Investigating the discrepancy property of de Bruijn sequences

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Abstract

The discrepancy of a binary string refers to the maximum (absolute) difference between the number of ones and the number of zeroes over all possible substrings of the given binary string. We provide an investigation of the discrepancy of over a dozen simple constructions of de Bruijn sequences as well as de Bruijn sequences based on linear feedback shift registers whose feedback polynomials are primitive. Furthermore, we demonstrate constructions that attain the lower bound of $\Theta(n)$ and a new construction that attains the previously known upper bound of $\Theta(\frac{2^n}{\sqrt{n}})$. This extends the work of Cooper and Heitsch [*Discrete Mathematics*, 310 (2010)].

1 Introduction

Let $\mathbf{B}(n)$ denote the set of binary strings of length n. A *de Bruijn sequence* is a circular string of length 2^n that contains every string in $\mathbf{B}(n)$ as a substring. Thus, each length-n substring must occur exactly once. As an example,

$$000000\underline{111111011110011101011}1000110110100110010110000101010001001 \tag{1}$$

is a de Bruijn sequence of order $n = 6$; it contains each length-6 binary string as a substring when viewed circularly. There is an extensive literature on de Bruijn sequences motivated in part by their random-like properties. As articulated by Golomb [19], de Bruijn sequences have the following properties:

- they are *balanced*, as they contain the same number of 0s and 1s;
- they satisfy a *run property*, as there are an equal number of contiguous runs of 0s and 1s of the same length in the sequence;
- they satisfy a *span-n property*, as they contain every distinct length n binary string as a substring.

From the example in (1) for $n = 6$, note that there are exactly 2^{n-1} 0s and 1s respectively; there are 2^{n-2} contiguous runs of 0s and 1s respectively; and by definition, it contains every distinct length n binary string as a substring.

Despite these properties, many de Bruijn sequences display other properties that are far from random. For instance, consider the greedy prefer-1 construction [24]. After starting with an initial seed, successive bits are appended by always trying a 1 first. Only if adding a 1 results in repeating a length-n substring will a 0 be appended instead. The resulting de Bruijn sequence for $n = 6$ is the one obtained in (1) by rotating the

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initial prefix of 0s to the suffix. As one would expect, it has more 1s than 0s at the start of the sequence. One measure that accounts for this is the *discrepancy*, which is informally defined to be the maximum absolute difference between the number of 0s and 1s in any substring of a given sequence.

To formally define discrepancy, we first introduce some notation. Let w be a binary string. Let $|w|_a$ denote the number of occurrences of the symbol a in w. Let $\mathbf{C}(w)$ denote the set of all substrings of w where we interpret w as a circular string. For example, taking $w = 110$ we have that $|w|_1 = 2$ and $\mathbf{C}(w) = \{\epsilon, 0, 1, 11, 10, 01, 110, 101, 011\}$. Then the *discrepancy* $\mathrm{disc}(w)$ of w is defined as

$$\mathrm{disc}(w) = \max_{u \in \mathbf{C}(w)} \Big| \, |u|_1 - |u|_0 \, \Big|.$$

The discrepancy of the sequence in (1) is $|17 - 5| = 12$ as witnessed by the underlined substring. The sequences generated by the prefer-1 construction are known to have discrepancy $\Theta(\frac{2^n \log n}{n})$ [6] with an exact formulation based on the Fibonacci and Lucas numbers [7]. In contrast, the expected discrepancy of a random sequence of length 2^n is $\Theta(2^{n/2}\sqrt{\log n})$ [6]. Letting $\mathcal{D}$ be a de Bruijn sequence of order n, $\mathrm{disc}(\mathcal{D})$ is bounded below by n since $\mathcal{D}$ must contain 1^n as a substring. In other words, $\mathrm{disc}(\mathcal{D}) \in \Omega(n)$. By putting upper bounds on character sums of non-linear recurrence sequences, an upper bound of $\mathrm{disc}(\mathcal{D}) \in O(2^n/\sqrt{n})$ can be obtained; see page 1131 in [4] for an explicit calculation. One of the main results of this paper is to build on the preliminary work by the authors [16] to demonstrate de Bruijn sequence constructions with discrepancies that obtain these asymptotic lower and upper bounds.

Some applications in pseudo-random bit generation require de Bruijn sequences that do not have large discrepancy. For example, when used as a carrier signal, a de Bruijn sequence with a large discrepancy causes spectral peaks that could interfere with devices operating at these frequencies [26]. Similar measures described as "balance" and "uniformity" are discussed in [21]. However, they focus only on $n = 2$ and instead vary the size of the alphabet. They explain that de Bruijn sequences with good balance and uniformity are useful in the planning of reaction time experiments [11, 34]. De Bruijn sequences with high discrepancy necessarily have poor balance and uniformity.

In this paper, we extend the work initiated by Cooper and Heitsch [6] providing a more complete analysis of discrepancy for a wide variety of de Bruijn sequence constructions.

1. We evaluate the discrepancies of an additional 12 simple/interesting de Bruijn sequence constructions up to $n = 30$.

2. We evaluate the discrepancies of all de Bruijn sequences obtained from linear feedback shift registers (LFSRs) based on primitive polynomials up to $n = 25$.

3. We demonstrate de Bruijn sequences constructions with discrepancy that attain the asymptotic lower bound of $\Theta(n)$.

4. We present a new de Bruijn sequence construction with discrepancy that attains the asymptotic upper bound of $\Theta\big(\frac{2^n}{\sqrt{n}}\big)$.

The remainder of this paper is presented as follows. In Section 1.1 we present background definitions and notation. In Section 1.2 we present an overview of our experimental results. They are partitioned into five groups which are further analyzed in Sections 2, 3, 4, 5, and 6. We conclude in Section 7 with open problems and future avenues of research.

1.1 Background and notation

Let $\alpha = a_1 a_2 \cdots a_n$ be a binary string in $\mathbf{B}(n)$. A *feedback function* is a function f that maps $\mathbf{B}(n)$ to $\{0,1\}$. A *feedback shift register* (FSR) is a function on $\mathbf{B}(n)$ that maps a string α to $a_2 a_3 \cdots a_n f(\alpha)$, where $f(\alpha)$ is feedback function. If f is linear, then an FSR is called a *linear feedback shift register* (LFSR). The following four "simple" LFSRs are presented on page 171 in the third edition of the classic work by Golomb [20]. We follow their notation letting the operator $\oplus$ represent addition modulo 2.

- The *pure cycling register* (PCR) has feedback function $f(\alpha) = a_1$.
- The *complemented cycling register* (CCR) has feedback function $f(\alpha) = 1 \oplus a_1$.
- The *pure summing register* (PSR) has feedback function $f(\alpha) = a_1 \oplus a_2 \oplus \cdots \oplus a_n$.
- The *complemented summing register* (CSR) has feedback function $f(\alpha) = 1 \oplus a_1 \oplus a_2 \oplus \cdots \oplus a_n$.

In addition to these four LFSRs, there is one other LFSR that relates to the de Bruijn sequences under investigation [3, 29].

- The *pure run-length register* (PRR) has feedback function $f(\alpha) = a_1 \oplus a_2 \oplus a_n$.

Let 0^n denote n copies of 0 concatenated together. When the feedback function of an LFSR is based on a primitive polynomial (discussed further in Section 6), then its corresponding LFSR produces a *maximal-length sequence* or *m-sequence*, which is a de Bruijn sequence without the 0^n substring. Adding an additional 0 before the substring $0^{n-1}1$ in an m-sequence yields a regular de Bruijn sequence.

