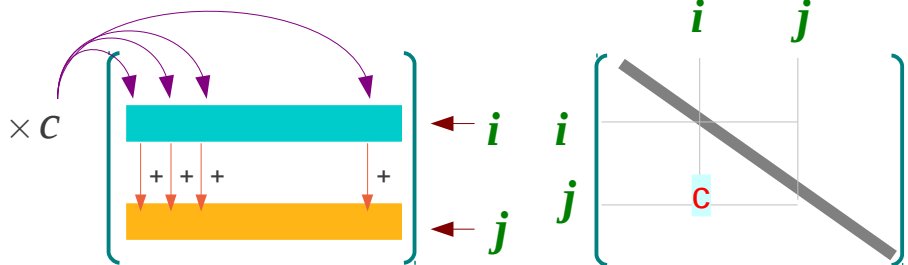
$$\begin{vmatrix} ca_{11} & ca_{12} & ca_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\mathbf{B}) = \begin{vmatrix} ca_{11} & ca_{12} & ca_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} \det(\mathbf{B}) &= b_{11}C_{11} + b_{12}C_{12} + b_{13}C_{13} \\ &= c \cdot a_{11}C_{11} + c \cdot a_{12}C_{12} + c \cdot a_{13}C_{13} \\ &= c(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) \\ &= c \cdot \det(\mathbf{A}) \end{aligned}$$

# Elementary Matrix and Determinant (4)

Add a multiple of one row to another



$$\det(\mathbf{B}) = \det(\mathbf{A})$$

$$\begin{vmatrix} a_{11} + c a_{21} & a_{12} + c a_{22} & a_{13} + c a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\det(\mathbf{B})$

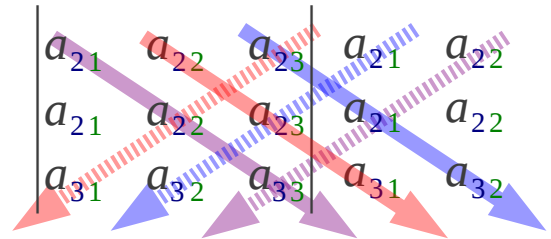
$$\begin{vmatrix} a_{11} + c a_{21} & a_{12} + c a_{22} & a_{13} + c a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\mathbf{B}) = b_{11}C_{11} + b_{12}C_{12} + b_{13}C_{13}$$

$$\begin{aligned} &= (a_{11} + c a_{21})C_{11} + (a_{12} + c a_{22})C_{12} + (a_{13} + c a_{23})C_{13} \\ &= (a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) \rightarrow \det(\mathbf{A}) \\ &\quad + c(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}) \rightarrow 0 \end{aligned}$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}) = 0$$



# Determinant of Diagonal Matrix

Lower Triangular Matrix

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & 0 \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$\det(\mathbf{A}) = a_{11} a_{22} a_{33}$$

Upper Triangular Matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{matrix}$$

