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Research article

Pointwise-in-time α-robust error estimate of the ADI difference scheme for three-dimensional fractional subdiffusion equations with variable coefficients

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Abstract: In this paper, a fully-discrete alternating direction implicit (ADI) difference method is proposed for solving three-dimensional (3D) fractional subdiffusion equations with variable coefficients, whose solution presents a weak singularity at $t = 0$. The proposed method is established via the L1 scheme on graded mesh for the Caputo fractional derivative and central difference method for spatial derivative, and an ADI method is structured to change the 3D problem into three 1D problems. Using the modified Grönwall inequality we prove the stability and α-robust convergence. The results presented in numerical experiments are in accordance with the theoretical analysis.

Keywords: three-dimensional fractional equation; ADI scheme; variable coefficient; α-Robust; L1 scheme; stability and convergence
Mathematics Subject Classification: 65M15, 65M06

1. Introduction

This paper focuses on the 3D subdiffusion equations with variable coefficients

$$D_t^\alpha u - a_1 u_{xx} - a_2 u_{yy} - a_3 u_{zz} + bu = f(x,y,z,t), \quad (x,y,z) \in \Lambda, t \in (0,T], \tag{1.1}$$

$$u(x,y,z,0) = g(x,y,z), \quad (x,y,z) \in \Lambda, \tag{1.2}$$

$$u(x,y,z,t) = \varphi(x,y,z,t), \quad (x,y,z) \in \partial\Lambda, \quad t \in (0,T] \tag{1.3}$$

in which: a_1, a_2, a_3, and b are positive constants; f, φ, and g are smooth functions; and $\Lambda = (0, \Upsilon_x) \times (0, \Upsilon_y) \times (0, \Upsilon_z)$ is a rectangular field bounded by $\partial\Lambda$. The Caputo fractional derivative can be defined via

$$D_t^\alpha u(\cdot,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} \frac{\partial u(\cdot,\xi)}{\partial \xi} d\xi, \alpha \in (0,1).$$

The fractional subdiffusion equation with variable coefficients is derived from the classical partial differential equation. Many scholars have studied the existence and uniqueness of the solution of this equation [1,2]. The high-dimensional subdiffusion problem can accurately describe the mechanical and physical processes with non-locality and spatial global correlation, it has become an important tool to describe a variety of complex mechanical and physical behaviors [3–5], and many numerical schemes have been proposed in [6–15]. Among them, the ADI method is favored by many scholars. Tian and Ge [16] offered the fourth-order ADI difference method for 2D unsteady convection diffusion problems. Wang et al. [17,18] considered the 2D fractional integro-differential equation by using an ADI compact difference method. Zhang et al. [19] provided the 2D semilinear multidelay parabolic equations by using Crank-Nicolson ADI compact difference schemes. Zhang et al. [20] considered the 2D space-fractional nonlinear Ginzburg-Landau equation via linearized ADI difference schemes. Wang et al. [21,22] presented an analysis of the convergence of the α-robust H^1-norm for the ADI differece scheme for the 2D time fractional diffusion equation (TFDE). Chen et al. [23,24] considered the ADI difference scheme for a class of two-dimensional multi-term TFDEs and gave the point error estimation. Qiao et al. [25–27] presented the ADI difference scheme for 2D TFDE with a weakly singular kernel. Zhang et al. [28] presented an efficient ADI difference scheme for the nonlocal evolution problem in 3D space. Zhou et al. [29,30] solved the 3D TFDE via the ADI method.

Along with the previous research, many scholars considered the subdiffusion equation with variable coefficients. Ngondiep [31] discussed a class of convection-diffusion-reaction equations with variable coefficients and gave a two-level fourth-order scheme. Wei et al. [32] gave the anisotropic finite element method (FEM) for 2D time fractional variable coefficient diffusion equations on graded meshes. Zeng and Tan [33] proposed a two-grid FEM for nonlinear TFDEs with variable coefficients. Ma et al. [34] presented the L1 robust analysis of the fourth-order block-centered FDM for 2D TFDEs with variable coefficient. As we all know, the study of the variable coefficient subdiffusion equation is almost based on one or two dimensions. It is not easy to solve three-dimensional fractional subdiffusion equations with variable coefficients due to the existence of the weak singularity near the initial time. In this paper, we put forward an α-robust L1 ADI difference scheme on graded mesh to overcome these difficulties. The L1 scheme on graded meshes is used to discretize the Caputo fractional derivative, and the difference method is used to approximate the spatial derivative. Using the modified Grönwall inequality, we prove the stability and α-robust convergence of the proposed method.

In this article, the main achievements are as follows:

- We construct an ADI difference method for the 3D subdiffusion equation with variable coefficients. Most previous work has focused on 1D or 2D problems.
- We introduce a new norm to prove the stability and convergence of the proposed method.
- The α-robust convergence is proved strictly, which is able to avoid the blow-up phenomenon in the case of $\alpha \to 1^-$.

The structure of this paper is as follows. In Section 2, the ADI scheme is developed for the 3D variable coefficient subdiffusion problems. In Section 3, we demonstrate the stability and convergence of the method. In Section 4, the three-dimensional numerical example is given to verify the effectiveness and accuracy of the proposed method. We conclude the paper in Section 5.

2. Derivation of ADI scheme

Set $h_x = \frac{\Upsilon_x}{\ell_1}$, $h_y = \frac{\Upsilon_y}{\ell_2}$, $h_z = \frac{\Upsilon_z}{\ell_3}$ for the positive integers ℓ_1, ℓ_2, and ℓ_3. Let $N \geq 1$ be a positive integer number and set $t_n = T(n/N)^\gamma$ for $n = 0, 1, \cdots, N$, $\gamma \geq 1$. Denote time step $\iota_n = t_n - t_{n-1}$ for $1 \leq n \leq N$.

Set $\overline{\Lambda}_h = \{(x_i, y_r, z_k)|0 \le i \le \ell_1, 0 \le r \le \ell_2, 0 \le k \le \ell_3\}$, and $\Lambda_h = \overline{\Lambda}_h \cap \Lambda$, $\partial\Lambda_h = \Lambda_h \cap \partial\Lambda$. Let u^n_{irk} be the approximation solution of (1.1)-(1.3).

For $n = 1, 2, \cdots, N$, by the L1 scheme [35, 36], denote

$$\begin{aligned} D^\alpha_N u^n_{irk} :&= \frac{1}{\Gamma(1-\alpha)} \sum_{q=0}^{n-1} \frac{u^{q+1}_{irk} - u^q_{irk}}{\iota_{q+1}} \int_{t_q}^{t_{q+1}} (t_n - \xi)^{-\alpha} d\xi \\ &= p_{n,1} u^n_{irk} - p_{n,n} u^0_{irk} - \sum_{q=1}^{n-1} (p_{n,q} - p_{n,q+1}) u^{n-q}_{irk} \end{aligned} \tag{2.1}$$

where

$$p_{n,1} = \frac{\iota_n^{-\alpha}}{\Gamma(2-\alpha)}, \quad p_{n,q} = \frac{(t_n - t_{n-q})^{1-\alpha} - (t_n - t_{n-q+1})^{1-\alpha}}{\iota_{n-q+1}\Gamma(2-\alpha)}, \quad 1 \le q \le n.$$

