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by

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Dedicated to Professor Nigel Hitchin on his 60th birthday with great respect and admiration

Abstract

We study the stabilizer orbit of the coadjoint action of the Virasoro algebra on its dual. The vector field associated to the stabilizer orbit is called the projective vector field and the equation associated to this is called the projective vector field equation. At first we study the Riccati and higher Riccati equations associated to this equation. We obtain the solutions of these special higher Riccati equations in terms of the solutions of ordinary Riccati equation. We also derive Painlevé II equation ($\alpha = 2$) from the second order Riccati equation. Using the geometrical relation between the projective vector field equation and Hill's equation we obtain the solutions of various anharmonic oscillators. Solutions of the Ermakov-Pinney equation, Kummer-Schwarz equation Emden-Fowler and Painlevé II are given in terms of global projective connection. In the second half of the paper we derive generalized Chazy equation for dihedral triangle case, Chazy class XII equation and Painlevé II ($\alpha = -1/2$) from the second and third order Riccati equations. The relation between the Riccati and the projective vector field equations is explored via invariant methods.

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1 Introduction

In our earlier paper [16] we investigated finite dimensional integrable Hamiltonian systems associated to the stabilizer orbit of the coadjoint action of the Virasoro algebra. We studied several well known integrable ordinary differential equations [for example, see 3] including the celebrated Painlevé II equation. The vector field connected to the stabilizer space of the Virasoro orbit is known as projective vector field [17,19].

The importance of the projective vector field has been taken seriously in our earlier paper [16]. We have explored its connection to various integrable anharmonic oscillators. We show that using Kirillov's superalgebra [22,23] it is possible to describe the solution of the integrable systems associated to the stabilizer orbit. We formulate the solutions of Ermakov-Pinney equation [6,28], Painlevé II [20] etc. in terms of the projective vector field and its square roots in our earlier paper.

It is well known that a large number of integrable partial differential equations are connected to Virasoro orbit [27,30,31] . These mostly follow from the Euler–Poincaré flows with respect to various metrics on the Virasoro space. Naturally one would like to investigate the role the stabilizer set of Virasoro orbit. Since this should be the most natural place to tap for various integrable nonlinear ODEs and Painlevé equations.

The Painlevé equations have played a significant role in integrable systems. They arise from similarity reductions of classical soliton equations and as monodromy preserving deformation equations associated with linear systems of ordinary differential equations with rational coefficient [4,20]. Painlevé's original motivation was to search for new special functions. It is known that a large family of classical special functions are associated with a linear ordinary differential equation with polynomial or rational coefficients, for examples, Gauss' hypergeometric functions, Kummer's confluent hypergeometric functions, and various special functions with the name of Airy, Bessel, Hermite, etc.

Chazy [8,9] attempted to generalize the work of Painlevé to third order differential equations. In an attempt to classify the third-order ODEs $y"' = F(x, y, y', y")$ with F polynomial in y, y', $y"$ having the Painlevé property, Chazy introduced 13 classes of reduced equations. Chazy's work is closely related to the theory of modular functions. Modular functions are an important family of special functions that satisfy a third order differential quation. It is known [2,18] that classically known generalizations of the Chazy equation and Darboux-Halphen system are reductions of the self-dual Yang-Mills (SDYM) equations with an infinite-dimensional gauge algebra. It has been shown in [1] that the Darboux-Halphen system reduces to the generalized Chazy equation. Recently, Clarkson and Olver [10] has expressed the general solution of classical Chazy equation as a ratio of two solutions to a hypergeometric equation. In fact, they have shown that this equation can be reduced to Riccati equation. Thus Chazy equations are always fascinating equations and can be considered to be a close analogue of Painlevé equations of third order differential equations. In fact, Chazy's third order equation has a special solution that is also related to the sixth Painlevé equation. These special solutions are known as Picard solutions of the sixth Painlevé equation.

Later, Bureau [7] extended Painlevés first objective, and gave a partial classification of fourth-order equations. In recent years Cosgrove [12] presented at first in a superb paper the results on the Painlevé classification of the fourth- and fifth-order ODEs of the reduced forms $y^{(4)} = Ayy" + B(y')^2 + Cy^3$ and $y^{(5)} = Ayy"' + By'y" + Cy^2y'$. The list of the fourth-order ODEs contains six equations, F-I, $\cdots$, F-VI, including the Bureau barrier equation F-II which fails some Painlevé tests. The known equations F-I, F-III, F-IV, F-V are group-invariant reductions of known soliton equations. It was shown in [12] that for some parameter values, solutions of F-III and F-IV are related to each other by the Bäcklund transformation coming from the Miura transformation between the Kaup-Kupershmidt and Sawada-Kotera equations. It also demonstrated by Cosgrove that the F-I can be integrated in terms of the classical Painlevé-IV functions in the generic case and in terms of the Painlevé-II and Painlevé-I functions for the special parameter values. The integrability status of F-II is still unknown to us. The list of the fifth-order ODEs contains equations Fif-I, Fif-II, Fif-III, which are group-invariant reductions of known soliton equations, and a new equation Fif-IV. Most recently Cosgrove [13] completed the Painlevé classification of fourth-order differential equations in the polynomial class, Bureau symbol P1, that was begun in his earlier paper, where the subcase having Bureau symbol P2 was treated.

In this paper we will study the Chazy and Bureau symbol P1 class of systems from higher Riccati equations. All these higher Riccati equations or Riccati chain [14,15] play a very important role in our paper. It is known that all the higher order Riccati equations shares almost all the properties of Riccati equation and some special higher Riccati equations associated to the stabilizer set play a significant role to understand the geometrical origin of the Bureau and Chazy equations.

1.1 Motivation

The relation between the Riccati equation and the Painlevé is straight forward. The *special function solutions* of a Painlevé equation is obtained from the Riccati equation

$$v_x = p_2 v^2 + p_1 v + p_0$$

for some function p_2, p_1, p_0. Let us demonstrate its connection to Painlevé II.

Differentiating the Riccati equation yields

$$v_{xx} = p_2' v^2 + 2p_2 v v' + p_1' v + p_1 v' + p_0'$$

$$= 2p_2^2v^3 + (p_2' + 3p_1p_2)v^2 + (p_1' + 2p_0p_2 + p_1^2)v + p_1' + p_1p_0$$
$$= 2v^3 + xv + \alpha.$$

Thus we obtain

$$p_2(x) = \pm 1, \qquad p_1(x) = 0, \qquad p_0(x) = \pm\frac{1}{2}x, \qquad \alpha = \pm 12.$$

This connection between a Riccati equation and a Painlevé transedents contains reach geometry. We will explore this geometrical relation. But the main thrust of the paper is to explore coonection between the higher Riccati equations and the higher Painlevé type equations.