For the remainder of the paper, when discussing a specific algorithm for constructing a de Bruin sequence we will put it in bold, e.g., the greedy **Prefer-1** construction.

1.2 The discrepancy of de Bruijn sequence constructions up to $n = 25$

In Table 1 we present exact discrepancies for 13 de Bruijn sequence constructions for values of n between 10 and 25. The results are partitioned into the following four groups based on increasing discrepancy. A larger table up to $n = 30$ is provided in the appendix.

Group 1: Constructions based on the CCR.

Group 2: Constructions based on the PRR, including the greedy prefer-same (**Prefer-same**) and prefer-opposite (**Prefer-opposite**) constructions.

Group 3: Constructions based on the PCR, including the **Prefer-1** construction. Table 1 also shows a random entry based on taking the average discrepancy of 10000 randomly generated[1] sequences of length 2^n.

Group 4: Constructions based on joining smaller weight-range cycles, including one based on the PSR/CSR.

Details about the constructions from each group are presented in their respective upcoming sections. Implementations for each of these constructions can be found at `http://debruijnsequence.org`. Each de Bruijn sequence can be generated in $O(n)$ time or better per bit using only $O(n)$ space.

[1] The sequences were generated in `C` using the `srand` and `rand` functions.

	(Group 1)				(Group 2)		
n	**Huang**	**CCR2**	**CCR3**	**CCR1**	**Pref-same**	**Lex-comp**	**Pref-opposite**
10	12	13	13	16	24	24	27
11	13	14	15	18	29	29	34
12	15	16	16	22	35	35	43
13	16	17	18	23	43	43	52
14	18	19	20	30	48	48	63
15	19	21	21	29	59	59	74
16	21	22	23	36	68	68	87
17	22	24	25	37	79	79	100
18	24	26	26	43	88	88	115
19	25	27	28	43	103	103	130
20	27	29	30	52	114	114	147
21	28	31	31	50	127	127	164
22	30	32	33	59	142	142	183
23	31	34	35	59	155	155	202
24	33	36	36	67	172	172	223
25	35	37	38	66	187	187	244

	(Group 3)					(Group 4)	
n	**PCR4**	**Random**	**PCR3**	**PCR2**	**Prefer-1/PCR1**	**Cool-lex**	**Weight-range**
10	29	50	75	101	120	131	131
11	41	71	141	180	222	257	257
12	51	101	248	321	416	468	468
13	70	143	468	587	784	801	930
14	85	203	850	1065	1488	1723	1723
15	110	288	1604	1974	2824	3439	3439
16	175	407	2965	3632	5376	6443	6443
17	246	575	5594	6785	10229	11452	12878
18	326	815	10461	12635	19484	24319	24319
19	462	1157	19765	23746	37107	48629	48629
20	730	1634	37243	44585	71250	92388	92388
21	954	2311	70575	84270	138332	167975	184766
22	1327	3264	133737	159281	268582	352727	352727
23	1820	4565	254322	302449	521553	705443	705443
24	2684	6252	484172	574819	1012795	1352090	1352090
25	3183	9192	924071	1096009	1966813	2496163	2704168

Table 1: Discrepancies of de Bruijn sequence constructions of order n ordered by increasing discrepancy and partitioned into four groups.

We also consider a fifth group of de Bruijn sequences, where each sequence corresponds to a unique primitive polynomial.

Group 5: Constructions based on primitive polynomials and their corresponding LFSRs.

By generating all primitive polynomials of degree n and their corresponding LFSRs, we compute the minimum, the maximum, and average discrepancies for their corresponding de Bruijn sequences in Table 2. Any related m-sequence known to be generated by such an LFSR can be completely determined after reading only $2n$ bits [25]. Thus, their application for generating pseudo-random numbers is limited. Further discussion is given in Section 6.

1.3 Computing the discrepancy of a de Bruijn sequence

Given a de Bruijn sequence of order n, how quickly can the discrepancy be calculated? De Bruijn sequences are exponentially long with respect to their order. Thus, it is natural to try to find a fast algorithm to compute the discrepancy of de Bruijn sequences.

	(Group 5)				
n	**min**	**avg**	**Random**	**max**	**LFSRs**
10	36	41	50	46	60
11	51	58	71	68	176
12	72	84	101	106	144
13	97	118	143	144	630
14	141	167	203	206	756
15	200	236	288	294	1800
16	280	335	407	432	2048
17	391	473	575	625	7710
18	544	669	815	860	7776
19	775	947	1157	1262	27594
20	1066	1341	1634	1842	24000
21	1500	1896	2311	2619	84672
22	2128	2681	3264	3634	120032
23	3009	3793	4565	5326	356960
24	4236	5362	6252	7545	276480
25	5905	7586	9192	11291	1296000

Table 2: The minimum, average, and maximum discrepancies of de Bruijn sequence constructions of order n based on primitive polynomials and their corresponding LFSRs. The number of LFSRs based on primitive polynomials is given in the final column.

A naïve way to calculate the discrepancy of a de Bruijn sequence $\mathcal{D}$ is to consider every substring u of $\mathcal{D}$ and compute $\Big|\, |u|_1 - |u|_0 \,\Big|$, keeping track of the maximum value. In a circular string of length m, there are m substrings of length 1, m substrings of length 2, and more generally m substrings of length j. Therefore, there are $\Theta(m^2)$ substrings in a length-m circular string. For every substring u that the algorithm visits, the absolute difference between the number of 0s and the number of 1s in u is computed, which takes $\Theta(|u|)$ time. Thus, when given a de Bruijn sequence of order n as input, this algorithm runs in $\Theta(2^{3n})$ time. A slight upgrade from this naïve approach is obtained by observing that every substring of a circular string w is a prefix of a rotation of w. For every rotation of w, we scan from left to right one bit at a time while keeping track of the number of 0s, the number of 1s, and the maximum absolute difference between them. This algorithm runs in $\Theta(2^{2n})$ time when given a de Bruijn sequence of order n as input.

We show that the discrepancy of a de Bruijn sequence of order n can be calculated in $\Theta(2^n)$ time. First, we define some notation. Let w be a binary string and let

$$d_0(w) = \max_{w=uv} \big(|u|_0 - |u|_1\big)$$

and

$$d_1(w) = \max_{w=uv} \big(|u|_1 - |u|_0\big).$$

In other words, $d_0(w)$ (resp., $d_1(w)$) denotes the maximum difference between the numbers of 0s and 1s (resp., 1s and 0s) in any prefix of w.

Lemma 1.1 *Let $\mathcal{D}$ be a de Bruijn sequence of order n. There exists a word y such that $\mathcal{D} = xyz$ and* $\operatorname{disc}(\mathcal{D}) = \Big|\, |y|_1 - |y|_0 \,\Big|$.

Proof. Suppose we cannot write $\mathcal{D} = xyz$ where $\operatorname{disc}(\mathcal{D}) = \Big|\, |y|_1 - |y|_0 \,\Big|$ for any strings x, y, z. Then we must have $\operatorname{disc}(\mathcal{D}) = \Big|\, |zx|_1 - |zx|_0 \,\Big|$ for some choice of x, z. However, since $\mathcal{D}$ is a de Bruijn sequence, it must contain the same number of 1s as 0s. Thus, $\operatorname{disc}(\mathcal{D}) = \Big|\, |zx|_1 - |zx|_0 \,\Big| = \Big|\, |y|_1 - |y|_0 \,\Big|$, which contradicts our initial assumption. □

Lemma 1.2 *Let $\mathcal{D}$ be a de Bruijn sequence of order n. Then* $\mathrm{disc}(\mathcal{D}) = d_0(\mathcal{D}) + d_1(\mathcal{D})$.