Thus, the problem (1.1)-(1.3) can be approximated by

$$D^\alpha_N u^n_{irk} - a_1 \delta^2_x u^n_{irk} - a_2 \delta^2_y u^n_{irk} - a_3 \delta^2_z u^n_{irk} + b u^n_{irk} = f^n_{irk},$$
$$(x_i, y_r, z_k) \in \Lambda_h, 1 \le n \le N, \tag{2.2}$$
$$u^0_{irk} = g(x_i, y_r, z_k), \quad (x_i, y_r, z_k) \in \overline{\Lambda}_h, \tag{2.3}$$
$$u^n_{irk} = \varphi(x_i, y_r, z_k, t_n), \quad (x_i, y_r, z_k) \in \partial\Lambda_h, 1 \le n \le N \tag{2.4}$$

where

$$\delta^2_x u^n_{irk} = \frac{u^n_{i+1,r,k} - 2u^n_{irk} + u^n_{i-1,r,k}}{h^2_x},$$
$$\delta^2_y u^n_{irk} = \frac{u^n_{i,r+1,k} - 2u^n_{irk} + u^n_{i,r-1,k}}{h^2_y},$$
$$\delta^2_z u^n_{irk} = \frac{u^n_{i,r,k+1} - 2u^n_{irk} + u^n_{i,r,k-1}}{h^2_z}.$$

Set $S_n = d^{-1}_{n,1} = \iota^\alpha_n \Gamma(2-\alpha)$, and to establish the ADI scheme, add a small term

$$\left(\frac{a_1 a_2 S^2_n \delta^2_x \delta^2_y}{1 + bS_n} + \frac{a_1 a_3 S^2_n \delta^2_x \delta^2_z}{1 + bS_n} + \frac{a_2 a_3 S^2_n \delta^2_y \delta^2_z}{1 + bS_n} - \frac{a_1 a_2 a_3 S^3_n \delta^2_x \delta^2_y \delta^2_z}{(1 + bS_n)^2}\right) D^\alpha_N u^n_{ijk} \tag{2.5}$$

onto the left side of Equation (2.2), which is order $O(N^{-2\alpha})$. Then, one has

$$\left(I + \frac{a_1 a_2 S^2_n \delta^2_x \delta^2_y}{1 + bS_n} + \frac{a_1 a_3 S^2_n \delta^2_x \delta^2_z}{1 + bS_n} + \frac{a_2 a_3 S^2_n \delta^2_y \delta^2_z}{1 + bS_n} - \frac{a_1 a_2 a_3 S^3_n \delta^2_x \delta^2_y}{(1 + bS_n)^2}\right) D^\alpha_N u^n_{irk}$$
$$- a_1 \delta^2_x u^n_{irk} - a_2 \delta^2_y u^n_{irk} - a_3 \delta^2_z u^n_{irk} + b u^n_{irk} = f^n_{irk},$$
$$(x_i, y_r, z_k) \in \Lambda_h, 1 \le n \le N, \tag{2.6}$$
$$u^n_{irk} = \varphi(x_i, y_r, z_k, t_n), \quad (x_i, y_r, z_k) \in \partial\Lambda_h, 1 \le n \le N, \tag{2.7}$$
$$u^0_{irk} = g(x_i, y_r, z_k), \quad (x_i, y_r, z_k) \in \overline{\Lambda}_h. \tag{2.8}$$

Multiplying both sides of (2.6) by S_n, we get

$$\begin{aligned}
&(I+\frac{a_1a_2S_n^2\delta_x^2\delta_y^2}{1+bS_n}+\frac{a_1a_3S_n^2\delta_x^2\delta_z^2}{1+bS_n}+\frac{a_2a_3S_n^2\delta_y^2\delta_z^2}{1+bS_n}-\frac{a_1a_2a_3S_n^3\delta_x^2\delta_y^2}{(1+bS_n)^2})u_{irk}^n\\
&-a_1S_n\delta_x^2u_{irk}^n-a_2S_n\delta_y^2u_{irk}^n-a_3S_n\delta_z^2u_{irk}^n+bS_nu_{irk}^n\\
=&S_nf_{irk}^n+S_n(I+\frac{a_1a_2S_n^2\delta_x^2\delta_y^2}{1+bS_n}+\frac{a_1a_3S_n^2\delta_x^2\delta_z^2}{1+bS_n}+\frac{a_2a_3S_n^2\delta_y^2\delta_z^2}{1+bS_n}\\
&-\frac{a_1a_2a_3S_n^3\delta_x^2\delta_y^2\delta_z^2}{(1+bS_n)^2})\left(p_{n,n}u_{irk}^0+\sum_{q=1}^{n-1}(p_{n,q}-p_{n,q+1})u_{irk}^{n-q}\right).
\end{aligned}$$

Then, rewriting Eqs. (2.6)-(2.8) in an ADI form:

$$\begin{aligned}
&(I-\frac{a_1S_n}{1+bS_n}\delta_x^2)(I-\frac{a_2S_n}{1+bS_n}\delta_y^2)(I-\frac{a_3S_n}{1+bS_n}\delta_z^2)u_{irk}^n=F_{irk}^n,\\
&(x_i,y_r,z_k)\in\Lambda_h,1\le n\le N, &(2.9)\\
&u_{irk}^0=g(x_i,y_r,z_k),\quad (x_i,y_r,z_k)\in\overline{\Lambda}_h, &(2.10)\\
&u_{irk}^n=\varphi(x_i,y_r,z_k,t_n),\quad (x_i,y_r,z_k)\in\partial\Lambda_h,1\le n\le N &(2.11)
\end{aligned}$$

in which

$$\begin{aligned}
F_{irk}^n=&\frac{S_n}{1+bS_n}(I+\frac{a_1a_2S_n^2\delta_x^2\delta_y^2}{1+bS_n}+\frac{a_1a_3S_n^2\delta_x^2\delta_z^2}{1+bS_n}+\frac{a_2a_3S_n^2\delta_y^2\delta_z^2}{1+bS_n}\\
&-\frac{a_1a_2a_3S_n^3\delta_x^2\delta_y^2\delta_z^2}{(1+bS_n)^2})\left(p_{n,n}u_{irk}^0+\sum_{q=1}^{n-1}(p_{n,q}-p_{n,q+1})u_{irk}^{n-q}\right)+\frac{S_n}{1+bS_n}f_{irk}^n.
\end{aligned}$$

Set $u_{irk}^{n-\frac{1}{3}}=(I-\frac{a_3S_n}{1+bS_n}\delta_z^2)u_{irk}^n$ and $u_{irk}^{n-\frac{2}{3}}=(I-\frac{a_2S_n}{1+bS_n}\delta_y^2)u_{irk}^n$. To get the solution $\{u_{irk}^n\}$ of problem (2.6)-(2.8), we just need to solve three of the one-dimensional equations.