$$\det(\mathbf{A}) = a_{11} a_{22} a_{33}$$

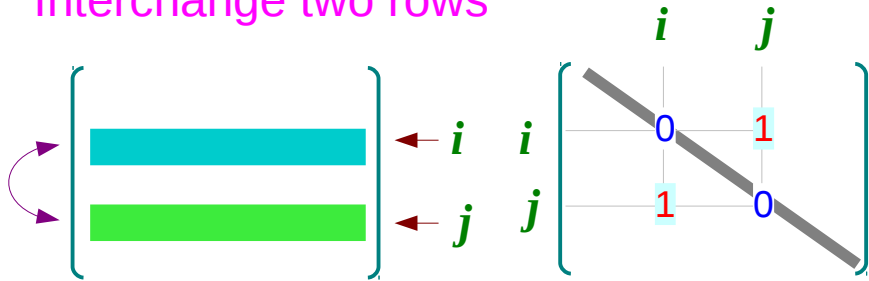
Diagonal Matrix

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{matrix}$$

$$\det(\mathbf{A}) = a_{11} a_{22} a_{33}$$

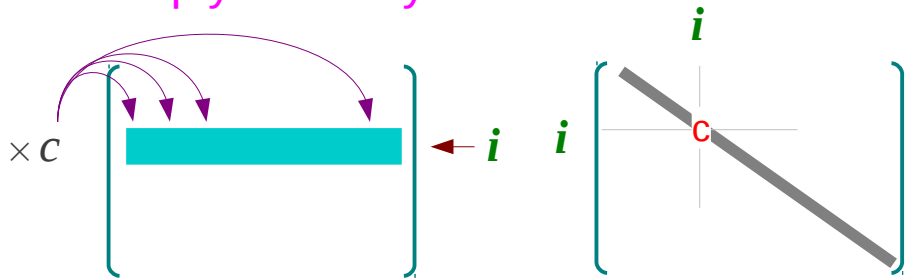
# Determinant of an Elementary Matrix

Interchange two rows



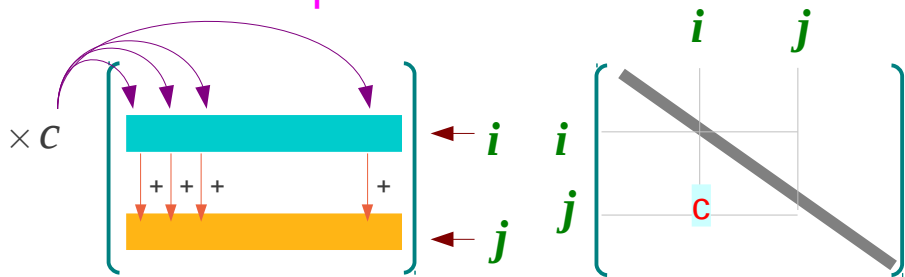
$$\det(\mathbf{E}_k) = -1$$

Multiply a row by a nonzero constant



$$\det(\mathbf{E}_k) = c$$

Add a multiple of one row to another



$$\det(\mathbf{E}_k) = 1$$

# Properties of Determinants

$$\det(k\mathbf{A}) = k^n \det(\mathbf{A})$$

$$\det(\mathbf{A}+\mathbf{B}) \neq \det(\mathbf{A})+\det(\mathbf{B})$$

$$\det(\mathbf{A}\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$$

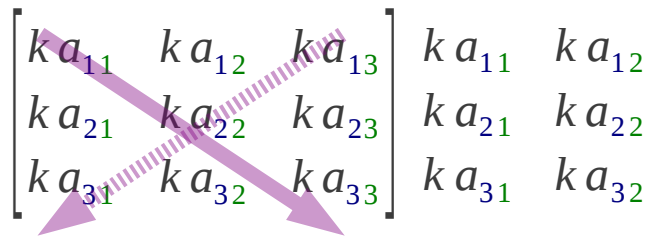


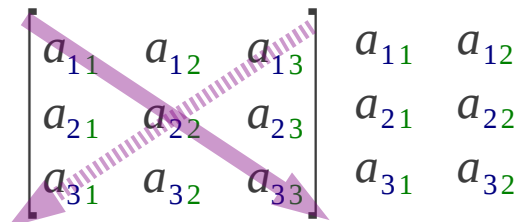
# Proof of $\det(kA) = k^n \det(A)$

$$\det(kA) = k^n \det(A)$$

$$\det(A+B) \neq \det(A) + \det(B)$$

$$\det(AB) = \det(A)\det(B)$$


$$\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix} \begin{matrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \\ ka_{31} & ka_{32} \end{matrix}$$


$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

# Proof of $\det(A+B) \neq \det(A) + \det(B)$

$$\det(kA) = k^n \det(A)$$

$$\det(A+B) \neq \det(A) + \det(B)$$

$$\det(AB) = \det(A)\det(B)$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix} \begin{matrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{matrix}$$

$A \quad n \times n$

$$\begin{bmatrix} \text{ } & \text{ } & \text{ } \\ \$ & \$ & \$ \\ \text{ } & \text{ } & \text{ } \end{bmatrix}$$

$B \quad n \times n$

$$\begin{bmatrix} \text{ } & \text{ } & \text{ } \\ \# & \# & \# \\ \text{ } & \text{ } & \text{ } \end{bmatrix}$$

$$C = A + B$$



$$\det(C) = \det(A) + \det(B)$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix} \begin{matrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 2a_{21} & 2a_{22} \\ 2a_{31} & 2a_{32} \end{matrix}$$

# Proof of $\det(AB) = \det(A) \det(B)$ (1)

$E_k$  (Elementary Matrices)

Three elementary matrices are shown as 2x2 grids with a diagonal line from top-left to bottom-right. The first matrix has a 0 in the top-left and a 1 in the bottom-right, with a 1 in the top-right and a 0 in the bottom-left. The second matrix has a 'c' in the top-left and a 1 in the bottom-right. The third matrix has a 1 in the top-left and a 'c' in the bottom-right, with a 1 in the top-right and a 0 in the bottom-left. Rows are labeled 'i' and 'j', and columns are labeled 'i' and 'j'.