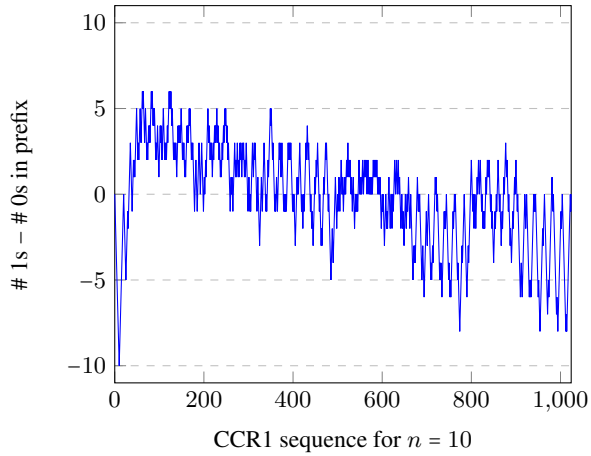
Proof. Let γ_i denote the length-i prefix of $\mathcal{D}$. Let $j_0, j_1, \ldots, j_{2^n}$ denote a sequence of integers such that $j_0 = 0$ and $j_i = |\gamma_i|_1 - |\gamma_i|_0$ for all $i \in \{1, 2, \ldots, 2^n\}$. Since $\mathcal{D}$ is a de Bruijn sequence, it has an equal number of 1s and 0s. Thus, $j_{2^n} = 0$. By Lemma 1.1, we can write $\mathcal{D} = xyz$ for strings x, y, z such that $\mathrm{disc}(\mathcal{D}) = \Big|\, |y|_1 - |y|_0 \,\Big|$. The number of 1s in y is equal to the number of 1s in xy minus the number of 1s in x. The same is true for the number of 0s in y. Therefore, $|y|_1 - |y|_0 = j_{|xy|} - j_{|x|}$. The value $\Big|\, j_{|xy|} - j_{|x|} \,\Big|$ is maximized when either $j_{|x|}$ is as large as possible and $j_{|xy|}$ is as small as possible, or when $j_{|x|}$ is as small as possible and $j_{|xy|}$ is as large as possible. In the former case, the value corresponds to $d_1(\mathcal{D}) - (-d_0(\mathcal{D}))$, and in the latter case the value corresponds to $|-d_0(\mathcal{D}) - d_1(\mathcal{D})|$. In both cases the value simplifies to $d_1(\mathcal{D}) + d_0(\mathcal{D})$. □

Corollary 1.3 *Let $\mathcal{D}$ be a de Bruijn sequence of order n. The discrepancy of $\mathcal{D}$ can be calculated in $\Theta(|\mathcal{D}|)$ time.*

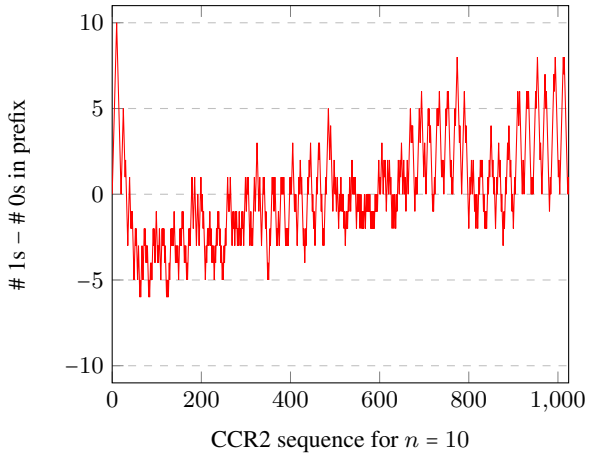
2 Group 1: CCR-based constructions

In this section we consider the four de Bruijn sequence constructions in Group 1 based on the CCR. The sequences generated by the constructions **CCR1**, **CCR2**, and **CCR3** are based on shift-rules presented in [18]. The sequences generated by the **CCR2** and **CCR3** constructions can also be constructed by concatenation approaches [17] described later in this section; the equivalence of the shift-rules to their respective concatenation constructions has been confirmed up to $n = 30$, though no formal proof has been given. The **Huang** construction is a shift-rule based construction in [22]. Since every de Bruijn sequence of order n contains the substring 0^n, a lower bound on discrepancy is clearly n. In this section we prove that two aforementioned concatenation based constructions have discrepancy at most $2n$, and thus attain the smallest possible asymptotic discrepancy of $\Theta(n)$.

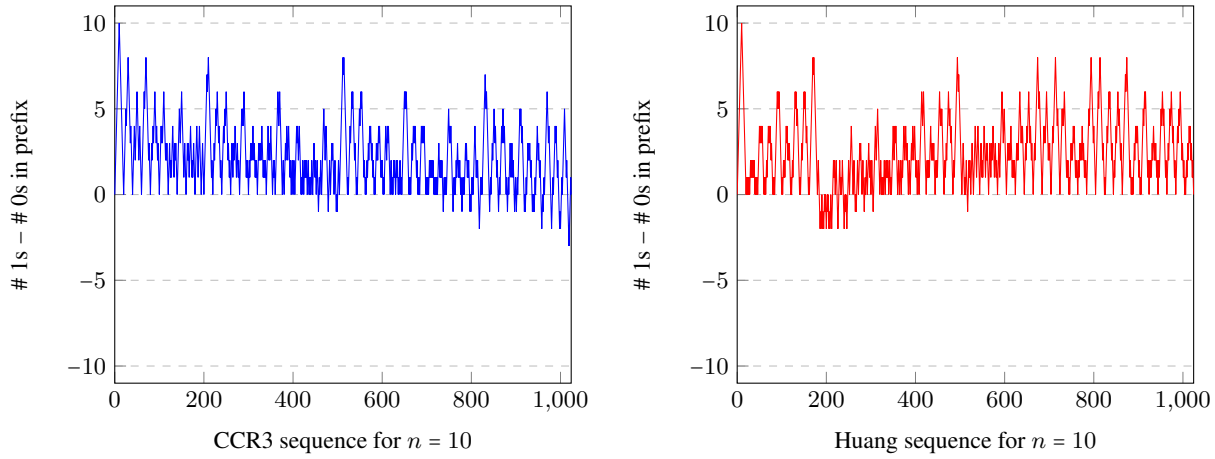
To get a better feel for these four de Bruijn sequence constructions, the following graphs illustrate the running difference between the number of 1s and the number of 0s in each prefix of the given de Bruijn sequence. The examples are for $n = 10$, so the de Bruijn sequences have length $2^{10} = 1024$.



CCR1 sequence for $n = 10$



CCR2 sequence for $n = 10$



Recall that the CCR is a feedback shift register with feedback function $f(a_1a_2\cdots a_n) = a_1 + 1 \pmod 2$. $\mathbf{B}(n)$ is partitioned into equivalence classes of strings, called *co-necklaces*, by the orbits of f. For example, the following four columns are the co-necklace equivalence classes for $n = 5$:

00000	**00010**	**00100**	**01010**
00001	00101	01001	10101
00011	01011	10011	
00111	10111	00110	
01111	01110	01101	
11111	11101	11011	
11110	11010	10110	
11100	10100	01100	
11000	01000	11001	
10000	10001	10010	

The *periodic reduction* of a string α, denoted by $\mathrm{pr}(\alpha)$, is the shortest prefix β of α such that $\alpha = \beta^j$ for some $j \geq 1$. In [17], the following two de Bruijn sequence constructions **CCR2** and **CCR3** concatenate the periodic reductions of $\alpha\overline{\alpha}$ for given representatives α of each co-necklace equivalence class.