In this paper we study certain well-known ordinary diferential equations, higher order Riccati equations or Riccati chains, Painlevé II and Chazy equation XII [12] from the perspective of Virasoro orbit. In particular, we show that all these equations are related to the stabilizer set of Virasoro orbit. Let $f(x)\frac{d}{dx} \in Vect(S^1)$ be a vector field on a circle and $(udx^2, 1)$ be its dual. Then $f(x)\frac{d}{dx}$ is called projective vector field or $f(x)\frac{d}{dx} \in Stab\ (udx^2, 1)$ if and only if

$$f_{xxx} + 4uf_x + 2u_xf = 0. \tag{1}$$

This equation is known as projective vector field equation [19,17]. The importance of equation (1) is immense and this is as fundamental as KdV equation in a Virasoro orbit. A large number of ODEs are associated to this equation. The importance of this equation was unvieled by Kirillov while studying the classification of coadjoint orbits of orientation preserving group of diffeomorphism $Diff^1(S^1)$. So sometimes equation (1) is also known as Kirillov equation.

Equation (1) popped up in various places of integrable systems. For example in an interesting paper Rogers et al. [29] demonstrated that the extended Pinney equation

$$y_{xx} + u(x)y = \frac{\alpha}{y^3} - \frac{3}{\alpha}y^4y_x - \frac{1}{4\alpha^2}u^9, \qquad \alpha \neq 0, \tag{2}$$

can be recasted to projective field equation via

$$y = (\frac{\alpha}{2}\frac{y^2}{\int y^2\, dx})^{1/4}.$$

In fact relation between the Pinney equation and equation (1) is well known and this has appeared in various places. Geometrically, solutions of the Pinney equation are given by a global projective vector field. In the language of differential equation. The general solution of the Pinney equation can be expressed as a superposition of solutions of the linear Schrödinger equation. Our goal is to study equation (1) and show that many integrable ODEs including the extended Pinney equation follows from (1).

In this paper we explore that the projective vector field equation also plays a very vital role to understand the Pinney [28,29], generalized Pinney type $0+1$ dimensional integrable systems [5,6]. We obtain a series of Riccati equations, called Riccati chain from equation (1). These higher order Riccati equations play a important role in our paper.

The paper is **organized** as follows: We give all definitions of Virasoro algebra, coadjoint action and projective vector field in Section 2. We describe Kirillov's superalgebra in Section 3. We describe Riccati chain and its connection to Virasoro orbit in Section 4. Section 5 is devoted to anharmonic oscillators, Painlevé II and its connection to projective vector field equation. We also obtain the solutions of the Ermakov-Pinney equation and the Kummer-Schwarz equation in terms of global projective vector field. Section 6 is devoted to Painlevé II and Chazy equation. We explore the relation between higher Riccati and the Bureau symbol P1 in Section 7. The relation between Riccati and Eqn. (1) is explored from invariant method [26] in Section 8.

1.1.1 Main result

In this paper we bring three sets of idea together. We unveil the relation between the projective vector field equation associated to the stabilizer set of Virasoro orbit and various integrable anharmonic oscillator, Painlevé equations, Chazy equation and Riccati chain.

Our work yields following results:

■ We show that many integrable ODEs like the Pinney, extended Pinney, the Duffing–van der Pol oscillator etc. are associated the stabilizer set of the Virasoro orbit. In fact their solutions can be expressed in terms of the projective vector field.

■ We obtain the Riccati chain or higher order Riccati equations associated to projective vector field. We compute the solution of this special higher order Riccati equations in terms of ordinary equation. We show that Painlevé II for parameter $\alpha = 2$ follows from the second order special Riccati equation.

■ We derive the solutions of the Ermakov-Pinney equation and the Kummer-Schwarz equation in terms of global projective vector field. We also derive another Painlevé II for parameter $\alpha = -1/2$. These two Painlevé II equations are connected by Bäcklund transformations as shown by Clarkson et al. [10].

■ We show that generalized Chazy equation, Chazy class XII can be obtained from the second and third Riccati equations, which in turn connected to stabilizer set of Virasoro orbit. In particular, we derive the generalized Chazy equation and Chazy class XII from third order Riccati equation.

■ We demonstrate that two equations from the list of Cosgrove on Painlevé classification of fourth-order differential equations with Bureau symbol P1 follows from our construction.

■ We demonstrate further connection between Riccati and projective vector field equation from the invariants method.

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2 Recap: Virasoro orbit and projective structure

Consider the Lie algebra of vector fields on S^1, $Vect(S^1)$. The dual of this algebra is identified with space of quadratic differential forms $u(x)dx^{\otimes 2}$ by the following pairing,

$$< u(x), f(x) >= \int_0^{2\pi} u(x)\ f(x)dx,$$

where $f(x)\frac{d}{dx} \in Vect(S^1)$. The Virasoro algebra Vir has a unique nontrivial central extension by means of $\mathbf{R}$

$$0 \longrightarrow \mathbf{R} \longrightarrow Vir \longrightarrow Vect(S^1)$$

described by the Gelfand-Fuchs cocycle $\omega_1(f,g) = \frac{1}{2}\int_{S^1} f'g''dx$.

The elements of Vir can be identified with the pairs (2π periodic function, real number). The commutator in Vir takes the form

$$[(f(x)\frac{d}{dx}, a), (g(x)\frac{d}{dx}, b)] = ((fg' - gf')\frac{d}{dx}, \int_{S^1} \frac{1}{2} f'g''\ dx).$$

The dual space Vir^* can be identified to the set $\{(\mu, udx^2) \mid \mu \in \mathbf{R}$.

A pairing between a point $(\lambda, f(x)\frac{d}{dx}) \in Vir$ and a point (μ, udx^2) is given by

$$\lambda\mu + \int_{S^1} f(x)u(x)\ dx.$$

Lemma 2.1

$$ad^*_{(\lambda, f(x)\frac{d}{dx})}(\mu, udx^2) \ = \ \frac{1}{2}\mu f''' + 2f'u + 2fu'.$$

Proof: It follows from the definition

$$\begin{aligned} < ad^*_{(\lambda,f)}(\mu,u),(\nu,g) > \ &= \ < (\mu,u), ad_{(\lambda,\ f)}(\nu,g) > \\ &= \ < (\mu,u), (\frac{1}{2}\int_{S^1} f'g''dx, [f\frac{d}{dx}, g\frac{d}{dx}]) > . \\ &= \int_{S^1} u(fg' - f'g)dx + \frac{1}{2}\mu \int_{S^1} f'g''. \end{aligned}$$

□

Corollary 2.2 *The stabilizer space of the action of* $f\frac{d}{dx} \in Vect(S^1)$ *on the space of third-order differential operators of special type is given by*

$$f''' + 2u'f + 4uf' = 0,$$

or

$$ff'' + 2uf^2 - \frac{1}{2}(f')^2 = c, \tag{3}$$

where c is a constant.

Let Ω be the cotangent bundle of S^1. Let $\Omega^{\pm 1/2}$ be the square root of the tangent and cotangent bundle of S^1 respectively.