$$\det(E_k B) = \det(E_k) \det(B)$$

$A$   $n \times n$  : invertible

$$AA^{-1} = A^{-1}A = I$$



$$\det(A) \neq 0$$

$A$   $n \times n$  : invertible


$$AA^{-1} = A^{-1}A = I$$

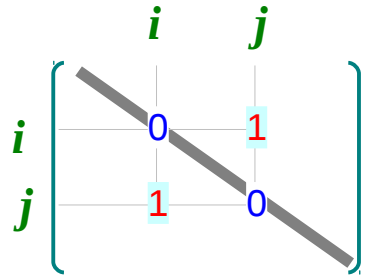


$$\det(A^{-1}) = \frac{1}{\det(A)}$$

# Proof of $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$ (2)

$\mathbf{E}_k$  (Elementary Matrices)

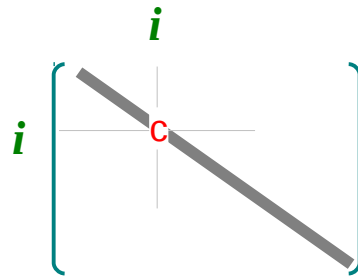


$$\det(\mathbf{E}_k \mathbf{B}) = \det(\mathbf{E}_k) \det(\mathbf{B})$$


$$\det(\mathbf{E}_k \mathbf{B}) = -\det(\mathbf{B})$$

$$\det(\mathbf{E}_k) = -1$$

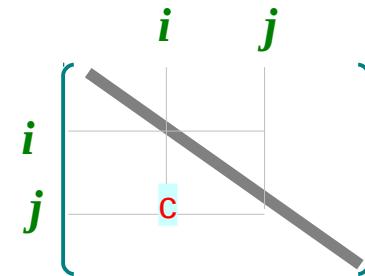
$$\det(\mathbf{E}_k \mathbf{B}) = \det(\mathbf{E}_k) \det(\mathbf{B})$$



$$\det(\mathbf{E}_k \mathbf{B}) = c \cdot \det(\mathbf{B})$$

$$\det(\mathbf{E}_k) = c$$

$$\det(\mathbf{E}_k \mathbf{B}) = \det(\mathbf{E}_k) \det(\mathbf{B})$$



$$\det(\mathbf{E}_k \mathbf{B}) = \det(\mathbf{B})$$

$$\det(\mathbf{E}_k) = 1$$

$$\det(\mathbf{E}_k \mathbf{B}) = \det(\mathbf{E}_k) \det(\mathbf{B})$$

# Proof of $\det(AB) = \det(A) \det(B)$ (3)

$A$   $n \times n$  : invertible

$$AA^{-1} = A^{-1}A = I$$



$$\det(A) \neq 0$$

$$E_r \cdots E_2 E_1 A = R$$

Reduced Row Echelon Form

$$\det(E_r) \cdots \det(E_2) \det(E_1) \det(A) = \det(R)$$

non-zero

$A$   $n \times n$  : invertible



$$R = I \quad \det(R) = 1 (\neq 0)$$

$$\det(A) \neq 0$$



$$\det(R) \neq 0$$

No zero row  $R = I$

$A$   $n \times n$  : invertible

# Proof of $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$ (4)

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$$

$$\begin{array}{l} \mathbf{A} \ n \times n \text{ : not invertible} \\ \det(\mathbf{A}) = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} \mathbf{AB} \ n \times n \text{ : not invertible} \\ \det(\mathbf{AB}) = 0 \end{array}$$

$$\begin{array}{l} \mathbf{A} \ n \times n \text{ : invertible} \end{array} \quad \Rightarrow \quad \begin{array}{l} \mathbf{A} = \mathbf{E}_r \cdots \mathbf{E}_2 \mathbf{E}_1 \\ \mathbf{AB} = \mathbf{E}_r \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{B} \end{array}$$

$$\det(\mathbf{AB}) = \det(\mathbf{E}_r) \cdots \det(\mathbf{E}_2) \det(\mathbf{E}_1) \det(\mathbf{B})$$

$$\det(\mathbf{AB}) = \det(\mathbf{E}_r \cdots \mathbf{E}_2 \mathbf{E}_1) \det(\mathbf{B})$$

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$$

# Proof of $\det(AB) = \det(A) \det(B)$ (5)

$\mathbf{A}$   $n \times n$  : invertible

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$



$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$\det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{I})$$

$$\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$$

# Computing $A \cdot \text{adj}(A)$ – diagonal elements

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose



Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\det(\mathbf{A}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\det(\mathbf{A}) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\det(\mathbf{A}) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$