CCR2

1. Let the representative for each co-necklace equivalence class of order n be its lexicographically smallest string.
2. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ denote these representatives in colex order.
3. **Output**: $\mathrm{pr}(\alpha_1\overline{\alpha_1}) \cdot \mathrm{pr}(\alpha_2\overline{\alpha_2}) \cdots \mathrm{pr}(\alpha_m\overline{\alpha_m})$.

For $n = 5$, the representatives for this algorithm are the bolded strings in the equivalence classes above and **CCR2** produces:

$$0000011111 \cdot 0010011011 \cdot 0001011101 \cdot 01.$$

CCR3

1. Let the representative for each co-necklace equivalence class of order n be the string obtained by taking the lexicographically smallest string, removing its largest prefix of the form 0^j, and then appending 1^j to the end.
2. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ denote these representatives in lexicographic order.
3. **Output**: $\mathrm{pr}(\alpha_1\overline{\alpha_1}) \cdot \mathrm{pr}(\alpha_2\overline{\alpha_2}) \cdots \mathrm{pr}(\alpha_m\overline{\alpha_m})$.

For $n = 5$, the representatives for this algorithm are the underlined strings in the equivalence classes above and **CCR3** produces:

$$1001101100 \cdot 10 \cdot 1011101000 \cdot 1111100000.$$

We now prove that the discrepancy resulting from these two de Bruijn sequence constructions is at most $2n$.

Lemma 2.1 *Consider a sequence of binary strings $\alpha_1, \alpha_2, \ldots, \alpha_m$ where each α_i has the same number of 0s as 1s and has discrepancy at most n. Then $\mathrm{disc}(\alpha_1\alpha_2\cdots\alpha_m) \leq 2n$.*

Proof. Let $\mathcal{S} = \alpha_1\alpha_2\cdots\alpha_m$. By Lemma 1.1 there exists a shortest substring $y = u\alpha_{i+1}\alpha_{i+2}\cdots\alpha_{j-1}v$ of $\mathcal{S}$ such that $\mathrm{disc}(\mathcal{S}) = \Big|\, |y|_1 - |y|_0 \,\Big|$ where u is a suffix of α_i and v is a prefix of α_j. Since the number of 0s and 1s is the same in each α_k and $\mathrm{disc}(\alpha_k) \leq n$, $\Big|\, |y|_1 - |y|_0 \,\Big| = \Big|\, |uv|_1 - |uv|_0 \,\Big| \leq 2n$. □

Theorem 2.2 *The de Bruijn sequences constructed by* **CCR2** *and* **CCR3** *have discrepancy at most $2n$.*

Proof. Given a length n binary string α, $\alpha\overline{\alpha}$ has the same number of 0s and 1s and has discrepancy at most n. These properties also hold for $\mathrm{pr}(\alpha\overline{\alpha})$ by definition of the periodic reduction. Thus, by Lemma 2.1, the sequences constructed by **CCR2** and **CCR3** have discrepancy at most $2n$. □

Interestingly, from Table 1, these two concatenation-based constructions do not demonstrate the smallest discrepancy for $n \leq 30$. The construction by Huang [22], which is based on a cycle-joining approach, demonstrates slightly smaller discrepancy. In particular the author states:

> *"It seems clear that the sequences produced by our algorithm have a relatively good characteristic of local 0-1 balance in comparison with the ones produced by the 'prefer one' algorithm."*

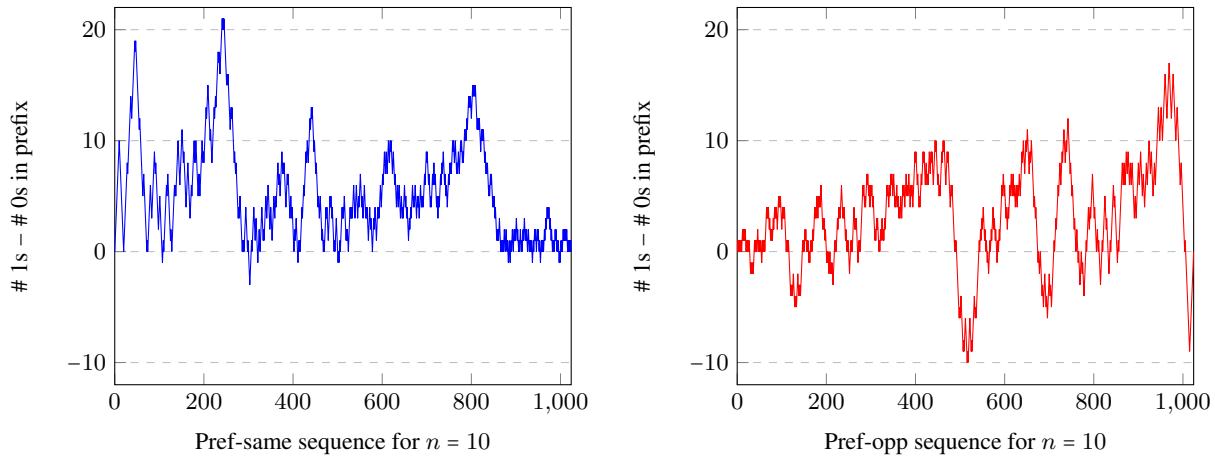
So the author indicates that their construction may have small discrepancy, however no analysis is provided.

3 Group 2: PRR-based constructions

In this section we consider the three de Bruijn sequence constructions in Group 2 based on the PRR. The **Pref-same** [3, 10, 13] and the **Pref-opposite** [2] are greedy constructions based on the last bit of the sequence as it is constructed. They have the downside of requiring an exponential amount of memory. The **Lex-comp** construction is obtained by concatenating lexicographic compositions. It was an attempt to generate

the sequence generated by the **Pref-same** construction without using exponential space and it the resulting sequences were conjectured to be the same for a very long prefix [14]. In fact, it attains the same discrepancy as the **Pref-same** for all values of n tested. Recently, it was demonstrated that the **Pref-same** and the **Pref-opposite** sequences can be generated in $O(n)$ time per bit using only $O(n)$ space by applying the PRR [29]. There is also a PRR-based construction that produces an equivalent sequence as **Lex-comp** for large n, but there is no formal proof showing they are equivalent.

To get a better feel for the two greedy de Bruijn sequence constructions, the following graphs illustrate the running difference between the number of 1s and the number of 0s in each prefix of the given de Bruijn sequence. The examples are for $n = 10$, so the de Bruijn sequences have length $2^{10} = 1024$.



In the following table we study some experimental results for the **Pref-same** construction. In particular, for $10 \le n \le 25$ we compute the maximum difference between the number of 1s and the number of 0s along with the maximum difference between the number 0s and the number of 1s, over all prefixes of each **Pref-same** de Bruijn sequence of order n. Adding these two values together, we get the discrepancies shown in Table 1.

n	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
max(#1s − #0s)	21	26	31	36	43	50	57	64	73	82	91	100	111	122	133	144
max(#0s − #1s)	3	3	4	7	5	9	11	15	15	21	23	27	31	33	39	43
discrepancy	24	29	35	43	48	59	68	79	88	103	114	127	142	155	172	187

Interestingly, the values in the row max(#1s − #0s) are equivalent to the known sequence A008811 in the Online Encyclopedia of Integer Sequences (OEIS) [1] offset by four positions. The sequence enumerates the "Expansion of $x(1+x^4)/((1-x)^2(1-x^4))$" and the provided formula demonstrates that each value is $\Theta(n^2)$. More specifically the values match the sequence for $6 \le n \le 30$, though we have no intuition as to why this is the case. This leads to the following conjecture.