Definition 2.3 *A projective connection on the circle is a linear second-order differential operator*

$$\Delta : \Gamma(\Omega^{-\frac{1}{2}}) \longrightarrow \Gamma(\Omega^{\frac{3}{2}})$$

such that the symbol of

1. Δ *is the identity and*
2. $\int_{S^1}(\Delta s_1)s_2 = \int_{S^1} s_1(\Delta s_2)$ *for all* $s_i \in \Gamma(\Omega^{-\frac{1}{2}})$.

Let us take $s = \psi(x)dx^{-\frac{1}{2}} \in \Gamma(\Omega^{-\frac{1}{2}})$, then $\Delta s \in \Gamma(\Omega^{3/2})$ is locally described by

$$\Delta s = (a\psi'' + b\psi' + c\psi)dx^{\frac{3}{2}}.$$

From the definition of the projective connection condition (1) implies $a = 1$ and condition (2) implies $b = 0$, hence projective connection can be identified with the Hill operator

$$\Delta^{(2)} \equiv \Delta = \frac{d^2}{dx^2} + u(x).$$

Definition 2.4 *A vector field* $v = f(x)\frac{d}{dx}$ *is called a projective vector field which keeps fixed a given projective connection* $\Delta = \frac{d^2}{dx^2} + u(x)$

$$\mathcal{L}_v \Delta s = \Delta(\mathcal{L}_v s), \tag{4}$$

for all $s \in \Gamma(\Omega^{-\frac{1}{2}})$, *where* $\mathcal{L}_v$ *is the Lie derivative of* v.

Proposition 2.5 *A projective vector field* $v = f\frac{d}{dx} \in \Gamma(\Omega^{-1})$ *satisfies*

$$f''' + 4f'u + 2fu' = 0.$$

Hence equation (1) is called projective vector field equation.

Lemma 2.6 *If ψ_1 and ψ_2 are the solutions of*

$$\Delta\psi \; = \; (\frac{d^2}{dx^2} + u)\psi \; = \; 0, \tag{5}$$

then the product $\psi_i\psi_j \in \Gamma(\Omega^{-1})$ satisfies equation $f''' + 2u'f + 4uf' = 0$ and traces out a three-dimesional spaces of solution.

The sections of $\Gamma(\Omega^{-\frac{1}{2}})$ which satisfy the equation (5) are not functions but the square root of a *projective* vector field, since $\psi \in \Omega^{-1/2}$, the space of scalar densities of weight $-1/2$, square root of $f \in Vect(S^1)$.

3 Kirillov Superalgebra and Stabilizer Orbit

We define

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1,$$

where we denote $\mathcal{G}_0 \equiv Vect(S^1)$ and $\mathcal{G}_1 \equiv \Omega^{-1/2}(S^1)$. $\mathcal{G}$ forms a super Lie algebra on $S^{1,1}$ [9,10,12] and $\mathcal{G}_1$ is the super-partner of $\mathcal{G}_0$. This is asserted since $\mathcal{G}_1$ is the $\mathcal{G}_0$ module and it is compatible with the structure of $\mathcal{G}_0$ module and satisfies $\mathcal{G}_1 \times \mathcal{G}_1 \longrightarrow \mathcal{G}_0$. A typical element of $\mathcal{G}$ would be

$$f(x)\frac{d}{dx} + \psi(x)\sqrt{\frac{d}{dx}},$$

and the super Lie Bracket is given by

$$[(f_1, \psi_1), (f_2, \psi_2)] = ([f_1, f_2] + \psi_1\psi_2, \{f_1, \psi_2\} + \{\psi_1, f_2\}).$$

Definition 3.1 *A superprojective vector field is a pair $(f\frac{d}{dx}, \psi\sqrt{\frac{d}{dx}})$ which satisfies*

$$f''' + 4f'u + 2fu' \; = \; 0$$

and

$$\psi'' + u\psi = 0.$$

In this realization $(f(x)\frac{d}{dx} \oplus \psi(x)\sqrt{\frac{d}{dx}})$, i.e. $(f(x), \psi(x))$, forms a super Lie algebra. $(f(x), \psi(x))$ satisfies

$$f(x + 2\pi) = f(x)$$

$$\psi(x + 2\pi) = \pm\psi(x).$$

When it is in the '+ ' sector, it is called the Ramond sector super Lie algebra and the'−' sector is known as Neveu-Schwarz sector.

The cocycle may be extended to this superalgebra via

$$c(\psi_1, \psi_2) = \int_{S^1} \psi_1' \psi_2' dx. \tag{6}$$

We concentrate on the coadjoint action of the odd (or Fermionic) part of the Ramond and Neveu-Schwarz superalgebras.

Proposition 3.2 *Let* $\hat{\xi} = (\xi(x)\sqrt{\frac{d}{dx}}, a)$ *and* $\hat{u} = (u(x)dx^2, c)$*. Then the coadjoint action of* $\hat{\xi}$ *on* $\hat{u}(x)$ *yields*

$$ad^*_{\hat{\xi}}\, \hat{u}(x) = (-c\frac{d^2}{dx^2} + u(x))\, \xi. \tag{7}$$

Sketch of Proof: It is clear that

$$[\xi(x)\sqrt{\frac{d}{dx}}, \eta(x)\sqrt{\frac{d}{dx}}] \;:=\; \xi(x)\eta(x)\;\frac{d}{dx}.$$

Thus

$$\begin{aligned}
&\langle ad^*_{\xi(x)\sqrt{\frac{d}{dx}}} u(x)dx^2, \eta(x)\sqrt{\frac{d}{dx}}\rangle \\
&= \langle u(x)dx^2, [\xi(x)\sqrt{\frac{d}{dx}}, \eta(x)\sqrt{\frac{d}{dx}}]\rangle \\
&= \langle (u(x)dx^2, c), (\xi\eta\frac{d}{dx}, \int_{S^1} \xi'\eta'\; dx\rangle \\
&= \langle (-c\xi'' + u(x)\xi\;, 0), \hat{\eta}\sqrt{\frac{d}{dx}}\rangle
\end{aligned}$$

□

Thus the Hamiltonian operator corresponding to "Fermionic" part of the Kirillov's superalgebra is

$$\mathcal{O}_{Fer} = -c\frac{d^2}{dx^2} + u(x). \tag{8}$$

4 Stabilizer set, Riccati chain and other integrable systems

Consider the stabilizer equation of the odd part of Kirillov's superalgebra which coincides with the Hill's equation

$$a\psi_{xx} + u\psi = 0,$$

where a is a constant.

We make the change of variables

$$p(x) = \frac{\psi_x}{a\psi}.$$

Then

$$p_x = \frac{\psi_{xx}}{a\psi} - \frac{\psi_x^2}{a\psi^2}.$$

Thus after substituting this into Hill's equation, we obtain the celebrated Riccati equation

$$p_x + a\,p^2 + u = 0. \tag{9}$$

Thus it is readily clear that the Riccati equation under a Cole-Hopf transformation is connected to the stabilizer orbit of the "Fermionic" part of the Kirillov's superalgebra.