First, for fixed $r\in\{1,2,\cdots,\ell_2-1\}$ and $k\in\{1,2,\cdots,\ell_3-1\}$, we can solve $\{u_{irk}^{n-\frac{2}{3}}\}$ in the x-direction as follows:

$$\begin{cases}(I-\frac{a_1S_n}{1+bS_n}\delta_x^2)u_{irk}^{n-\frac{2}{3}}=F_{irk}^n\quad\text{for}\quad 1\le i\le\ell_1-1,\\ u_{0rk}^{n-\frac{2}{3}}=(I-\frac{a_2S_n}{1+bS_n}\delta_y^2)(I-\frac{a_3S_n}{1+bS_n}\delta_z^2)u_{0rk}^n,\\ u_{\ell_1,r,k}^{n-\frac{2}{3}}=(I-\frac{a_2S_n}{1+bS_n}\delta_y^2)(I-\frac{a_3S_n}{1+bS_n}\delta_z^2)u_{\ell_1,r,k}^n.\end{cases}\tag{2.12}$$

Then, for fixed $i\in\{1,2,\cdots,\ell_1-1\}$ and $k\in\{1,2,\cdots,\ell_3-1\}$, we can solve $\{u_{irk}^{n-\frac{1}{3}}\}$ in the y-direction as follows:

$$\begin{cases}(I-\frac{a_2S_n}{1+bS_n}\delta_y^2)u_{irk}^{n-\frac{1}{3}}=u_{irk}^{n-\frac{2}{3}}\quad\text{for}\quad 1\le r\le\ell_2-1,\\ u_{i0k}^{n-\frac{1}{3}}=(I-\frac{a_3S_n}{1+bS_n}\delta_z^2)u_{i0k}^n,\quad u_{i,\ell_2,k}^{n-\frac{1}{3}}=(I-\frac{a_3S_n}{1+bS_n}\delta_z^2)u_{i,\ell_2,k}^n.\end{cases}\tag{2.13}$$

At last, for $i\in\{1,2,\cdots,\ell_1-1\}$ and $r\in\{1,2,\cdots,\ell_2-1\}$, we utilize $\{u_{irk}^{n-\frac{2}{3}},u_{irk}^{n-\frac{1}{3}}\}$ in the z-direction to solve

$$\begin{cases}(I-\frac{a_3S_n}{1+bS_n}\delta_z^2)u_{irk}^n=u_{irk}^{n-\frac{1}{3}}\quad\text{for}\quad 1\le k\le\ell_3-1,\\ u_{ir0}^n=\varphi(x_i,y_r,z_0,t_n),\quad u_{i,r,\ell_3}^n=\varphi(x_i,y_r,z_{\ell_3},t_n)\end{cases}\tag{2.14}$$

where the solution $\{u_{ijk}^n\}$ is attained.

3. Analysis of stability and convergence

For grid function $u = \{u_{irk}^n | 0 \le i \le \ell_1, 0 \le r \le \ell_2, 0 \le k \le \ell_3, 0 \le n \le N\}$, we define

$$\|u^n\|^2 = h_x \sum_{i=1}^{\ell_1-1} h_y \sum_{r=1}^{\ell_2-1} h_z \sum_{k=1}^{\ell_3-1} (u_{irk}^n)^2,$$

$$\|\delta_x u^n\|^2 = h_x \sum_{i=1}^{\ell_1} h_y \sum_{r=1}^{\ell_2-1} h_z \sum_{k=1}^{\ell_3-1} (\delta_x u_{i-\frac{1}{2},r,k}^n)^2,$$

$$\|\delta_x \delta_y u^n\|^2 = h_x \sum_{i=1}^{\ell_1} h_y \sum_{r=1}^{\ell_2} h_z \sum_{k=1}^{\ell_3-1} (\delta_x \delta_y u_{i-\frac{1}{2},r-\frac{1}{2},k}^n)^2,$$

$$\|\delta_x \delta_y \delta_z u^n\|^2 = h_x \sum_{i=1}^{\ell_1} h_y \sum_{r=1}^{\ell_2} h_z \sum_{k=1}^{\ell_3} (\delta_x \delta_y \delta_z u_{i-\frac{1}{2},r-\frac{1}{2},k-\frac{1}{2}}^n)^2,$$

$$\|\delta_x \delta_y \delta_z^2 u^n\|^2 = h_x \sum_{i=1}^{\ell_1} h_y \sum_{r=1}^{\ell_2} h_z \sum_{k=1}^{\ell_3-1} (\delta_x \delta_y \delta_z^2 u_{i-\frac{1}{2},r-\frac{1}{2},k}^n)^2,$$

$$\|\Delta_h u^n\|^2 = h_x \sum_{i=1}^{\ell_1-1} h_y \sum_{r=1}^{\ell_2-1} h_z \sum_{k=1}^{\ell_3-1} (\Delta_h u_{irk}^n)^2$$

where $\delta_x u_{i-\frac{1}{2},r,k} = \frac{1}{h_x}(u_{irk} - u_{i-1,r,k})$. The norms $\|\delta_y u^n\|$, $\|\delta_z u^n\|$, $\|\delta_y \delta_z u^n\|$, $\|\delta_x \delta_z u^n\|$, $\|\delta_x \delta_z \delta_y^2 u^n\|$ and $\|\delta_y \delta_z \delta_x^2 u^n\|$ can be defined similarly.

Set $\overline{U} = \{u_{irk} | u_{irk} = 0 \text{ if } (x_i, y_r, z_k) \in \partial\Lambda_h \text{ and } (x_i, y_r, z_k) \in \overline{\Lambda_h}\}$, for $\forall u, w \in \overline{U}$, we define

$$L_1 = \frac{a_1 S_n}{1 + bS_n}, L_2 = \frac{a_2 S_n}{1 + bS_n}, L_3 = \frac{a_3 S_n}{1 + bS_n},$$

$$\begin{aligned}
\|\nabla_h u^n\|^2 =& \|\delta_x u^n\|^2 + \|\delta_y u^n\|^2 + \|\delta_z u^n\|^2, \\
\|u^n\|_{H^1}^2 =& \|u^n\|^2 + \|\nabla_h u^n\|^2, \\
\left\|\delta_x^2 u^n \omega^n\right\|_B^2 =& (L_1 L_2 \left\|\delta_x^2 \delta_y u^n\right\| \left\|\delta_x^2 \delta_y \omega^n\right\| + L_1 L_3 \left\|\delta_x^2 \delta_z u^n\right\| \left\|\delta_x^2 \delta_z \omega^n\right\| \\
&+ L_2 L_3 \left\|\delta_x \delta_y \delta_z u^n\right\| \left\|\delta_x \delta_y \delta_z \omega^n\right\| \\
&+ L_1 L_1 L_2 L_3 \left\|\delta_x^2 \delta_y \delta_z u^n\right\| \left\|\delta_x^2 \delta_y \delta_z \omega^n\right\|), \\
\|\delta_x u^n\|_B^2 =& L_1 L_2 \left\|\delta_x^2 \delta_y u^n\right\|^2 + L_1 L_3 \left\|\delta_x^2 \delta_z u^n\right\|^2 \\
&+ L_2 L_3 \left\|\delta_x \delta_y \delta_z u^n\right\|^2 + L_1 L_2 L_3 \left\|\delta_x^2 \delta_y \delta_z u^n\right\|^2, \\
\|u^n\|_A^2 =& L_1 \|\delta_x u^n\|^2 + L_2 \left\|\delta_y u^n\right\|^2 + L_3 \|\delta_z u^n\|^2 \\
&+ L_1(1 + bS_n) \|\delta_x u^n\|_B^2 + L_2(1 + bS_n) \left\|\delta_y u^n\right\|_B^2 \\
&+ L_3(1 + bS_n) \|\delta_z u^n\|_B^2.
\end{aligned}$$

3.1. Stability analysis

This section mainly discusses the stability analysis of ADI difference scheme (2.6)-(2.8).

Lemma 3.1. *For grid functions $u, \omega \in \overline{U}$, one has*

$$-h_x h_y h_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}(u_{irk}^n + (1+bS_n)(L_1L_2\delta_x^2\delta_y^2u_{irk}^n + L_1L_3\delta_x^2\delta_z^2u_{irk}^n$$
$$+ L_2L_3\delta_y^2\delta_z^2u_{irk}^n - L_1L_2L_3\delta_x^2\delta_y^2\delta_z^2u_{irk}^n))$$
$$(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)\omega_{irk}^n \le \|u^n\|_A\|\omega^n\|_A \tag{3.1}$$

where the inequality is equal when $\omega = u$.