Conjecture 3.1 *The de Bruijn sequences constructed by the* **Pref-same** *and* **Lex-comp** *algorithms have discrepancy* $\Theta(n^2)$.

A similar analysis was performed for sequences generated by the **Pref-opposite** construction.

n	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
max(#1s − #0s)	10	13	17	21	26	31	37	43	50	57	65	73	82	91	101	111
max(#0s − #1s)	17	21	26	31	37	43	50	57	65	73	82	91	101	111	122	133
discrepancy	27	34	43	52	63	74	87	100	115	130	147	164	183	202	223	244

Remarkably, observe that the two middle rows are a shift from each other by two positions. Just as interesting, the sequences also correspond to a known sequence in OEIS [1], namely A033638. Specifically, the row max(#1s− #0s) corresponds to this sequence shifted by four positions. The sequence does not match for $n < 10$, but we have verified it matches for $10 \le n \le 30$. The sequence corresponds to "quarter squares plus 1", and by applying the appropriate shifts, the discrepancy for the **Prefer-opposite** sequence of order n, for $10 \le n \le 30$ is given by:

$$\left\lfloor \frac{(n-4)^2}{4} \right\rfloor + \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 2.$$

This leads to the following conjecture.

Conjecture 3.2 *The de Bruijn sequence constructed by the* **Pref-opposite** *algorithm has discrepancy* $\Theta(n^2)$.

We conclude this section with an observation regarding the **Pref-opposite** de Bruijn sequence: For $2 \le n \le 25$, each sequence has the following suffix where $j = \lceil n/3 \rceil$:

$$0^j 1^{n-j} \cdot 0^{j-1} 1^{n-j+1} \cdots 01^{n-1} \cdot 10^{n-1}.$$

For example, when $n = 10$, the **Pref-opposite** de Bruijn sequence has suffix

$$0000001111 \cdot 0000\underline{11111 \cdot 0000111111 \cdot 0001111111 \cdot 0011111111 \cdot 0111111111 \cdot 1}000000000,$$

and the underlined substring has $5+6+7+8+10$ ones and $4+3+2+1$ zeros. A slight rearrangement gives a lower bound of $(5-1)+(6-2)+(7-3)+(8-4)+10 = 4 \cdot 4 + 10 = 26$ for the discrepancy of the sequence. The actual discrepancy is 27. More generally, if this suffix is indeed a suffix for each **Pref-opposite** de Bruijn sequence, then a lower bound on its discrepancy will be

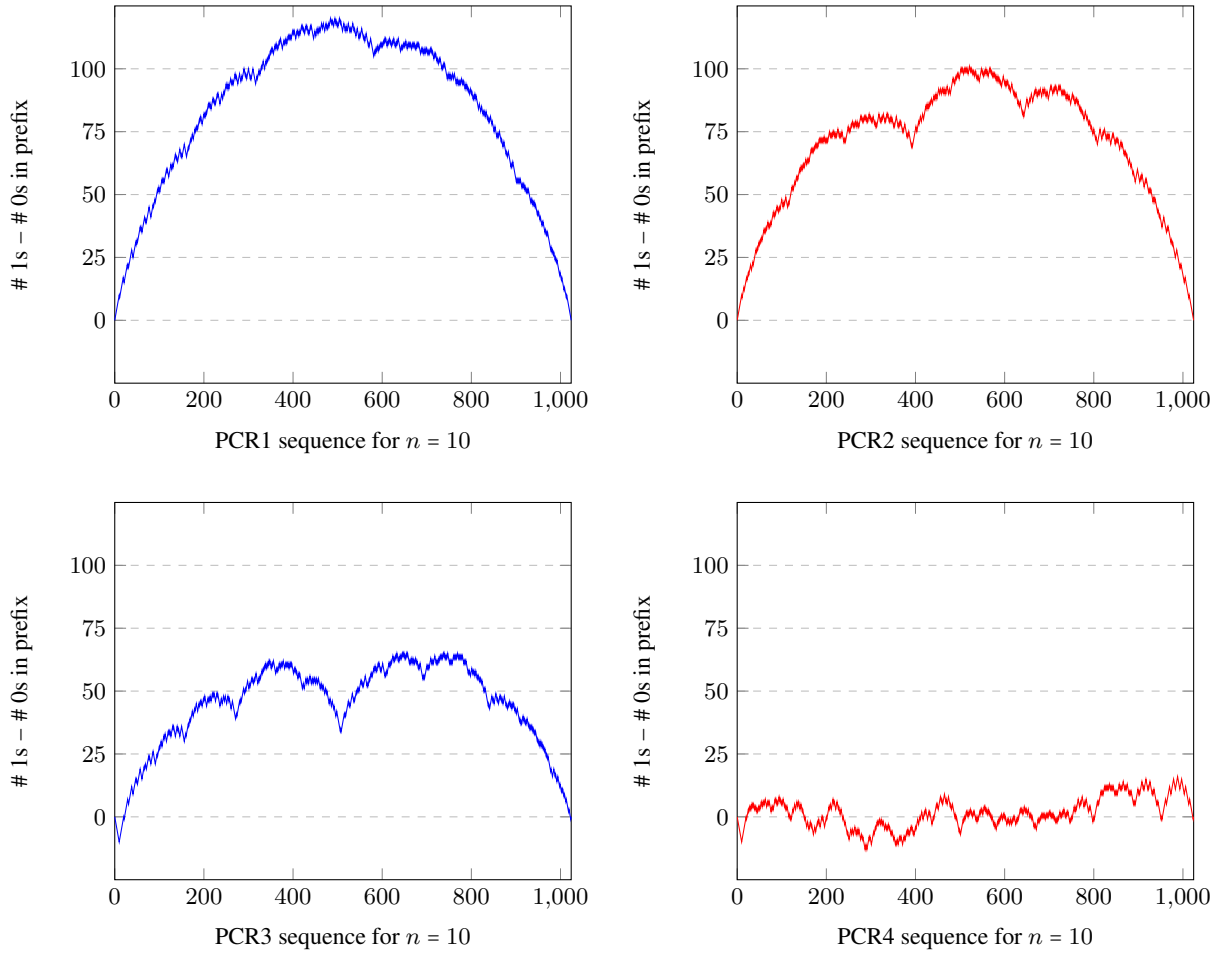
$$(\lceil n/2 \rceil - 1)(\lfloor n/2 \rfloor - 1) + n \;=\; \Omega(n^2).$$

4 Group 3: PCR-based constructions

In this section we consider the four de Bruijn sequence constructions in Group 3 based on the PCR. The constructions **PCR1**, **PCR2**, **PCR3**, and **PCR4** are based on shift-rules presented in [18]. Like the other shift-rule constructions, these four rules result from joining smaller cycles based on the underlying feedback function; depending on how the "bridge states" are defined leads to the different shift-rules. The sequences generated by **PCR1** are the same as the ones generated by the prefer-0 greedy construction; they are the complements of the sequences generated by **PCR1**, and so they have the same. The sequences generated by **PCR1** can also be generated by two necklace concatenation constructions, one based on lexicographic order [15], and another taking a recursive approach [27].

The sequences generated by **PCR2** are the same as the ones generated by a necklace concatenation construction based on colex order [8, 9]. The **PCR3** is based on a general approach in [23] and revisited in [33].

To get a better feel for these four de Bruijn sequence constructions, the following graphs illustrate the running difference between the number of 1s and the number of 0s in each prefix of the given de Bruijn sequence. The examples are for $n = 10$, so the de Bruijn sequences have length $2^{10} = 1024$.