There are some interesting featues of the Riccati equation. If *one solution* of a Riccati equation is known, then we can get immediately general solutions of the whole family of Riccati equations obtained from the original one under the change of variables

$$\hat{p} = \frac{a(x)p + b(x)}{c(x)p + d(x)}. \tag{10}$$

It is also interesting to note that for the Riccati equation, if we know any three solutions p_1, p_2, p_3, we can construct all other solutions p using a simple formula known as *cross ratio.*

Lemma 4.1 *Given a triple (p_1, p_2, p_3) of distinct points in $\mathbf{R}P^1$, there is a unique projective linear transformation μ mapping (p_1, p_2, p_3) onto $(0, 1, \infty)$. It is given by the formula*

$$\mu(p) = \frac{(p - p_1)(p_2 - p_3)}{(p_1 - p_2)(p_3 - p)}, \tag{11}$$

where $\mu(p)$ is called the cross-ratio of the quadruple (p, p_1, p_2, p_3).

Corollary 4.2 *Let $\mathbb{S}$ be the solution space of Riccati equation passing through three distinct points p_1, p_2, p_3. Then $\mathbb{S}$ is the set of all points $p \in \mathbf{R}P^1$ such that*

$$p = \frac{k(p_1 - p_2) + p_1(p_2 - p_3)}{k(p_1 - p_2) + (p_2 - p_3)} \qquad k \in \mathbf{R}P^1. \tag{12}$$

This formula is called superposition formula.

4.1 Higher order Riccati and projective vector field equation

We wish to explore the connection between the projective vector field equation and the second-order Riccati equation.

Assume that

$$v = \frac{f_x}{f} \tag{13}$$

where f satisfies projective vector field equation

$$f_{xxx} + 4uf_x + 2u_x f = 0.$$

We obtain from equation (12)

$$\frac{f_{xxx}}{f} = v_{xx} + 3vv_x + v^3$$

and, after substituting the results above in Eqn. (1), it takes form

$$v_{xx} + 3vv_x + v^3 + 4uv + 2u_x = 0, \tag{14}$$

which is a particular case of the second-order Riccati equation. The coefficients are fixed by the projective vector field equation.

It is quite natural to search for the Riccati analogue of Lemma 2.6. In other words, we seek to find the relation between the solutions of the ordinary Riccati equation and the second order Riccati equation associated to projective vector field.

Proposition 4.3 1. *The projective vector field equation is equivalent to a particular form of second order Riccati equation* $v_{xx} + 3vv_x + v^3 + 4uv + 2u_x = 0$*, where* $v = f_x/f$*.*

2. *Suppose* $p(x) = p_1$ *be the solution of the Riccati equation. Then the second order Riccati satisfies* $v(x) = 2p_1$*.*

Proof: By direct computation one can check this result.
□

Therefore the above result yields the correspondences between solutions of the second-order Riccati and ordinary Riccati equation. At this stage we must give the definition of Riccati chains. In fact all the higher order Riccati equations satisfy most of the properties of the Riccati equation .

Definition 4.4 *Let* L *be the following differential operator*

$$L = \frac{d}{dx} + v(x).$$

The nth–order equation of the Riccati chain is given by the following formula

$$L^n v(x) + \sum_{j=1}^{n-1} \alpha_j(x)(L^{j-1}v(x)) + \alpha_0(x) = 0, \tag{15}$$

where n is an integer characterizing the order of the Riccati equation in the chain and $\alpha_j(x)$, $j = 0, 1, \cdots N$, are arbitrary functions.

The lowest-order equations in the chain after the ordinary Riccati equation are:

$$n = 2, \qquad v_{xx} + 3v(x)v_x + v^3(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0 \tag{16}$$

$$\begin{aligned} n = 3, \qquad & v_{xxx} + 4vv_{xx} + 3v_x^2 + 6v^2 v_x + \alpha_2(x)v_x \\ & + v^4(x) + \alpha_2 v^2(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0. \end{aligned} \tag{17}$$

$$\begin{aligned} n = 4 \qquad & v_{xxxx} + 5vv_{xxx} + 10v_x v_{xx} + 15vv_x^2 + 10v^2 v_{xx} + 10v^3 v_x + v^5 \\ + \alpha_3(x)(v_{xx} & + 3v(x)v_x + v^3(x)) + \alpha_2(V^2 + v_x) + \alpha_1(x)v(x) + \alpha_0(x) = 0. \end{aligned} \tag{18}$$

Corollary 4.5 *The Burgers hierarchy is defined as*

$$v_{t_n} = L^n v(x) \qquad \textit{where} \qquad L = \partial_x + v \tag{19}$$

Hence the Riccati chain reduces to stationary Burgers hierarchy for all $\alpha_i = 0$. Incidentally this coincides with the famous Faá di Bruno polynomials defined by

$$v^{(j+1)} = (\partial_x + v)v^{(j)} \tag{20}$$

Therefore the ordinary Riccati equation $v_x + v^2 + u = 0$ can be writen in the form

$$v^{(2)} + uv^{(0)} = 0 \qquad \textit{where} \qquad v^{(2)} = v_x + v^2. \tag{21}$$

Remark The second Riccati equation can be also expressed in terms of Faa di Bruno polynomial. It is given by

$$v^{(3)} + \alpha_1 v^{(1)} + \alpha^{(0)} v^{(0)} = 0. \tag{22}$$

The special second Riccati equation coincides with the $n = 2$ member of the Riccati chain when $\alpha_1 = 4u$ and $\alpha_0(x) = 2u_x$.

4.2 Construction and solutions of special higher order Riccati equations

In this section we consider a special class of Riccati equations whose solutions can be expressed in terms of ordinary Riccati equation. Let us consider the action of $Vect(S^1)$ on $\Delta^{(n)}$, defined by

$$\Delta^{(n)} = \frac{d^n}{dx^n} + \alpha_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \cdots + \alpha_1\frac{d}{dx} + \alpha_0 \tag{23}$$

Definition 4.6 *The $Vect(S^1)$ action on Δ^n is defined by*

$$[\mathcal{L}_v, \Delta^{(n)}] := \mathcal{L}_v^{-(n+1)/2} \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathcal{L}_v^{(n-1)/2}. \tag{24}$$

We consider a "special" fourth order differential equation

$$\Delta^{(4)} = \partial_x^4 + 10u\partial_x^2 + 10u'\partial_x + 9u^2 + 3u'' \tag{25}$$

It is special in this sense that the operator (25) satisfies most of the properties of projective connections on a circle.

Claim 4.7 *Let $f(x)\frac{d}{dx}$ be a projective vector field. The action of a vector field $f(x)\frac{d}{dx} \in Vect(S^1)$ on $\Delta^{(4)}$ yields*

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta^{(4)}] = 0.$$

Proof: It is easy to check that if f satisfies equation (1) then above result follows immediately.

□

Proposition 4.8 *Let ψ_1 and ψ_2 be the solution of Hill's operator. The equation $f'''' + 10uf'' + 10u'f' + (9u^2 + 3u'')f = 0$ traces out a four dimensional spaces of solution spanned by*

$$\{\psi_1^3,\ \psi_1^2\psi_2,\ \psi_1\psi_2^2,\ \psi_2^3\}.$$

Proof: By direct computation.