Proof. Using the Cauchy-Schwartz inequality, one has

$$\begin{aligned}
&- h_x h_y h_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}(u_{irk}^n + (1+bS_n)(L_1L_2\delta_x^2\delta_y^2u_{irk}^n + L_1L_3\delta_x^2\delta_z^2u_{irk}^n\\
&\quad + L_2L_3\delta_y^2\delta_z^2u_{irk}^n - L_1L_2L_3\delta_x^2\delta_y^2\delta_z^2u_{irk}^n))(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)\omega_{irk}^n\\
\le& L_1\|\delta_x u^n\|\,\|\delta_x\omega^n\| + L_1(1+bS_n)\left\|\delta_x^2u^n\omega^n\right\|_B^2 + L_2\left\|\delta_y u^n\right\|\left\|\delta_y\omega^n\right\|\\
&\quad + L_2(1+bS_n)\left\|\delta_y^2u^n\omega^n\right\|_B^2 + L_3\|\delta_z u^n\|\,\|\delta_z\omega^n\| + L_3(1+bS_n)\left\|\delta_z^2u^n\omega^n\right\|_B^2\\
\le& [L_1\|\delta_x u^n\|^2 + L_2\left\|\delta_y u^n\right\|^2 + L_3\|\delta_z u^n\|^2 + L_1(1+bS_n)\|\delta_x u^n\|_B^2\\
&\quad + L_2(1+bS_n)\left\|\delta_y u^n\right\|_B^2 + L_3(1+bS_n)\|\delta_z u^n\|_B^2]^{\frac{1}{2}}\\
&\quad [L_1\|\delta_x\omega^n\|^2 + L_2\left\|\delta_y\omega^n\right\|^2 + L_3\|\delta_z\omega^n\|^2 + L_1(1+bS_n)\|\delta_x\omega^n\|_B^2\\
&\quad + L_2(1+bS_n)\left\|\delta_y\omega^n\right\|_B^2 + L_3(1+bS_n)\|\delta_z\omega^n\|_B^2]^{\frac{1}{2}}\\
\le& \|u^n\|_A\|\omega^n\|_A.
\end{aligned}$$

The proof is completed.

For $n = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, n-1$, define a positive sequence

$$\varrho_{n,n} = 1, \varrho_{n,j} = \sum_{q=1}^{n-j}\frac{1}{p_{n,q} - p_{n,q+1}}\varrho_{n-q,j} > 0.$$

Lemma 3.2. *[37, 38] Assume that the sequences $\{m_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ are nonnegative, and assume that the grid function $\{u^n : n = 0, 1, \cdots, N\}$ satisfies $u_0 \ge 0$ and*

$$(D_N^{\alpha}u^n)u^n \le m^n u^n + (g^n)^2, \quad n = 1, 2, \cdots, N.$$

Then

$$u^n \le u^0 + S_n\sum_{j=1}^{n}\varrho_{n,j}(m_j + g_j) + \max_{1\le j\le n}\left\{g^j\right\}.$$

Lemma 3.3. *[39] When $n = 1, 2, \cdots, N$, one has*

$$S_n\sum_{j}^{n}\varrho_{n,j} \le Ct_n^{\alpha}.$$

Next, the stability theorem is presented.

Theorem 3.4. *For $n = 1, 2, \cdots, N$, the solution u_{irk}^n of ADI scheme (2.6)-(2.8) satisfies*

$$\|u^n\|_A \le \|u^0\|_A + C(t_n^\alpha + 1)\max_{1\le j\le n}\{\|f^j\|\}.$$

Proof. Multiplying both sides of $-h_xh_yh_z(L_1\delta_x^2+L_2\delta_y^2+L_3\delta_z^2)u_{irk}^n$, and summing over i, r, k for $(x_i, y_r, z_k) \in \Lambda_h$, we have

$$\begin{aligned}
&- p_{n,1}h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}[(u_{irk}^n + (1+bS_n)(L_1L_2\delta_x^2\delta_y^2u_{irk}^n + L_1L_3\delta_x^2\delta_z^2u_{irk}^n\\
&+ L_2L_3\delta_y^2\delta_z^2u_{irk}^n - L_1L_2L_3\delta_x^2\delta_y^2\delta_z^2u_{irk}^n))(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n]\\
&+ \frac{1+bS_n}{S_n}\left\|(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n\right\|^2\\
&- h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}(bu_{irk}^n)(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n\\
=&- p_{n,n}h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}[(u_{irk}^0 + (1+bS_n)(L_1L_2\delta_x^2\delta_y^2u_{irk}^0 + L_1L_3\delta_x^2\delta_z^2u_{irk}^0\\
&+ L_2L_3\delta_y^2\delta_z^2u_{irk}^0 - L_1L_2L_3\delta_x^2\delta_y^2\delta_z^2u_{irk}^0))(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n]\\
&+ \sum_{q=1}^{n-1}(p_{n,q} - p_{n,q+1})[-h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}(u_{irk}^{n-q}\\
&+ (1+bS_n)(L_1L_2\delta_x^2\delta_y^2u_{irk}^{n-q} + L_1L_3\delta_x^2\delta_z^2u_{irk}^{n-q} + L_2L_3\delta_y^2\delta_z^2u_{irk}^{n-q}\\
&- L_1L_2L_3\delta_x^2\delta_y^2\delta_z^2u_{irk}^{n-q}))(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n]\\
&- h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}f_{irk}^n(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n. \qquad (3.2)
\end{aligned}$$

Then, using the Cauchy-Schwartz inequality, Young's inequality, and Lemma 3.1, one has

$$\begin{aligned}
&p_{n,1}\|u^n\|_A^2 + \frac{1+bS_n}{S_n}\left\|(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n\right\|^2\\
&- h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}(bu_{irk}^n)(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n\\
\le& p_{n,n}\left\|u^0\right\|_A\|u^n\|_A + \sum_{q=1}^{n-1}(p_{n,q} - p_{n,q+1})\left\|u^{n-q}\right\|_A\|u^n\|_A\\
&- h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}f_{irk}^n(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n,
\end{aligned}$$

$$- h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}f_{irk}^n(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n$$

$$
\begin{aligned}
&\leq \|f^n\| \left\|(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n\right\| \\
&\leq \frac{S_n}{4(1+bS_n)} \|f^n\|^2 + \frac{1+bS_n}{S_n} \left\|(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n\right\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&- h_xh_yh_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}(bu_{irk}^n)(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)u_{irk}^n \\
=&h_xh_yh_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}(b\delta_x u_{irk}^n)(L_1\delta_x u_{irk}^n) \\
&+ h_xh_yh_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}(b\delta_y u_{irk}^n)(L_2\delta_y u_{irk}^n) \\
&+ h_xh_yh_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}(b\delta_z u_{irk}^n)(L_3\delta_z u_{irk}^n) > 0.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
p_{n,1}\|u^n\|_A^2 \leq & p_{n,n}\|u^0\|_A\|u^n\|_A + \sum_{q=1}^{n-1}(p_{n,q} - p_{n,q+1})\|u^{n-q}\|_A\|u^n\|_A \\
& + \frac{S_n}{4(1+bS_n)}\|f^n\|^2
\end{aligned}
$$

which obtains

$$
(D_N^\alpha\|u^n\|_A)\|u^n\|_A \leq C\|f^n\|^2. \tag{3.3}
$$

Finally, Lemma 3.2 and Lemma 3.3 are applied in (3.3), and the proof is completed.