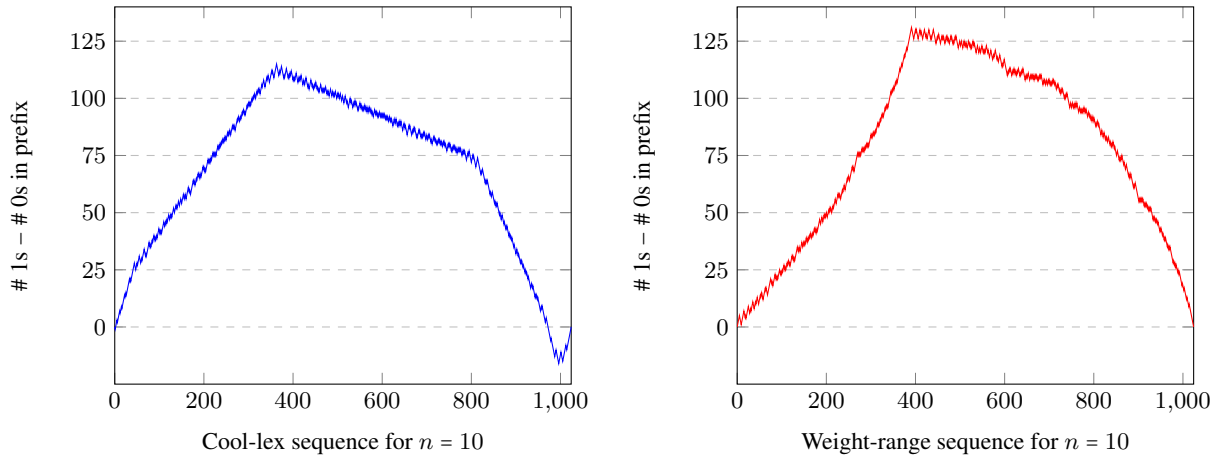


The discrepancy for the sequence generated by the **PCR1** construction has already been studied in [6] where they show that the discrepancy is $\Theta(\frac{2^n \log n}{n})$. The sequences generated by the **PCR2** and **PCR3** constructions appear to have a similar growth trajectories. More interesting are the sequences generated by the **PCR4** construction that, from Table 1, appear to have discrepancy that is closest to that of a random string. It would be interesting to do a more detailed investigation of this construction, which is based on a very simple successor rule.

5 Group 4: Weight range constructions and the PSR/CSR

In this section we consider two de Bruijn sequence constructions that join smaller cycles based on weight (number of 1s). In some related works the term density is also used to mean weight, so will use the variable d to indicate a weight. The **Cool-lex** construction [28], is a concatenation approach which is based on creating underlying cycles which contain all strings with weights d and $d+1$ given $0 \leq d < n$. Then, appropriate such cycles can be joined together to obtain a de Bruijn sequence [31]. By the nature of how the cycles are joined, most length-n substrings in the first half of the resulting de Bruijn sequence have weight less than or equal to $n/2$. Similarly, most length-n substrings in the latter half of the sequence have weight greater than or equal to $n/2$. Thus, as one would expect, the resulting de Bruijn sequence has a very large discrepancy. The **Weight-range** construction is a new construction presented in this section. Its resulting de Bruijn sequence has discrepancy that attains the asymptotic upper bound of $\Theta(2^n/\sqrt{n})$.

To get a better feel for these two de Bruijn sequence constructions, the following graphs illustrate the running difference between the number of 1s and the number of 0s in each prefix of the given de Bruijn sequence. The examples are for $n = 10$, so the de Bruijn sequences have length $2^{10} = 1024$.



Notice that if we had shifted the starting position of the **Cool-lex** sequence the profile of the graph would be very similar to that the **Weight-range** sequence. In fact, the discrepancies of the two sequences are the same except when $n \bmod 4 \equiv 1$ (see Table 1). This will be discussed more after we present the **Weight-range** construction.

A *minimum weight de Bruijn sequence* is a cyclic sequence that contains each binary string of length n with weight at least d exactly once. A *maximum weight de Bruijn sequence* is defined similarly where the weight of each string is at most d. A construction for the former sequence is given in [32]; it is constructed by concatenating the periodic reduction of each necklace of weight $\geq d$ when the necklaces are listed in lexicographic order. Let the resulting sequence be denoted by $\mathcal{D}_d(n)$.

Remark 5.1 *For any $d < n$, $\mathcal{D}_d(n)$ begins with $0^{n-d}1^d$ and ends with 1^n.*

By complementing the bits in $\mathcal{D}_d(n)$, we obtain a maximum weight de Bruijn sequence with weight at most $n - d$. Denote this sequence by $\overline{\mathcal{D}}_d(n)$. From the previous remark, it begins with $1^{n-d}0^d$ and ends with 0^n.

Example 1 The necklaces of length 6 with weight $d \geq 3$ in lexicographic order are:

$$000111, 001011, 001101, 001111, 010101, 010111, 011011, 011111, 111111.$$

Concatenating together their periodic reductions we obtain the minimum weight de Bruijn sequence $\mathcal{D}_3(6)$.

$$000111 \cdot 001011 \cdot 001101 \cdot 001111 \cdot 01 \cdot 010111 \cdot 011 \cdot 011111 \cdot 1$$

As further examples,

$$\mathcal{D}_4(6) = 001111 \cdot 010111 \cdot 011 \cdot 011111 \cdot 1$$

and

$$\overline{\mathcal{D}}_4(6) = 110000 \cdot 101000 \cdot 100 \cdot 100000 \cdot 0.$$

From the above example observe that:

- $\mathcal{D}_3(6)$ contains all binary strings of length 6 with weight greater than or equal to 3,
- $\overline{\mathcal{D}}_4(6)$ contains all binary strings of length 6 with weight less than or equal to 2,
- The length $n-1$ prefix of $\overline{\mathcal{D}}_4(6)$, namely 11000, appears in the wraparound of $\mathcal{D}_3(6)$.

Let $\mathcal{D}^r_d(n)$ denote the sequence $\mathcal{D}_d(n)$ with the suffix 1^{d-1} rotated to the front. Then by applying the Gluing Lemma [31], the following is a de Bruijn sequence of order 6:

$$\underbrace{11000010100001001000000}_{\overline{\mathcal{D}}_4(6)} \cdot \underbrace{11000111001011001101001111010101110110111 1}_{\mathcal{D}^r_3(6)}.$$

Applying this strategy more generally, let $\mathcal{DB}_{max}(n)$ denote the de Bruijn sequence obtained by joining two such smaller cycles.

Weight-range construction

$$\mathcal{DB}_{max}(n) = \overline{\mathcal{D}}_d(n) \cdot \mathcal{D}^r_{d'}(n),$$

where $d = \lfloor n/2 \rfloor + 1$ and $d' = \lceil n/2 \rceil$.

A complete C implementation to construct $\mathcal{DB}_{max}(n)$ is given in the Appendix[2].

The following technical lemma leads to a lower bound for the discrepancy of $\mathcal{DB}_{max}(n)$.