□

We consider the third-order Riccati equation as

$$v_{xxx} + 4vv_{xx} + 3v_x^2 + 6v^2v_x + 10uv_x + v^4 + 10uv^2 + 10u_xv + 9u^2 + 3u_{xx} = 0. \tag{26}$$

This third-order Riccati equation is associated to operator (25) is also known as the projective third-order Riccati equation. By the Cole-Hopf transformation

$$v(x)f(x) = \frac{df}{dx}(x)$$

one can easily linearize all these higher order Riccati equations to obtian higher order linear equations. In other words the whole class of Riccati equations in Riccati chain linearizes to a linear ordinary differential equation with variable coefficients

$$\frac{d^n f}{dx^n} + \sum_{j=0}^{n-2} \alpha_j \frac{d^{n-2} f}{dx^{n-2}} = 0. \tag{27}$$

It is clear that for *special third-order Riccati equation* we identify $\alpha_2 = 10u$, $\alpha_3 = 10u_x$ and $\alpha_3 = 9u^2 + 3u_{xx}$.

By simple inspection and also from direct computation one can check that

Proposition 4.9 *Let $p(x) = p_1$ be the solution of the Riccati equation. Then the solution of the projective third order Riccati equation (18) is given by $q(x) = 3p_1$*

4.2.1 Kolchin closed and homogeneous differential polynomials

A differential polynomial $P \in Der(\psi_1, \psi_2)$ is ∂ - homogeneous if there is a positive integer n such that for all λ,

$$P(\lambda\psi_1, \lambda\psi_2) = \lambda^n P(\psi_1, \psi_2).$$

Definition 4.10 *A subset of $\mathbf{R}P^1(Der(\psi_1, \psi_2))$ is called Kolchin closed if it is the set of zeros of a finite set of ∂ - homogeneous differential polynomials in $Der(\psi_1, \psi_2)$. The Kolchin closed set on a projective line is defined by the differential polynomial*

$$Wr(\psi_1, \psi_2) = \psi_2\psi_1' - \psi_2'\psi_1.$$

Remark Notice that the above definition of Kolchin closed is analogous to the Zariski closed for homogeneous polynomials. A subset of $\mathbf{R}P^1$ is *Zariski closed* if it is set of zeros of a finite set of homogeneous polynomials.

4.3 Finite-gap potential and generalized Schwarzian equation

In this Section we study the connection between the modified Schwarzian equation and projective vector field. This equation appears various places in integrable systems.

Let us consider a polynomial generalized potential

$$u(x, \lambda) = \lambda^n + u_1\lambda^{n-1} + \cdots + u_n. \tag{28}$$

Hence, the projective vector field equation becomes

$$f_{xxx} + 4u(x, \lambda) f_x + 2u_x(x, \lambda) f = 0. \tag{29}$$

A generalized potential $u(x,\lambda)$ is called N-phase potential if (29) has a solution which is a polynomial in λ of degree N, i.e.,

$$f(x,\lambda) = \lambda^N + f_1\lambda^{N-1} + \cdots + f_N.$$

It is easy to transform the projective vector field equation to

$$\frac{f''}{f} + 4u(x,\lambda) - \frac{f'^2}{f^2} = \frac{Wr^2}{f^2} \tag{30}$$

where the constant part can be fixed by the Wronskian of its partner equation

$$\psi'' + u(x,\lambda)\psi = 0.$$

Proposition 4.11 *Let* ψ_1 *and* ψ_2 *be the solutions of* $\psi'' + u(x,\lambda)\psi = 0$. *Let us define*

$$v_i = \frac{\psi_1\psi_2 \mp Wr}{2\psi_1\psi_2}, \tag{31}$$

where $Wr = \psi_1\psi_2' - \psi_2\psi_1'$ *is the Wronskian. Then* q_i *maps the Riccati equation*

$$v_{ix} + v_i^2 + u(x,\lambda) = 0$$

to

$$ff'' + 2u(x,\lambda)f^2 - \frac{1}{2}(f')^2 = Wr^2.$$

Proof: By substituting v_i into the Riccati equation one obtains the proof.
□

Suppose we assume $Wr(\psi_1,\psi_2) = \lambda$. Thus corresponding to (30) we obtain the following modified Schwarzian derivative equation

$$u(x,\lambda) = \frac{1}{2}\frac{g_{xx}}{g} - \frac{3}{4}\frac{g_x^2}{g^2} + \lambda^2 g^2. \tag{32}$$

This equation is called the modified Schwarzian equation by Kartashova and Shabat [21]. This equation has a profound application in integrable systems.

Then Eqn. (32) has unique asymptotic solution represented by formal Laurent series, such that

$$g(x,\lambda) = 1 + \sum_{l=1}^{\infty} \lambda^{-l} g_l(x),$$

where coefficients g_l are different polynomials in all $u_1, \cdots, u_n$.

5 Connection to Painlevé II and various anharmonic oscillators

Last section has been devoted to study Riccati chains related to projective vector field equation. These are special classes of higher Riccati equations the coefficients of which are governed by all (higher) projective connections.

In this Section we use various members of the Riccati chains to explore the connection between stabilizer set of the Virasoro orbit and the various anharmonic oscillators, for example, Embden equation, van der Pol Oscillator equation, Ermakov-Pinney equation etc. We also derive the Painlevé II equation from the second Riccati equation. Thus, in this section we establish a systematic method to obtain the solutions a class of integrable ODEs via projective vector field and its global analogue.

5.1 Nonlinear oscillator equations

We start with an easy example. Consider the ordinary Riccati equation associated to Hill's equation

$$v_x + v^2 + u = 0.$$

Assume v takes the following form

$$v = \frac{y_x}{y} + W(y(x))). \tag{33}$$

It is straight forward to check that y satisfies

$$y_{xx} + (2W + yW')y_x + (W^2 + u)y = 0, \tag{34}$$

where W' denotes the variational derivative or Frechet derivative with respect to y. Let us give a few examples.

Case I Equation (34) boils down to the generalized Emden - type equation

$$y_{xx} + 3cyy_x + c^2y^3 + uy = 0 \tag{35}$$

for $W = cy$.

Case II Let $W = ky^4$. We have a freedom to choose u also. We set $u = c(x) + k_1y^{-4}$. Substituting these values of W and u in Eqn. (34) we obtain the extended Pinney equation

$$y_{xx} + 6ky^4y_x + k^2y^9 + c(x)y + \frac{k_1}{y^3} = 0.$$

The standard form of the Pinney equation is

$$y_{xx} + c(x)y + \frac{k_1}{y^3} = 0,$$

where k_1 is a constant which can be normalized to $k_1 = \mp 1$.

Since the solution of the projective vector field equation is spanned by

$$Span(\psi_1^2, \psi_2^2, \psi_1\psi_2),$$

naturally, an arbitrary solution of projective vector field equation is given by

$$\Psi = A\psi_1^2 + 2B\psi_1\psi_2 + C\psi_2^2, \tag{36}$$

an arbitrary linear combination of basis vectors. This is periodic and hence a global solution of the projective vector field equation. This Ψ is called *global projective vector field*. After renaming the constants we can express the solution of the Pinney equation in terms of the square root of the global projective vector field. We will consider this case in next section.