3.2. H^1-norm convergence analysis

Lemma 3.5. *[24] Assume that $|\partial_t^p u(x,y,z,t)| \leq C(1+t^{\alpha-1})$, $p = 0,1,2$. Then, for $(x_i, y_r, z_k, t_n) \in \Lambda$, we have*

$$
|D_N^\alpha u_{irk}^n - D_t^\alpha u(x_i, y_r, z_k, t_n)| \leq C t_n^{-\alpha} N^{-\min\{\gamma\alpha, 2-\alpha\}}. \tag{3.4}
$$

Lemma 3.6. *[40] For grid function $v \in v_h$, one has $\| v \| \leq \frac{Ł_1Ł_2}{\sqrt{6(Ł_1^2+Ł_2^2)}} \| \nabla_h v \|$.*

Define

$$
e_{irk}^n := u(x_i, y_r, z_k, t_n) - u_{irk}^n, \quad 1 \leq n \leq N, \tag{3.5}
$$

$$
\begin{aligned}
R_t u_{irk}^n :=& (I + (1+bS_n)(L_1L_2\delta_x^2\delta_y^2 + L_1L_3\delta_x^2\delta_z^2 + L_2L_3\delta_y^2\delta_z^2 \\
& - L_1L_2L_3\delta_x^2\delta_y^2\delta_z^2))D_N^\alpha u_{irk}^n - D_t^\alpha u(x_i, y_r, z_k)
\end{aligned} \tag{3.6}
$$

and

$$
R_s u_{irk}^n \quad := \quad [a_1u_{xx} - a_2u_{yy} - a_3u_{zz}](x_i, y_r, z_k, t_n) - (a_1\delta_x^2 + a_2\delta_y^2 + a_3\delta_z^2)u_{irk}^n. \tag{3.7}
$$

According to the definition of the small term (2.5), one has

$$(1 + bS_n)(L_1L_2\delta_x^2\delta_y^2 + L_1L_3\delta_x^2\delta_z^2 + L_2L_3\delta_y^2\delta_z^2 - L_1L_2L_3\delta_x^2\delta_y^2\delta_z^2)D_N^\alpha u_{irk}^n \\ = O(N^{-2\alpha}). \tag{3.8}$$

Using a Taylor expansion, one has

$$\|R_su^n\| = O(h_x^2 + h_y^2 + h_z^2).$$

Applying Lemma 3.5 and Eq. (3.8), one has

$$\|R_tu^i\| \le C(N^{-2\alpha} + t_n^{-\alpha}N^{-\min\{\gamma\alpha,2-\alpha\}}). \tag{3.9}$$

We now show the H^1-norm convergence analysis.

Theorem 3.7. *Suppose that* $u(x, y, z, \cdot) \in C^{4,4,4}(\Lambda \cap \partial\Lambda)$, $|\partial_t^p u(x, y, z, t)| \le C(1 + t^{\alpha-1})$, $p = 0, 1, 2$. *one has*

$$\|e^n\|_{H^1} \le C(h_x^2 + h_y^2 + h_z^2 + N^{-min\{2\alpha,2-\alpha,\gamma\alpha\}}) \tag{3.10}$$

where C is an α*-robust constant that does not blow up as* $\alpha \to 1^-$.

Proof. Subtracting Eqs. (1.1)-(1.3) from Eqs. (2.6)-(2.8), the error equations are presented via

$$(I + (1 + bS_n)(L_1L_3\delta_x^2\delta_z^2 + L_2L_3\delta_y^2\delta_z^2 - L_1L_2L_3\delta_x^2\delta_y^2\delta_z^2))D_N^\alpha e_{irk}^n \\ - (a_1\delta_x^2 + a_2\delta_y^2 + a_3\delta_z^2)e_{irk}^n + be_{irk}^n = R_tu_{irk}^n + R_su_{irk}^n, \\ (x_i, y_r, z_k) \in \Lambda_h, \quad 1 \le n \le N, \tag{3.11}$$

$$e_{irk}^0 = 0, \quad (x_i, y_r, z_k) \in \overline{\Lambda}_h, \tag{3.12}$$

$$e_{irk}^n = 0, \quad (x_i, y_r, z_k) \in \partial\Lambda_h, \quad 1 \le n \le N. \tag{3.13}$$

Multiplying both sides of (3.11) by $-h_xh_yh_z(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)e_{irk}^n$ and summing over i, r, k for $(x_i, y_r, z_k) \in \Lambda_h$, one has

$$\begin{aligned} &p_{n,1}\|e^n\|_A^2 + \frac{1 + bS_n}{S_n}\left\|(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)e_{irk}^n\right\|^2 \\ &- h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}(be_{irk}^n)(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)e_{irk}^n \\ \le &p_{n,n}\left\|e^0\right\|_A\|e^n\|_A + \sum_{p=1}^{n-1}(p_{n,p} - p_{n,p+1})\left\|e^{n-p}\right\|_A\|e^n\|_A \\ &- h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}R_tu_{irk}^n(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)e_{irk}^n \\ &- h_xh_yh_z\sum_{i=1}^{\ell_1-1}\sum_{j=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1}R_su_{irk}^n(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)e_{irk}^n. \end{aligned}$$

Since

$$
\begin{aligned}
& - h_x h_y h_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1} R_t u_{irk}^n (L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2) e_{irk}^n \\
\le & \|\nabla_h R_t u^n\| \left\|(L_1\delta_x + L_2\delta_y + L_3\delta_z)e^n\right\| \\
\le & \|\nabla_h R_t u^n\| \, \|e^n\|_A ,
\end{aligned}
$$

$$
\begin{aligned}
& - h_x h_y h_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1} R_s u_{irk}^n (L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2) e_{irk}^n \\
\le & \|R_s u^n\| \left\|(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)e^n\right\| \\
\le & \frac{S_n}{4(1+bS_n)} \|R_s u^n\|^2 + \frac{1+bS_n}{S_n} \left\|(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2)e^n\right\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
& - h_x h_y h_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1} (b e_{irk}^n)(L_1\delta_x^2 + L_2\delta_y^2 + L_3\delta_z^2) e_{irk}^n \\
= & h_x h_y h_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1} (b\delta_x e_{irk}^n)(L_1\delta_x e_{irk}^n) \\
& + h_x h_y h_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1} (b\delta_y e_{irk}^n)(L_2\delta_y e_{irk}^n) \\
& + h_x h_y h_z \sum_{i=1}^{\ell_1-1}\sum_{r=1}^{\ell_2-1}\sum_{k=1}^{\ell_3-1} (b\delta_z e_{irk}^n)(L_3\delta_z e_{irk}^n) > 0.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
p_{n,1}\|e^n\|_A^2 \le & p_{n,n}\|e^0\|_A\|e^n\|_A + \sum_{q=1}^{n-1}(p_{n,q} - p_{n,q+1})\|e^{n-q}\|_A\|e^n\|_A \\
& + \|\nabla_h R_t u^n\| \cdot \|e^n\|_A + \frac{S_n}{4(1+bS_n)}\|R_s u^n\|^2.
\end{aligned}
$$