Lemma 5.2 *A maximum weight de Bruijn sequence of order n and maximum weight d has $\binom{n-1}{d}$ more 0s than 1s.*

Proof. By definition, a maximum weight de Bruijn sequence of order n and maximum weight d contains every binary string of length n with weight at most d as a substring exactly once. Since each bit in this sequence belongs to n different strings the total number of 1s in the sequence is

$$\begin{aligned} ones &= \frac{1}{n}\sum_{j=0}^{d} j\binom{n}{j} \\ &= \frac{0}{n}\binom{n}{0} + \frac{1}{n}\binom{n}{1} + \frac{2}{n}\binom{n}{2} + \cdots + \frac{d}{n}\binom{n}{d} \\ &= 0 + \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{d-1}, \end{aligned}$$

and the total number of 0s is

$$\begin{aligned} zeros &= \frac{1}{n}\sum_{j=0}^{d} (n-j)\binom{n}{j} \\ &= \frac{n}{n}\binom{n}{0} + \frac{n-1}{n}\binom{n}{1} + \frac{n-2}{n}\binom{n}{2} + \cdots + \frac{n-d}{n}\binom{n}{d} \\ &= \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{d}. \end{aligned}$$

Thus $zeros - ones = \binom{n-1}{d}$. □

[2]It is also available at `http://debruijnsequence.org`.

Theorem 5.3 *The de Bruijn sequence $\mathcal{DB}_{max}(n)$ has discrepancy at least $\binom{n-1}{\lfloor n/2 \rfloor} + \lfloor \frac{n}{2} \rfloor$.*

Proof. Let $d = \lfloor n/2 \rfloor + 1$ and $d' = \lceil n/2 \rceil$. Recall that $\overline{\mathcal{D}}_d(n)$ is a maximum weight de Bruijn sequence with maximum weight $n - d$. Thus, by Lemma 5.2, it has $\binom{n-1}{n-d} = \binom{n-1}{n-(\lfloor n/2 \rfloor + 1)} = \binom{n-1}{\lfloor n/2 \rfloor}$ more 0s than 1s. Consider $\overline{\mathcal{D}}_d(n)$ with its prefix of 1^{n-d} removed. The resulting string, which is a substring of $\mathcal{DB}_{max}(n)$, has $\binom{n-1}{\lfloor n/2 \rfloor} + (n - d)$ more 0s than 1s. When n is odd we have $n - d = n - \lfloor n/2 \rfloor - 1 = \lfloor \frac{n}{2} \rfloor$ and thus $\mathcal{DB}_{max}(n)$ has discrepancy at least $\binom{n-1}{\lfloor n/2 \rfloor} + \lfloor \frac{n}{2} \rfloor$. When n is even, we additionally add the length $n - 1$ prefix of $\mathcal{D}^r_{d'}(n)$ which has more 0s than 1s (exactly one more). Since $n - d + 1 = n - (\lfloor n/2 \rfloor - 1) + 1 = \lfloor \frac{n}{2} \rfloor$ (when n is even) this again means that $\mathcal{DB}_{max}(n)$ has discrepancy at least $\binom{n-1}{\lfloor n/2 \rfloor} + \lfloor \frac{n}{2} \rfloor$. □

By applying Stirling's approximation to $\binom{n-1}{\lfloor n/2 \rfloor}$ we obtain the following corollary.

Corollary 5.4 *The discrepancy of the de Bruijn sequence $\mathcal{DB}_{max}(n)$ attains the asymptotic upper bound of $\Theta(\frac{2^n}{\sqrt{n}})$.*

Observe from Table 1 that the discrepancy of $\mathcal{DB}_{max}(n)$ is exactly $\binom{n-1}{\lfloor n/2 \rfloor} + \lfloor \frac{n}{2} \rfloor$ for $10 \leq n \leq 25$. This leads to the following conjecture.

Conjecture 5.5 *The de Bruijn sequence $\mathcal{DB}_{max}(n)$ has discrepancy equal to $\binom{n-1}{\lfloor n/2 \rfloor} + \lfloor \frac{n}{2} \rfloor$, and moreover, it is the maximum possible discrepancy over all de Bruijn sequences of order n.*

As noted earlier, the discrepancy of the **Cool-lex** construction matches the discrepancy for the **Weight-range** construction for $10 \leq n \leq 25$, except for when $n \bmod 4 \equiv 1$ (see Table 1). As illustration, the **Cool-lex** construction first constructs cycles of the following weights for $n = 6, 7, 8, 9$:

- $n = 6$: (0,1,2), (3,4), (5,6)
- $n = 7$: (0,1), (2,3), (4,5), (6,7)
- $n = 8$: (0,1,2), (3,4), (5,6), (7,8)
- $n = 9$: (0,1), (2,3), (4,5), (6,7), (8,9)

before joining them together one at a time. Note when $n = 9$, strings with weights 4 and 5 are grouped together before the smaller cycles are joined together. This causes a reduction in the discrepancy compared to the **Weight-range** construction. It is possible, however, to tweak the **Cool-lex** implementation so the discrepancies are equivalent. For instance for $n = 9$, the smaller cycles with weights $(0, 1, 2), (3, 4), (5, 6), (7, 8, 9)$ could be joined together instead.

Recently, a shift-rule construction based on the PSR and CSR has been discovered to generate the same sequence as **Cool-lex** [30]. A discussion on generating de Bruijn sequences applying the PSR and CSR is also given in [12]; it describes joining small cycles together in the same manner as **Cool-lex**. Thus, we anticipate the resulting sequences would obtain a similar discrepancy profile.

6 Group 5: LFSR constructions based on primitive polynomials

In this section we consider de Bruijn sequences that can be generated for a specific n by a primitive polynomial of degree n. As discussed by Golomb [20], a primitive polynomial of the form $g(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ over GF(2) corresponds to a feedback function of the form $f(a_1 a_2 \cdots a_n) = c_n a_1 \oplus c_{n-1} a_2 \oplus \cdots \oplus c_1 a_n$.

Example 2 The primitive polynomial $1 + x^2 + x^5$ of degree 5 over GF(2) corresponds to the feedback function $f(a_1a_2a_3a_4a_5) = a_1 + a_4$. If $a_1a_2a_3a_4a_5$ is initialized to 00001, then the LFSR with this feedback function produces the m-sequence

$$0000101011101100011111001101001$$

of length $2^5 - 1 = 31$ when outputting the value a_1 before each application of the LFSR. By prepending a 0 to the beginning of this m-sequence we obtain a de Bruijn sequence.

To obtain the data in Table 2, we generated all primitive polynomials of degree n for $n = 10, 11, \ldots, 25$ along with their corresponding LFSRs. The algorithm used to exhaustively list the primitive polynomials is based on the work in [5] and is available at `http://debruijnsequence.org/lfsr`. We seeded the LFSRs with $0^{n-1}1$ as described in the above example to obtain a de Bruijn sequence. We then computed the discrepancy of all such sequences. The number of primitive polynomials (and hence LFSRs) of degree n is given by sequence A011260 in the On-Line Encyclopedia of Integer Sequences [1]. This number is listed in the final column of Table 2.