# Computing $A \cdot \text{adj}(A)$ – off-diagonal elements (1)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

**A**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$\text{adj}(A)$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$0 = a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$$

$$0 = a_{31}C_{11} + a_{32}C_{12} + a_{33}C_{13}$$

# Computing $A \cdot \text{adj}(A)$ - off-diagonal elements (2)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} C'_{11} \\ C'_{12} \\ C'_{13} \end{bmatrix}$$

$$C_{11} = + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad C_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad C_{13} = + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

the same cofactor      the same cofactor      the same cofactor

$$C'_{11} = + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad C'_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad C'_{13} = + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

the redundant row of  $\mathbf{B}$

- linearly dependent
- $\det(\mathbf{B}) = 0$
- turned into a zero row
- $\det(\mathbf{R}) = 0$

$$E_r \cdots E_2 E_1 \mathbf{B} = \mathbf{R}$$

the cofactor along the 1<sup>st</sup> row of  $\mathbf{B}$

$$\mathbf{B} \rightarrow a_{21} C'_{11} + a_{22} C'_{12} + a_{23} C'_{13} = 0$$

$$\mathbf{A} \rightarrow = a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13}$$

an off-diagonal element of  $\mathbf{A} \cdot \text{adj}(\mathbf{A})$

# Result of $A \cdot \text{adj}(A)$

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose



Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix} = \det(A) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

cofactor expansion  
along the 1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup> rows

$$\mathbf{A} \text{adj}(\mathbf{A}) = \det(\mathbf{A}) \mathbf{I}$$

↑  
matrix

↑  
value

$$\mathbf{A} \left[ \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})} \right] = \mathbf{I}$$

$$\mathbf{A} [\mathbf{A}^{-1}] = \mathbf{I}$$

# Linear Equations

$$\text{(Eq 1)} \rightarrow a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$\text{(Eq 2)} \rightarrow a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$\vdots$                        $\vdots$                        $\vdots$                        $\vdots$

$$\text{(Eq 3)} \rightarrow a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

# Cramer's Rule (1) – solutions

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{x} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_n \end{pmatrix} = \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})}$$

$$\begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_n \end{pmatrix} = \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix}$$

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})}$$

$$\begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}$$

$$x_n = \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A})}$$

# Cramer's Rule (2) - determinants

$$\mathbf{A}_1 \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\det(\mathbf{A}_1) = b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1}$$

cofactor expansion along the first column

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})}$$

$$\mathbf{A}_2 \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix}$$

$$\det(\mathbf{A}_2) = b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2}$$

cofactor expansion along the second column

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})}$$

$$\mathbf{A}_n \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}$$

$$\det(\mathbf{A}_n) = b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn}$$

cofactor expansion along the last column

$$x_n = \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A})}$$

# Cramer's Rule (3) – inverse matrix

$$\mathbf{Ax} = \mathbf{b} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{\mathit{adj}(\mathbf{A})}{\mathit{det}(\mathbf{A})}\mathbf{b}$$

$$\mathbf{x} = \frac{1}{\mathit{det}(\mathbf{A})} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

the transposed matrix  
note reverse order index

$$\mathbf{x} = \frac{1}{\mathit{det}(\mathbf{A})} \begin{pmatrix} b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1} \\ b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2} \\ \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn} \end{pmatrix} = \frac{1}{\mathit{det}(\mathbf{A})} \begin{pmatrix} \mathit{det}(\mathbf{A}_1) \\ \mathit{det}(\mathbf{A}_2) \\ \vdots \\ \mathit{det}(\mathbf{A}_n) \end{pmatrix}$$

# Equivalent Statements

$A$  : invertible

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \begin{bmatrix} A^{-1} \\ \text{green square} \end{bmatrix} = \begin{bmatrix} A^{-1} \\ \text{green square} \end{bmatrix} \begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} = \begin{bmatrix} I_n \\ \text{identity matrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$Ax = 0$   
only the trivial solution

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \begin{bmatrix} x \\ \text{orange bar} \end{bmatrix} = \begin{bmatrix} 0 \\ \text{zero vector} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A$  the RREF is  $I_n$   
(Reduced Row Echelon Form)

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \xrightarrow{\text{Elem Row Op}} \begin{bmatrix} I_n \\ \text{identity matrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A$  can be written as a product of  $E_k$   
(Elementary Matrices)

$$\begin{matrix} i & j \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ j \end{matrix} \quad \begin{matrix} i \\ \begin{bmatrix} c \\ \text{row } i \end{bmatrix} \\ \end{matrix} \quad \begin{matrix} i & j \\ \begin{bmatrix} c \\ \text{row } j \end{bmatrix} \\ j \end{matrix}$$



## References

- [1] <http://en.wikipedia.org/>
- [2] Anton & Busby, "Contemporary Linear Algebra"
- [3] Anton & Rorres, "Elementary Linear Algebra"