That is,

$$
(D_N^\alpha \|e^n\|_A)\|e^n\|_A \le \|\nabla_h R_t u^n\| \cdot \|e^n\|_A + \frac{S_n}{4(1+bS_n)}\|R_s u^n\|^2. \tag{3.14}
$$

Thus

$$
\nabla_h R_t u^n(ð) = \int_0^1 [\nabla_h R_t u^n(ð - sh) + \nabla_h R_t u^n(ð + sh)](1-s)ds. \tag{3.15}
$$

By the Poincaré inequality, Eq. (3.9), and Lemma 3.5, we have

$$
\|\nabla_h R_t u^n\| \le C\|R_t u^n\|_2 \le (N^{-2\alpha} + t_n^{-\alpha} N^{-\min\{\gamma\alpha, 2-\alpha\}}). \tag{3.16}
$$

From (3.6), (3.14), and (3.16), using Lemma 3.2 and Lemma 3.3, and noting that $\|e^0\|_A = 0$, one has

$$\begin{aligned}\|e^n\|_A \leq & CS_n \sum_{i=1}^{n} \varrho_{n,i}(\|\nabla_h R_t u^i\| + \frac{S_n}{4(1+bS_n)}\|R_s u^i\|) \\ & + \max_{1\leq i\leq n}\left\{\frac{S_n}{4(1+bS_n)}\|R_s u^i\|\right\} \\ \leq & CS_n \sum_{i=1}^{n} \varrho_{n,i}(h_x^2 + h_y^2 + h_z^2 + t_n^{-\alpha} N^{-min\{2-\alpha,\gamma\alpha\}} + N^{-2\alpha}) \\ & + O(h_x^2 + h_y^2 + h_z^2) \\ \leq & C\left(h_x^2 + h_y^2 + h_z^2 + N^{-min\{2\alpha,2-\alpha,\gamma\alpha\}}\right). \end{aligned} \tag{3.17}$$

Finally, using Lemma 3.6 and the definition of $\|e^n\|_A$, the proof is completed.

4. Numerical experiment

Example 4.1. For problem (1.1)-(1.3), considering $\Upsilon_x = \Upsilon_y = \Upsilon_z = \pi$, $\ell_1 = \ell_2 = \ell_3 = \ell$, $T = 1$, $\varphi(x, y, z, t) = 0$, $g(x, y, z) = 0$, and

$$\begin{aligned} f(x,y,z,t) &= \Gamma(1+\alpha)\sin x \sin y \sin z + a_1 t^\alpha \sin x \sin y \sin z \\ & \quad + a_2 t^\alpha \sin x \sin y \sin z + a_3 t^\alpha \sin x \sin y \sin z + b t^\alpha \sin x \sin y \sin z. \end{aligned}$$

The exact solution is $u(x, y, z, t) = t^\alpha \sin x \sin y \sin z$. Set $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 1$. In Tables 1–3, we fix $\ell = 128$ to test the error and convergence in time. In Table 1, choosing different α, we give errors, convergence orders, and CPU time in time for $\gamma = \frac{2-\alpha}{\alpha}$. In Table 2, set $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 100$, and choosing different α we give errors, convergence orders and CPU time in time for $\gamma = \frac{2-\alpha}{\alpha}$. In Table 2, the convergence order goes down for $N = 32$ and $\alpha = 0.2$, which can be caused by the round-off error computing the coefficients of the L1 scheme (2.1), see Remark 2.1 in [41]. In Table 3, set $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 0.001$, and choosing different α we give errors, convergence orders, and CPU time in time for$\gamma = \frac{2-\alpha}{\alpha}$. Due to the influence of multiple variable coefficient values such as a_1, a_2, a_3, b, the degree of freedom of the equation fitting is higher, and there may be some differences for CPU times. From Tables 1–3, it is obtained that the order of convergence in time is ($N^{-min\{2\alpha,2-\alpha,\gamma\alpha\}}$), which is consistent with the theoretical analysis.

In Tables 4–6, we fix $N = 512$ to present errors and convergence in space for $\gamma = (2 - \alpha)/\alpha$. In Table 4, set $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 1$, and by choosing different α we give H^1-norm errors, convergence orders and CPU time in space. In Table 5, choosing different α we give H^1-norm errors, convergence orders and CPU time in space for $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 100$. In Table 6, choosing different α we give H^1-norm errors, convergence orders, and CPU time in space for the case where $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 0.001$. In Tables 4-6, we can find that the spatial convergence order is order 2, which is consistent with our theory. From Tables 1-6 we can notice that our scheme is computationally efficient and consistent with the analysis.

Table 1. The errors, convergence orders, and CPU time for $\ell = 128$ with $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 1$ in time.

N	$\alpha = 0.2$			$\alpha = 0.4$		
	Error	Order	CPU(s)	Error	Order	CPU(s)
4	3.0652e-1	-	48.83	2.2682e-1	-	51.04
8	2.5808e-1	0.2481	143.74	1.4655e-1	0.6301	145.86
16	2.1683e-1	0.2513	458.23	9.3775e-2	0.6442	448.40
32	1.7966e-1	0.2713	1613.40	5.8933e-2	0.6701	1563.70
N	$\alpha = 0.6$			$\alpha = 0.8$		
	Error	Order	CPU(s)	Error	Order	CPU(s)
4	1.5083e-1	-	51.43	9.5021e-2	-	51.82
8	7.2715e-2	1.0526	146.19	3.8592e-2	1.2999	146.22
16	3.4023e-2	1.1057	462.02	1.5557e-2	1.3108	452.52
32	1.5515e-2	1.1329	1559.60	8.3539e-3	1.2918	1586.70

Table 2. The errors, convergence orders, and CPU time in time for $\ell = 128$ with $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 100$

N	$\alpha = 0.2$			$\alpha = 0.4$		
	Error	Order	CPU(s)	Error	Order	CPU(s)
4	1.209e-3	-	51.55	1.164e-3	-	51.88
8	5.127e-4	1.2371	152.91	4.590e-4	1.3427	150.91
16	2.715e-4	0.9173	519.34	2.067e-4	1.1510	478.29
32	1.809e-4	0.5859	1727.10	1.085e-4	0.9297	1657.20
N	$\alpha = 0.6$			$\alpha = 0.8$		
	Error	Order	CPU(s)	Error	Order	CPU(s)
4	9.694e-4	-	51.79	6.210e-4	-	52.53
8	3.808e-4	1.3480	146.84	2.589e-4	1.2623	147.80
16	1.597e-4	1.2538	476.11	1.114e-4	1.2269	474.38
32	7.104e-5	1.1686	1654.50	4.799e-5	1.2144	1628.10

Table 3. The errors, convergence orders, and CPU time for $\ell = 128$ with $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 0.001$ in time.

N	$\alpha = 0.2$			$\alpha = 0.4$		
	Error	Order	CPU(s)	Error	Order	CPU(s)
4	7.0416e-1	-	48.81	4.7668e-1	-	54.38
8	6.0163e-1	0.2270	149.75	2.9985e-1	0.6688	147.75
16	4.9789e-1	0.2731	466.34	1.8209e-1	0.7196	471.73
32	4.0077e-1	0.3131	1692.90	1.0794e-1	0.7545	1693.90
N	$\alpha = 0.6$			$\alpha = 0.8$		
	Error	Order	CPU(s)	Error	Order	CPU(s)
4	2.8199e-1	-	58.09	1.6609e-1	-	53.54
8	1.2866e-1	1.1320	148.79	6.5972e-2	1.3321	148.34
16	5.6762e-2	1.1806	470.31	2.6820e-2	1.296	468.54
32	2.4623e-2	1.2049	1678.6	1.1297e-2	1.2479	1646.70

Table 4. H^1-norm errors, convergence orders, and CPU time for $N = 512$ with $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 1$ in space.