Below is a list of feedback functions for $n = 10, 11, \ldots, 25$ that generated de Bruin sequences with discrepancy closest to the corresponding entry for a random sequence.

n	**Feedback function**	**Random**	**Discrepancy**
10	$a_1 \oplus a_2 \oplus a_6 \oplus a_9$	50	46
11	$a_1 \oplus a_6 \oplus a_7 \oplus a_{10}$	71	68
12	$a_1 \oplus a_4 \oplus a_7 \oplus a_8 \oplus a_9 \oplus a_{12}$	101	99
13	$a_1 \oplus a_2 \oplus a_4 \oplus a_5 \oplus a_6 \oplus a_7 \oplus a_8 \oplus a_{10} \oplus a_{11} \oplus a_{12}$	143	143
14	$a_1 \oplus a_2 \oplus a_4 \oplus a_5 \oplus a_7 \oplus a_8 \oplus a_9 \oplus a_{13}$	203	203
15	$a_1 \oplus a_2 \oplus a_6 \oplus a_{11}$	288	287
16	$a_1 \oplus a_2 \oplus a_3 \oplus a_4 \oplus a_{13} \oplus a_{15}$	407	406
17	$a_1 \oplus a_3 \oplus a_5 \oplus a_8 \oplus a_9 \oplus a_{10} \oplus a_{11} \oplus a_{12} \oplus a_{13} \oplus a_{15} \oplus a_{16} \oplus a_{17}$	575	575
18	$a_1 \oplus a_4 \oplus a_5 \oplus a_7 \oplus a_{10} \oplus a_{11} \oplus a_{12} \oplus a_{15} \oplus a_{16} \oplus a_{17}$	815	814
19	$a_1 \oplus a_2 \oplus a_3 \oplus a_5 \oplus a_6 \oplus a_{10} \oplus a_{11} \oplus a_{14} \oplus a_{16} \oplus a_{17}$	1157	1157
20	$a_1 \oplus a_2 \oplus a_3 \oplus a_4 \oplus a_5 \oplus a_6 \oplus a_7 \oplus a_8 \oplus a_9 \oplus a_{10} \oplus a_{11} \oplus a_{12} \oplus a_{13} \oplus a_{15} \oplus a_{16} \oplus a_{17} \oplus a_{19} \oplus a_{20}$	1634	1633
21	$a_1 \oplus a_2 \oplus a_3 \oplus a_5 \oplus a_8 \oplus a_{11} \oplus a_{16} \oplus a_{19} \oplus a_{20} \oplus a_{21}$	2311	2311
22	$a_1 \oplus a_3 \oplus a_6 \oplus a_{12} \oplus a_{13} \oplus a_{15} \oplus a_{17} \oplus a_{18} \oplus a_{19} \oplus a_{20} \oplus a_{21} \oplus a_{22}$	3264	3264
23	$a_1 \oplus a_2 \oplus a_3 \oplus a_4 \oplus a_5 \oplus a_6 \oplus a_9 \oplus a_{10} \oplus a_{12} \oplus a_{14} \oplus a_{16} \oplus a_{17} \oplus a_{18} \oplus a_{23}$	4565	4565
24	$a_1 \oplus a_2 \oplus a_4 \oplus a_6 \oplus a_{12} \oplus a_{13} \oplus a_{14} \oplus a_{15} \oplus a_{16} \oplus a_{17} \oplus a_{18} \oplus a_{20} \oplus a_{21} \oplus a_{24}$	6252	6252
25	$a_1 \oplus a_8 \oplus a_9 \oplus a_{11} \oplus a_{12} \oplus a_{16} \oplus a_{20} \oplus a_{21} \oplus a_{24} \oplus a_{25}$	9192	9192

7 Future directions and open problems

In this paper, we investigated the discrepancies of 13 de Bruijn sequence constructions. We proved that two constructions attain the lower bound of $\Theta(n)$ and presented one new construction that attains the upper bound of $\Theta(\frac{2^n}{\sqrt{n}})$. It remains an interesting problem to demonstrate a generic construction for all n with discrepancy that is close to that of a random stream of bits of the same length. Some additional avenues of future research include the following.

1. Simplify the description of the **Huang** construction [22]. Does it have the smallest discrepancy over all de Bruijn sequences?

2. Answer the conjectures regarding the discrepancies for the greedy **Pref-same** and **Pref-opposite** constructions (Conjecture 3.1 and Conjecture 3.2).

3. Analyze the discrepancy of **PCR4** which had discrepancy closest to one we might expect from a random stream of bits.

4. Determine whether or not the maximum discrepancy of any de Bruijn sequence is $\binom{n-1}{\lfloor n/2 \rfloor} + \lfloor \frac{n}{2} \rfloor$ (Conjecture 5.5).

5. Generalize the investigation of discrepancy to de Bruijn sequences over an arbitrary alphabet size k.

6. Study the distribution of discrepancy over all possible de Bruijn sequences.

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A Table of discrepancies

	(Group 1)				(Group 2)		
n	**Huang**	**CCR2**	**CCR3**	**CCR1**	**Pref-same**	**Lex-comp**	**Pref-opposite**
10	12	13	13	16	24	24	27
11	13	14	15	18	29	29	34
12	15	16	16	22	35	35	43
13	16	17	18	23	43	43	52
14	18	19	20	30	48	48	63
15	19	21	21	29	59	59	74
16	21	22	23	36	68	68	87
17	22	24	25	37	79	79	100
18	24	26	26	43	88	88	115
19	25	27	28	43	103	103	130
20	27	29	30	52	114	114	147
21	28	31	31	50	127	127	164
22	30	32	33	59	142	142	183
23	31	34	35	59	155	155	202
24	33	36	36	67	172	172	223
25	35	37	38	66	187	187	244
26	36	39	40	77	208	208	267
27	38	41	42	74	224	224	290
28	40	43	43	85	246	246	315
29	41	44	45	84	264	264	340
30	43	46	47	94	286	286	367

			(Group 3)			(Group 4)	
n	**PCR4**	**Random**	**PCR3**	**PCR2**	**Prefer-1/PCR1**	**Cool-lex**	**Weight-range**
10	29	50	75	101	120	131	131
11	41	71	141	180	222	257	257
12	51	101	248	321	416	468	468
13	70	143	468	587	784	801	930
14	85	203	850	1065	1488	1723	1723
15	110	288	1604	1974	2824	3439	3439
16	175	407	2965	3632	5376	6443	6443
17	246	575	5594	6785	10229	11452	12878
18	326	815	10461	12635	19484	24319	24319
19	462	1157	19765	23746	37107	48629	48629
20	730	1634	37243	44585	71250	92388	92388
21	954	2311	70575	84270	138332	167975	184766
22	1327	3264	133737	159281	268582	352727	352727
23	1820	4565	254322	302449	521553	705443	705443
24	2684	6252	484172	574819	1012795	1352090	1352090
25	3183	9192	924071	1096009	1966813	2496163	2704168
26	4108	13074	1766284	2092284	3819605	5200313	5200313
27	5604	17933	3382851	4004050	7453523	10400613	10400613
28	7629	22672	6488970	7672443	14544826	20058314	20058314
29	10433	34591	12468181	14730243	28382864	37442182	40116614
30	13637	57357	23991972	28316271	55421919	77558775	77558775

B C program to construct maximum-discrepancy de Bruijn sequences

```
#include <stdio.h>
int n,c,a[100],first;

//-----------------------------------------------------------------------------
// Generate the lexicographically smallest universal cycle (de Bruijn sequence)
// for binary strings of length "n" with minimum weight "c" when comp = 0. When
// comp=1, it complements the bits producing a maximum weight n-c universal cycle
//-----------------------------------------------------------------------------
void Gen(int t, int p, int w, int comp) {
     int i;

     if (t > n) {
          if (n%p == 0) {
               if (first == 0) {
                    for (i=1; i<=c; i++) printf("0");
                    first = 1;
               }
               else {
                    for (i=1; i <= p; i++) printf("%d", (a[i]+comp) % 2);
               }
          }
     }
     else {
          // Append 0
          a[t] = 0;
          if (a[t-p] == 0 && c-w < n-t+1)  Gen(t+1,p,w,comp);

          // Append 1
          a[t] = 1;
          if (a[t-p] == 1) Gen(t+1,p,w+1,comp);
          else Gen(t+1,t,w+1, comp);
     }
}
//================================================
int main() {

     printf("Enter n: ");   scanf("%d", &n);

     c = n/2+1;
     for (int i=1; i<=c-1; i++) printf("1");
     a[0] = 0; first = 1;
     Gen(1,1,0,0);

     c = n-c+1; first = 0;
     Gen(1,1,0,1);
     printf("\n");
}
```