Case III Similarly one can derive van der Pol oscillator type systems. We substitute $W = cy^2$ in to equation (34), we obtain

$$y_{xx} + 4cy^2 y_x + c^2 y^5 + uy = 0. \tag{37}$$

Therefore we can say that equation (34) can be viewed as a "master equation" for various oscillator type nonlinear ODEs.

The solutions of these equations (35,37) can also be obtained from the linearization method. In other words solutions of (!, !!) can be expressed in terms of ψ_i, solutions of Hill's operator

$$\psi_i = ye^{\int_0^x W(y(x'))\, dx'}.$$

It is known that the basis of solutions of Hill's equation of type

$$\Delta\psi \equiv \frac{d^2}{dx^2} + u_1 = 0$$

explicitly:

1. $\psi_1 = \sin\sqrt{u_1}x, \qquad \psi_2 = \cos\sqrt{u_1}x \qquad u_1 > 0.$
2. $\psi_1 = 1, \qquad \psi_2 = 0 \qquad u_1 = 0.$
3. $\psi_1 = e^{\lambda x}, \qquad \psi_2 = e^{-\lambda x} \qquad u_1 = -\lambda^2 < 0.$

The Floquet matrix is ± 1 only if $u_1 = \frac{n^2}{4}$ for $n \in \mathbb{Z}$. Therefore it is not hard to get solutions of such integrable anharmonic oscillator equations.

5.2 Second order Riccati equation and Painlevé II equation

Consider the Airy differential equation [cf. 2, 20]

$$\psi_{xx} + x\psi = 0. \tag{38}$$

It is clear that for a special choice of $u = x$, the Hill's equation becomes the Airy differential equation.

Proposition 5.1 *Let $v = \frac{\psi_x}{\psi}$ satisfy the second Riccati equation. If ψ satisfies the Airy equation*

$$\psi'' - x\psi = 0,$$

then the second Riccati equation satisfies the Painlevé II equation

$$v'' = 2v^3 + xv + 2 \tag{39}$$

Proof: It is clear that $u = -x$. We make the change of variables

$$v = \frac{\psi_x}{\psi}. \qquad \text{Then } v_x = x - v^2.$$

When we substitute this result into second Riccati equation, it takes the form of equation (42).
□

Remark Let us briefly describe the connection between Painlevé II hierarchy and our approach. It is readily clear that the projective vector field equation is the stabilizer set of Virasoro orbit. In other words, it yields the second Hamiltonian structure of the KdV equation

$$\mathcal{O}_2 = \partial_x^3 + 4u\partial_x + 2u_x. \tag{40}$$

Using " frozen Lie-Poisson structure" we can define the first Hamiltonian structure of the KdV equation too. This satisfies famous Lenard scheme

$$\partial_x \mathcal{H}_{n+1} = (\partial_x^3 + 4u\partial_x + 2u_x)\mathcal{H}_n. \tag{41}$$

The mKdV hierarchy is obtained from the KdV hierarchy through Miura map $u = v_x - v^2$. Thus the Painlevé hierarchy defined by Clarkson et al. [10] is given as

$$P_{II}^n(v, \beta_n) \equiv (\frac{d}{dx} + 2v)\mathcal{H}_n(v_x - v^2) - vx - \beta_n = 0. \tag{42}$$

Remark Let us combine the second Riccati and the ordinary Riccati equation. This system can be reduced to the equation VI of the Painlevé-Gambier classification

$$v_{xx} + 3vv_x + v^3 = q(x)(v_x + v^2). \tag{43}$$

5.3 Global projective vector field and integrable systems

The global projective vector field also plays an important role to the solutions of some integrable systems. In this Section we demonstrate this property with some examples.

The simplest Ermakov system reads

$$\psi'' + u(x)\psi = \frac{\sigma}{\psi^3}. \tag{44}$$

Proposition 5.2 *If ψ_1 and ψ_2 satisfy Hill's equation then the square root of the global projective vector field, i.e.,*

$$\psi = \sqrt{A\psi_1^2 + 2B\psi_1\psi_2 + C\psi_2^2} \tag{45}$$

satisfies Ermakov equation

$$\psi'' + u(x)\psi = \frac{\sigma}{\psi^3}, \qquad \sigma = AC - B^2.$$

Proof: It follows from

$$\psi\psi'' + u\psi^2 + \psi'^2 = (A\psi_1' + B\psi_2')\psi_1' + (B\psi_1' + C\psi_2')\psi_2',$$

and unit Wronskian property.
□

Corollary 5.3 *The solution of the Kummer-Schwarz equation*

$$\frac{1}{2}\frac{f''}{f} - \frac{3}{4}(\frac{f'}{f})^2 + \sigma f^2 = u(x)$$

is given by

$$f(x) = (A\psi_1^2 + 2B\psi_1\psi_2 + C\psi_2^2)^{-1}, \tag{46}$$

where ψ_1 and ψ_2 satisfy the Hill's equation.

Remark: The relation between the Ermakov equation and the Kummer-Schwarz equation depicts the global version of the relation between Hill's equation and projective vector field equation.

5.3.1 Second type of Painlevé II and Generalized Emden-Fowler equation

In this section we derive second type of Painlevé II with a different parameter value ($\alpha = \frac{1}{2}$). Let us consider once again the Cole-Hopf transformation.

$$v = \frac{\psi_x}{\psi}.$$

Lemma 5.4 *If ψ satisfies the Ermakov equation then y satisfies*

$$v_{xx} + 6vv_x + 4v^3 + 4vu + u_x = 0. \tag{47}$$

This equation is called the generalized Embden-Fowler equation.

Proof: After differentiating twice the equation $y = \frac{\psi_x}{\psi}$ we obtain

$$v_{xx} + 2vv_x + u_x = -4v(\frac{\sigma}{\psi^4})$$

where $v_x + v^2 + u = \frac{\sigma}{\psi^4}$.
□

Let us consider the Airy differential equation

$$\psi_{xx} + x\psi = 0,$$

Proposition 5.5 *If ψ satisfies Airy equation*

$$\psi_{xx} + \frac{x}{2}\psi = 0,$$

then y satisfies Painlevé II equation

$$v_{xx} = 2v^3 + xv - \frac{1}{2} \tag{48}$$

Corollary 5.6 *The solutions of the Painlevé II*

$$v_{xx} = 2v^3 + xv - \frac{1}{2}$$

can be obtained in terms of following Riccati equation

$$\frac{dv}{dx} + v^2 + \frac{1}{2}x = 0,$$

which yields Airy equation under $v = \frac{\psi_x}{\psi}$.

Remark The Painlevé transcedents (P-II – P-VI) possess *Bäcklund transformations* which map solutions of a given Painlevé equation to solutions of the same Painlevé equation, but with different values of the parameters. Therefore two Painlevé equations for $\alpha = 2$ and $\alpha = -1/2$ are connected by Bäcklund transformations.