ℓ	$\alpha = 0.6$			$\alpha = 0.8$		
	Error	Order	CPU(s)	Error	Order	CPU(s)
4	7.2337e-2	-	31.74	6.9950e-2	-	46.11
8	1.9667e-3	1.8790	174.84	1.9154e-2	1.8604	290.63
16	4.7608e-3	2.0465	1080.70	4.8763e-3	1.9738	1023.10
32	7.8250e-4	2.6051	4364.90	1.0680e-3	2.1909	4603.40

Table 5. H^1-norm errors, convergence orders, and CPU time in space for $N = 512$ with $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 100$

ℓ	$\alpha = 0.6$			$\alpha = 0.8$		
	Error	Order	CPU(s)	Error	Order	CPU(s)
4	2.509e-3	-	31.23	2.511e-3	-	30.28
8	7.034e-4	1.8347	170.04	7.064e-4	1.8296	168.96
16	1.822e-4	1.9483	820.18	1.857e-4	1.9277	814.60
32	4.262e-5	2.0963	3704.50	4.621e-5	2.0064	3738.70

Table 6. H^1-norm errors, convergence orders, and CPU time for $N = 512$ with $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.5$, and $b = 0.001$ in space.

ℓ	$\alpha = 0.6$			$\alpha = 0.8$		
	Error	Order	CPU(s)	Error	Order	CPU(s)
4	9.8959e-2	-	31.55	9.2094e-2	-	30.58
8	2.6733e-2	1.8882	170.77	2.5135e-2	1.8734	170.73
16	6.4312e-3	2.0556	825.40	6.2566e-3	2.0163	815.74
32	1.0216e-4	2.6543	3759.60	1.2250e-3	2.3526	3760.00

5. Conclusions

In this paper, an ADI scheme is proposed to solve the 3D variable coefficient subdiffusion problem, and our theoretical analysis shows that the method is unconditionally stable, its temporal convergence order is order $\min\{2-\alpha, \gamma\alpha, 2\alpha\}$, and its spatial convergence order is order 2. We present numerical results when taking different coefficients, which show that our ADI scheme is consistent with the theoretical analysis and is very efficient in solving such problems. The ADI technique proposed can reduce the computational cost from spatial discretization. In the future, we will consider some fast and parallel numerical methods [42–44] to improve computational efficiency in time.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. A. Saadatmandi, M. Dehghan, M. R. Azizi, The Sinc-Legendre collocation method for a class of fractional convection-diffusion equations with variable coefficients, *Commun. Nonlinear. Sci.*, **17** (2012), 4125–4136. https://doi.org/10.1016/j.cnsns.2012.03.003
2. J. E. Restrepo, M. Ruzhansky, D. Suragan, Explicit solutions for linear variable-coefficient fractional differential equations with respect to functions, *Appl. Math. Comput.*, **403** (2021), 126177. https://doi.org/10.1016/j.amc.2021.126177

3. M. Abdulhameed, M. M. Muhammad, A. Y. Gital, D. G. Yakubu, I. Khan, Effect of fractional derivatives on transient MHD flow and radiative heat transfer in a micro-parallel channel at high zeta potentials, *Physica A*, **519** (2019), 42–71. https://doi.org/10.1016/j.physa.2018.12.019

4. C. D. Constantinescu, J. M. Ramirez, W. R. Zhu, An application of fractional differential equations to risk theory, *Financ. Stoch.*, **23** (2019), 1001–1024. https://doi.org/10.1007/s00780-019-00400-8

5. S. E. Chidiac, M. Shafikhani, Electrical resistivity model for quantifying concrete chloride diffusion coefficient, *Cement. Concrete. Comp.*, **113** (2020), 103707. https://doi.org/10.1016/j.cemconcomp.2020.103707

6. G. H. Gao, Z. Z. Sun, A compact finite difference scheme for the fractional sub-diffusion equations, *J. Comput. Phys.*, **230** (2011), 586–595. https://doi.org/10.1016/j.jcp.2010.10.007

7. X. H. Yang, H. X. Zhang, The uniform l1 long-time behavior of time discretization for time-fractional partial differential equations with nonsmooth data, *Appl. Math. Lett.*, **124** (2022), 107644. https://doi.org/10.1016/j.aml.2021.107644

8. C. J. Li, H. X. Zhang, X. H. Yang, A high-precision Richardson extrapolation method for a class of elliptic Dirichlet boundary value calculation, *Journal of Hunan University of Technology*, **38** (2024), 91–97. doi:10.3969/j.issn.1673-9833.2024.01.013

9. X. H. Yang, L. J. Wu, H, X, Zhang, A space-time spectral order sinc-collocation method for the fourth-order nonlocal heat model arising in viscoelasticity, *Appl. Math. Comput.*, **457** (2023), 128192. https://doi.org/10.1016/j.amc.2023.128192

10. Q. Q. Tian, X. H. Yang, H. X. Zhang, D. Xu, An implicit robust numerical scheme with graded meshes for the modified Burgers model with nonlocal dynamic properties, *Comput. Appl. Math.*, **42** (2023), 246. https://doi.org/10.1007/s40314-023-02373-z

11. W. Wan, H. X. Zhang, X. X. Jiang, X. H. Yang, A high-order and efficient numerical technique for the nonlocal neutron diffusion equation representing neutron transport in a nuclear reactor, *Ann. Nucl. Energy.*, **195** (2024), 110163. https://doi.org/10.1016/j.anucene.2023.110163

12. H. X. Zhang, X. H. Yang, Q. Tang, D. Xu, A robust error analysis of the OSC method for a multi-term fourth-order sub-diffusion equation, *Comput. Math. Appl*, **109** (2022), 180–190. https://doi.org/10.1016/j.camwa.2022.01.007

13. X. H. Yang, H. X. Zhang, Q. Zhang, G. W. Yuan, Simple positivity-preserving nonlinear finite volume scheme for subdiffusion equations on general non-conforming distorted meshes, *Nonlinear. Dynam.*, **108** (2022), 3859–3886. https://doi.org/10.1007/s11071-022-07399-2

14. J. W. Wang, H. X. Zhang, X. H. Yang, A predictor-corrector compact difference scheme for a class of nonlinear Burgers equations, *Journal of Hunan University of Technology*, **38** (2024), 98–104. doi:10.3969/j.issn.1673-9833.2024.01.014

15. X. H. Yang, Z. M. Zhang, On conservative, positivity preserving, nonlinear FV scheme on distorted meshes for the multi-term nonlocal Nagumo-type equations, *Appl. Math. Lett.*, **150** (2024), 108972. https://doi.org/10.1016/j.aml.2023.108972

16. Z. F. Tian, Y. B. Ge, A fourth-order compact ADI method for solving two-dimensional unsteady convection-diffusion problems, *Appl. Math. Lett.*, **198** (2007), 268–286. https://doi.org/10.1016/j.cam.2005.12.005

17. Z. B. Wang, D. K. Cen, Y. Mo, Sharp error estimate of a compact L1-ADI scheme for the two-dimensional time-fractional integro-differential equation with singular kernels, *Appl. Numer. Math.*, **159** (2021), 190–203. https://doi.org/10.1016/j.apnum.2020.09.006