6 Higher Riccati equations and Chazy equation

We use the higher order Riccati equation to obtain the Chazy equation. We use both the second-order Riccati and third order Riccati equations. We show that this equation leads to Chazy class XII equation.

Case I Let us study Chazy class XII equation. This follows readily from the third order Riccati equation.

Proposition 6.1 *Let*

$$v_{xxx} + 4vv_{xx} + 3v_x^2 + 6v^2v_x + 10uv_x + v^4 + 10uv^2 + 10u_xv + 9u^2 + 3u_{xx} = 0$$

be the special third order Riccati equation associated to fourth-order projective conection. This boils down to a special case of the Chazy equation XII for $u = v^2$, given by

$$v_{xxx} + 10vv_{xx} + 9v_x^2 + 36v^2v_x + 20v^4 = 0. \tag{49}$$

Case II We have already seen how ordinary Riccati equation can be effectively used to study certain transformation to determine various other nonlinear oscillator equations. We use similar technique for second order Riccati equation.

Proposition 6.2 *We define $v = \frac{y_x}{y} + g(x)$. If v satisfies the second order Riccati equation, then y satisfies*

$$y_{xxx} + 3gy_{xx} + (3g_x + 3g^2 + 4u)y_x + (3gg_x + g^3 + 4ug + 2u_x)y = 0. \tag{50}$$

Proof: One can check by direct computation that all the coefficients of y^{-3} and y^{-2} cancel and we obtain our desired result.
□

Corollary 6.3 *Suppose we take $g = 2y$ and $u = y_x$. Then equation (50) yields the Chazy XII type equation*

$$y_{xxx} + 8yy_{xx} + 10y_x^2 + 32y^2y_x + 8y^4 = 0 \tag{51}$$

Therefore we derive the Chazy equation XII from the second order Riccati equation.

6.1 Generalized Chazy equation

Recently Ablowitz et. al. studied a general class of Chazy equation, defined as

$$v_{xxx} - 2vv_{xx} + 3v_x^2 = \frac{4}{36 - n^2}(6v_x - v^2)^2. \tag{52}$$

This equation was first written down and solved by Chazy and is known today as the generalized Chazy equation. Clarkson and Olver showed that a necessary condition for the equation (52) to possess the Painlevé property is that the coefficient must be $\alpha = \frac{4}{36-n^2}$ with $1 < n \in \mathbb{N}$, provided that $n \neq 6$. It has been further shown in [11], the cases $n = 2, 3, 4$ and 5, correspond to the dihedral triangle, tetrahedral, octahedral and icosahedral symmetry classes.

It should be noted that the classical Darboux–Halphen system, which is also equivalent to the vacuum Einstein equations for Riemannian self dual Bianchi–IX metrics is equivalent to classical Chazy (also known as Chazy class III) equation

$$y_{xxx} - 2yy_{xx} + 3y_x^2 = 0 \tag{53}$$

The classical Chazy equation is such that the only singularity of its general solution is a movable noncritical natural boundary, a circle, whose centre and radius depend on three initial conditions of the Cauchy problem. Therefore, it shares lots of properties with Painlevé equation.

In this Section we will show that the generalized Chazy equation is a third Riccati equation in disguise. Let us assume all $\alpha_i = 0$. Therefore, the third Riccati equation becomes

$$v_{xxx} + 3v_x^2 + 4vv_{xx} + 6v^2v_x + v^4 = 0. \tag{54}$$

Proposition 6.4 *The third Riccati equation (54) is equivalent to*

$$v_{xxx} - 2vv_{xx} + 3v_x^2 = \frac{1}{8}(6v_x - v^2)^2. \tag{55}$$

Proof: Our proof follows from rescaling $v \to v/2$ and $x \to -x$.
□

Therefore, we obtain the generalized Chazy equation for $\alpha = \frac{1}{8}$, and this is related to $n = 2$ dihedraltriangle case.

Remark: Originally the link between classical Chazy equation and Riccati appeared in the work of Chazy. Most recently this has been investigated by Labrunie and Conte [24]. They noticed that Eqn. (52) admits a two parameter solution

$$v(x) = -6\frac{x - a}{(x - b)^2}, \tag{56}$$

where a, b are arbitrary parameters in the complex plane. Eliminating a and b between $v(x)$ calculated by (56) and its first two derivative yields

$$S \equiv 9v_{xx}^2 + 2(v^2 - 9v_x)vv_{xx} + 3(8v_x - v^2)v_x^2 = 0.$$

Then it can be shown $S_x - 2vS$ factorizes into second Riccati and Chazy class III equation. One of the basic needs in Galois theory is, obviously, factoring polynomials. This is a differential counter part. Usually, local differential Galois theory is studied for differential equations over $\mathbb{K}((x))$, that is, whose coefficients are (formal) meromorphic functions. I leave this to our alert readers.

Finally, we show that another Chazy equation, namely, Chazy -IV is connected to our programme.

Proposition 6.5 *The Chazy -IV equation*

$$v_{xxx} = -3vv_{xx} - 3v_x^2 - 3v^2v_x \tag{57}$$

is a derivative of second order Riccati equation for $\alpha_i = 0$.

7 Higher-order Painlevé equations and Bureau symbol P1

Most recently Cosgrove [13] has studied the more difficult subcase of the Painlevé classification of fourth-order differential equations in the polynomial class that was started in [12]. In his celebrated paper Cosgrove carried out Painlevé classification of differential equations of fourth order of the following form

$$\begin{aligned} y_{xxxx} = A(x)yy_{xxx} + B(x)y_xy_{xx} + C(x)y^2y_{xx} + D(x)yy_x^2 + E(x)y^3y_x + F(x)y^5 \\ +G(x)y_{xxx} + H(x)yy_{xx} + I(x)y_x^2 + J(x)y^2y_x + K(x)y^4 + L(x)y_{xx} + M(x)yy_x \\ +N(x)y^3 + P(x)y_x + Q(x)y^2 + R(x)y + S(x). \end{aligned} \tag{58}$$

The subcase treated here has Bureau symbol P1 and may be identified by its reduced equations which take the form

$$y_{xxxx} = Ayy_{xxx} + By_xy_{xx} + Cy^2y_{xx} + Dyy_x^2 + Ey^3y_x + Fy^5, \tag{59}$$

where $A, B, \cdots F$ are all constants, not all of them are zero.

Cosgrove presented the results of the Painlevé classification for fourth-order differential equations where the Bureau symbol is P1. He gave a long list of the equations F-VII – F-XVIII in this category. In addition to this list he added a non-Painlevé equation of similar shape which triggered interest among mathematicians. It is a simplest example beyond the Chazy-XII equation that can be studied via Clarkson and Olver method. It is known as Clarkson-Olver equation and defined by

$$\text{F-XIX}\ :\qquad (\frac{d}{dx}-\frac{4}{3}y)[y_{xxx}-2yy_{xx}+3y_x^2-\alpha(6y_x-y^2)^2]=0, \tag{60}$$

where α is an arbitrary constant and this has a very complicated singularity structures. The expression within the third brackets is already familiar to us - Chazy - XII. It is known that the Chazy - XII equation has a single-valued general solution when $\alpha = 4/(36-n^2)$ for $n \neq 1, 6$.

We derive three different equations from the list of Cosgrove on the Painlevé classification of fourth order equations with Bureau symbol P1.

Proposition 7.1 *The following two equations follow from the higher Riccati equations*

$$\begin{aligned} \textit{F-XII} \qquad v_{xxxx} &= -4vv_{xxx}-6v^2v_{xx}-4v^3v_x \\ &\quad -12vv_x^2-10v_xv_{xx}. \end{aligned} \tag{61}$$

$$\begin{aligned} \textit{F-XVI} \qquad v_{xxxx} &= -5vv_{xxx}-10v_xv_{xx}-15vv_x^2-10v^2v_{xx} \\ &-10v^3v_x-v^5+A(x)(v_{xxx}+4vv_{xx}+3y_x^2+6v^2v_x+v^4) \\ +B(x)(v_{xx}+3v(x)v_x+v^3(x)) &+C(x)(V^2+v_x)+D(x)v(x)+E(x)=0. \end{aligned} \tag{62}$$

Proof: A) The equation F-XII follows directly from the fourth order Riccati (R_4) and the third order Riccati (R_3) equations for all $\alpha_i = 0$ (also known as Burgers higer order flows). The F-XII fourth order equations with Bureau symbol P1 is given as

$$\text{F-XII}\ = R_4 - vR_3.$$

B) The F-XVI fourth order equations with Bureau symbol P1 is the combination of all higher order Riccati equations.
□

8 Invariants, Riccati and differential algebra

In the previous Section we have constructed Riccati chains from projective vector field equation. Some of the transformation we have made are not totally accidental. It has a deeper connection to some other exciting branches of mathematics. In this Section we shed some light on it.

For a differential equation in n-independent variables and one scalar dependent variable u, we consider the space $X \times V \ni (x, v)$, where $X = \mathbf{R}^n$ and $V = \mathbf{R}$. Suppose G is a Lie group acting on some open subset $M \subseteq X \times V$. Then the transformation by $\mathcal{G} \in G$ is

$$\mathcal{G}\cdot(x,v)=(\bar{x},\bar{v}), \qquad \bar{v}=\bar{v}(\bar{x}).$$

The vector field corresponding to g is

$$\chi = f^i(x,v)\partial_i + g(x,v)\frac{\partial}{\partial v}. \tag{63}$$

Given a smooth function $v = v(x)$, it induces a function $v^{(n)} = pr^{(n)}v(x)$, called the nth prolongation of v, where

$$pr^{(n)}v : \mathbf{R} \longrightarrow \mathbf{R}^{n+1}$$

is the vector consisting of all the derivatives of v of orders from 0 to n. The total space $X \times V^{(n)} \subseteq \mathbf{R}^{n+1}$, whose coordinates represent the independent varaible, dependent variable and the derivatives of v to order n is called the *nth order jet space* of the underlying space $X \times V$.

We are interested here on Riccati equation, so we concentrate on one dimension. In local coordinates, we write the group action infinitesinally

$$\bar{x} = x + \epsilon f(x,v) + O(\epsilon^2),$$

$$\bar{v} = x + \epsilon g(x,v) + O(\epsilon^2).$$

The vector field and corresponding to $\mathcal{G}$ is given by

$$\chi = f(x,v)\frac{\partial}{\partial x} + g(x,v)\frac{\partial}{\partial v}, \tag{64}$$

If we make a change of variables $\mathcal{G}(x,v) = (\bar{x},\bar{v})$, then first prolongation computes the relation between $d\bar{v}/d\bar{x}$ to dv/dx.

Lemma 8.1

$$\frac{d\bar{v}}{d\bar{x}} = (Dg - v_x Df) = g_x + (g_v - f_x)v_x - f_v v_x^2. \tag{65}$$

Proof: It follows straight from

$$\frac{d\bar{v}}{d\bar{x}} = \frac{d(v + \epsilon g + O(\epsilon^2)}{d(x + \epsilon f + O(\epsilon^2)}$$

$$= \frac{v_x + (g_x + g_v v_x)\epsilon + O(\epsilon^2)}{1 + (f_x + f_v v_x)\epsilon + O(\epsilon^2)}$$

$$= v_x + [g_x + (g_v - f_x)v_x - f_v v_x^2]\epsilon + O(\epsilon^2).$$

□

Definition 8.2 *The vector field corresponding to $pr^{(1)}\mathcal{G}$*

$$pr^{(1)}\chi = f(x,v)\frac{\partial}{\partial x} + g(x,v)\frac{\partial}{\partial v} + (Dg - u_x Df)\frac{\partial}{\partial v_x} \tag{66}$$

where

$$D = \frac{\partial}{\partial x} + v_x\frac{\partial}{\partial v}.$$

Proposition 8.3 *Suppose v satisfies Riccati equation*

$$v_x - e(x,v) \equiv v_x + v^2 + u = 0.$$

Then the Riccati equation remains invariant with respect to first prolongation $pr^{(1)}\chi$ for all $g(x,v) = a(x)v + b$ provided f satisfies projective vector field equation.

Outline of Proof: At first it is easy to check that

$$pr^{(1)}\chi(v_x - e(x,v)) = g_x + (g_v - f_x)e - f_v e^2 - f e_x - g e_v. \tag{67}$$

Let us substitute

$$e(x,v) = -(v^2 + u(x)), \qquad \text{and} \qquad g(x,v) = a(x)v + b$$

in equation (!). This yields

$$0 = [f_v]v^4 + [a(x) + f_x + 2f_v u]v^2 + [2b(x) - a_x]v$$
$$+[-b_x + (-a(x) + f_x)u - f_v u^2 - f u_x]$$

Solving recursive above equations we obtain our result.
□

Remark: One would obtain the same result if one starts from $v_x + Av^2 + Bv + C = 0$. In this case u must be expressed in terms of A, B and C and their derivatives.

9 Conclusion and Outlook

In this paper we have pursued integrable ODEs, Painleve II, Chazy XII, generalized Chazy equation, Bureau symbol P1 type systems from the study of the Virasoro orbit. We have shown that all these systems are connected to the stabilier set of Virasoro orbit. In particular, a large class of all these systems can be elegantly described by the projective vector field equation. Indeed we have shown in this paper that this equation can give an unified description of several $0+1$ dimensional integrable systems and their solutions can be elegantly described from the geometry of projective structures on circle.

There are several interesting questions popped up in this paper. It would be interesting to study the role of higher order Riccati equations to derive the modular functions. In fact, a number of interesting examples are derived in this paper which show that these symmetry algebras contain important information about the structure of the solution space to the differential equation which cannot be obtained from the standard Lie symmetries. The examples also suggest that these symmetry algebras may be the natural candidates for a differential Galois theory of nonlinear equations.

10 References

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