18. Z. B. Wang, Y. X. Liang, Y. Mo, A novel high order compact ADI scheme for two dimensional fractional integro-differential equations, *Appl. Numer. Math.*, **167** (2021), 257–272. https://doi.org/10.1016/j.apnum.2021.05.008

19. Q. F. Zhang, C. J. Zhang, L. Wang, The compact and Crank-Nicolson ADI schemes for two-dimensional semilinear multidelay parabolic equations, *J. Comput. Appl. Math.*, **306** (2016), 217–230. https://doi.org/10.1016/j.cam.2016.04.016

20. Q. F. Zhang, X. M. Lin, K. J. Pan, Y. Z. Ren, Linearized ADI schemes for two-dimensional space-fractional nonlinear Ginzburg-Landau equation, *Comput. Math. Appl.*, **80** (2020), 1201–1220. https://doi.org/10.1016/j.camwa.2020.05.027

21. Y. Wang, H. Chen, T. Sun, α-Robust H^1-norm convergence analysis of ADI scheme for two-dimensional time-fractional diffusion equation, *Appl. Numer. Math.*, **168** (2021), 75–83. https://doi.org/10.1016/j.apnum.2021.05.025

22. Y. Wang, B. Zhu, H. Chen, α-Robust H^1-norm convergence analysis of L1FEM-ADI scheme for 2D/3D subdiffusion equation with initial singularity, *Math. Method. Appl. Sci.*, **46** (2023), 16144–16155. https://doi.org/10.1002/mma.9442

23. D. Cao, H. Chen, Pointwise-in-time error estimate of an ADI scheme for two-dimensional multi-term subdiffusion equation, *J. Appl. Math. Comput.*, **69** (2023), 707–729. https://doi.org/10.1007/s12190-022-01759-2

24. K. Li, H. Chen, S. Xie, Error estimate of L1-ADI scheme for two-dimensional multi-term time fractional diffusion equation, *Netw. Heterog. Media.*, **18** (2023), 1454–1470. https://doi.org/10.3934/nhm.2023064

25. L. J. Qiao, D. Xu, Z. B. Wang, An ADI difference scheme based on fractional trapezoidal rule for fractional integro-differential equation with a weakly singular kernel, *Appl. Math. Comput.*, **354** (2019), 103–114. https://doi.org/10.1016/j.amc.2019.02.022

26. L. J. Qiao, J. Guo, W. L. Qiu, Fast BDF2 ADI methods for the multi-dimensional tempered fractional integrodifferential equation of parabolic type, *Comput. Math. Appl.*, **123** (2022), 89–104. https://doi.org/10.1016/j.camwa.2022.08.014

27. L. J. Qiao, D. Xu, A fast ADI orthogonal spline collocation method with graded meshes for the two-dimensional fractional integro-differential equation, *Adv. Comput. Math.*, **47** (2021), 64. https://doi.org/10.1007/s10444-021-09884-5

28. H. X. Zhang, Y. Liu, X. H. Yang, An efficient ADI difference scheme for the nonlocal evolution problem in three-dimensional space, *J. Appl. Math. Comput.*, **69** (2023), 651–674. https://doi.org/10.1007/s12190-022-01760-9

29. Z. Y. Zhou, H. X. Zhang, X. H. Yang, H^1-norm error analysis of a robust ADI method on graded mesh for three-dimensional subdiffusion problems, *Numer. Algorithms.*, (2023), 1–19. https://doi.org/10.1007/s11075-023-01676-w

30. Z. Y. Zhou, H. X. Zhang, X. H. Yang, J. Tang, An efficient ADI difference scheme for the nonlocal evolution equation with multi-term weakly singular kernels in three dimensions, *Int. J. Comput. Math.*, 2023, 1–18. https://doi.org/10.1080/00207160.2023.2212307

31. E. Ngondiep, A two-level fourth-order approach for time-fractional convection-diffusion-reaction equation with variable coefficients, *Commun. Nonlinear. Sci.*, **111** (2022), 106444. https://doi.org/10.1016/j.cnsns.2022.106444

32. Y. B. Wei, S. J. L*ü*, H. Chen, Y. M. Zhao, F. L. Wang, Convergence analysis of the anisotropic FEM for 2D time fractional variable coefficient diffusion equations on graded meshes, *Appl. Math. Lett.*, **111** (2021), 106604. https://doi.org/10.1016/j.aml.2020.106604

33. Y. H. Zeng, Z. J. Tan, Two-grid finite element methods for nonlinear time fractional variable coefficient diffusion equations, *Appl. Math. Comput.*, **434** (2022), 127408. https://doi.org/10.1016/j.amc.2022.127408

34. L. Ma, H. F. Fu, B. Y. Zhang, S. S. Xie, L1-robust analysis of a fourth-order block-centered finite difference method for two-dimensional variable-coefficient time-fractional reaction-diffusion equations, *Comput. Math. Appl.*, **148** (2023), 211–227. https://doi.org/10.1016/j.camwa.2023.08.020

35. Y. Lin, C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, *J. Comput. Phys.*, **225** (2007), 1533–1552. https://doi.org/10.1016/j.jcp.2007.02.001

36. Z. Z. Sun, X. Wu, A fully discrete difference scheme for a diffusion-wave system, *Appl. Numer. Math.*, **56** (2006), 193–209. https://doi.org/10.1016/j.apnum.2005.03.003

37. H. L. Liao, W. McLean, J. W. Zhang, A discrete Gronwall inequality with applications to numerical schemes for subdiffusion problems, *SIAM. J. Numer. Anal.*, **57** (2019), 218–237. https://doi.org/10.1137/16m1175742

38. C. B. Huang, M. Stynes, α-robust error analysis of a mixed finite element method for a time-fractional biharmonic equation, *Numer. Algorithms.*, **87** (2021), 1749–1766. https://doi.org/10.1007/s11075-020-01036-y

39. C. B. Huang, N. An, H. Chen, Optimal pointwise-in-time error analysis of a mixed finite element method for a multi-term time-fractional fourth-order equation, *Comput. Math. Appl.*, **135** (2023), 149–156. https://doi.org/10.1016/j.camwa.2023.01.028

40. Y. N. Zhang, Z. Z. Sun, Alternating direction implicit schemes for the two-dimensional fractional sub-diffusion equation, *J. Comput. Phys.*, **230** (2011), 8713–8728. https://doi.org/10.1016/j.jcp.2011.08.020

41. J, F. Zhou, X. M. Gu, Y. L. Zhao, H. Li, A fast compact difference scheme with unequal time-steps for the tempered time-fractional Black-Scholes model, *Int. J. Comput. Math.*, **2303** (2023), 10592. https://doi.org/10.1080/00207160.2023.2254412

42. J. Y. Shen, Z. Z. Sun, R. Du, Fast finite difference schemes for time-fractional diffusion equations with a weak singularity at initial time, *East Asian J. Appl. Math*, **8** (2018), 834–858. https://doi.org/10.13140/RG.2.2.18369.81767

43. Y. L. Zhao, X. M. Gu, A. Ostermann, A preconditioning technique for an all-at-once system from volterra subdiffusion equations with graded time steps, *J. Sci. Comput.*, **88** (2021), 11. https://doi.org/10.1007/s10915-021-01527-7

44. X. M. Gu, S. L. Wu, A parallel-in-time iterative algorithm for Volterra partial integro-differential problems with weakly singular kernel, *J. Sci. Comput.*, **417** (2020), 109576. https://doi.org/10.1016/j.jcp.2020.109576

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