Algorithms for Optimal Decisions
Tutorial 1
Answers

Exercise 1 *Show that the intersection S of any numbers of convex sets S_i is a convex set.*

Solution : Take any two elements x_1, x_2 from the intersection set $S = \cap_i S_i$.

- In order to prove that the intersection S is convex we need to prove that
$$\alpha x_1 + (1-\alpha)x_2 \in S, \quad \forall \alpha \in [0,1]. \tag{1}$$
- Since $x_1, x_2 \in S$ it follows that $x_1, x_2 \in S_i, \quad \forall i$.
- Because each S_i is a convex set then, for all i:
$$\alpha x_1 + (1-\alpha)x_2 \in S_i, \quad \forall \alpha \in [0,1]. \tag{2}$$
- Hence the point $\alpha x_1 + (1-\alpha)x_2$ belongs to all the sets S_i for $\forall \alpha \in [0,1]$. Consequently it belongs to the intersection $S = \cap_i S_i$ of all these sets.
- Therefore we proved that for any two elements x_1, x_2 in the intersection $S = \cap_i S_i$ and for any $\alpha \in [0,1]$ the following holds:
$$\alpha x_1 + (1-\alpha)x_2 \in S, \quad \forall \alpha \in [0,1]. \tag{3}$$
According to the definition of a convex set, the set $S = \cap_i S_i$ is also a convex set.

Exercise 2 *Show that if $f(x)$ and $g(x)$ are convex functions on a convex set S, then their sum*

$$h(x) = f(x) + g(x) \tag{4}$$

is also a convex function on S.

Solution : Take any two elements x_1, x_2 from the set S. To prove that the sum $f(x) + g(x)$ is a convex function we need to show that:

$$\begin{aligned} & f(\alpha x_1 + (1-\alpha)x_2) + g(\alpha x_1 + (1-\alpha)x_2) \leq \\ \leq \; & \alpha(f(x_1) + g(x_1)) + (1-\alpha)(f(x_2) + g(x_2)). \end{aligned} \tag{5}$$

- Since f and g are convex functions we have that for any two points $x_1, x_2 \in S$, and $\alpha \in [0, 1]$ the following holds:

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \tag{6}$$

$$g(\alpha x_1 + (1-\alpha)x_2) \leq \alpha g(x_1) + (1-\alpha) g(x_2). \tag{7}$$

- Adding (6) and (7) we have

$$\begin{aligned} & f(\alpha x_1 + (1-\alpha)x_2) + g(\alpha x_1 + (1-\alpha)x_2) \leq \\ \leq \; & \alpha f(x_1) + (1-\alpha) f(x_2) + \alpha g(x_1) + (1-\alpha) g(x_2) = \\ = \; & \alpha(f(x_1) + g(x_1)) + (1-\alpha)(f(x_2) + g(x_2)), \end{aligned}$$

  which shows that (5) holds and consequently the sum $f(x) + g(x)$ is a convex function.

Exercise 3 *Show that if $f(x)$ is a convex function, then the set*

$$L = \{x \in R^n \quad | \quad f(x) \leq b\} \tag{8}$$

is a convex set.

Solution : We need to prove that for every $x_1, x_2 \in L$ the point $\alpha x_1 + (1-\alpha)x_2$ is also in L.

- For any two elements $x_1, x_2 \in L$ we have:

$$\begin{aligned} f(x_1) &\leq b, \quad f(x_2) \leq b \\ \alpha f(x_1) &\leq \alpha b, \quad (1-\alpha) f(x_2) \leq (1-\alpha) b. \end{aligned}$$

- Adding the above two inequalities we have:

$$\alpha f(x_1) + (1-\alpha) f(x_2) \leq \alpha b + (1-\alpha) b = b. \tag{9}$$

- Since the function $f(x)$ is convex we also have:

$$f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2). \tag{10}$$

- From (9) and (10) it follows that

$$f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \leq b, \tag{11}$$

  which shows that the point $\alpha x_1 + (1-\alpha) x_2 \in L$ and consequently the set L is convex.

Exercise 4 *Consider the non–linear problem:*

$$\begin{array}{rrcl} \min\limits_{x} & f(x) & = & x_1^2 + x_2^2 - 4x_1 + 4 \\ s.t. & g_1(x) & = & x_1 - x_2 + 2 \geq 0 \\ & g_2(x) & = & -x_1^2 + x_2 - 1 \geq 0 \\ & g_3(x) & = & x_1 \geq 0 \\ & g_4(x) & = & x_2 \geq 0. \end{array} \tag{12}$$

1. *Show that the constraints define a convex set;*

2. *Show that the objective function $f(x)$ is convex.*

Solution :

(a) The feasible region (i.e. the set defined by the constraints of the problem) is convex because:

   (i) constraints $g_1(x), g_3(x)$ and $g_4(x)$ are linear and hence concave. (Remember that a linear function can be <u>both</u> concave and convex.)

   (ii) constraint $g_2(x)$ is non–linear. To check whether it is concave or not we need to find its Hessian matrix:

$$H_2 = \begin{bmatrix} \frac{\partial^2 g_2}{\partial x_1^2} & \frac{\partial^2 g_2}{\partial x_1 \partial x_2} \\ \frac{\partial^2 g_2}{\partial x_2 \partial x_1} & \frac{\partial^2 g_2}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \tag{13}$$

and show that it is negative semi–definite, i.e.

$$\forall v \in R^2, \quad v^t H_2 v \leq 0. \tag{14}$$

The matrix H_2 is negative semi–definite because for every vector $v^t = (v_1, v_2) \in R^2$ we have:

$$v^t H_2 v = (v_1, v_2) \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -2v_1^2 \leq 0. \tag{15}$$

Therefore all the functions $g_i, i = 1, 2, 3, 4$ which define the feasible region are concave functions. We have concave functions, and from the previous example, sets:

$$L_i = \{x \in R^n \quad | g_i(x) \geq 0\}, \quad i = 1, 2, 3, 4 \tag{16}$$

are convex (show it!). Feasible region $\mathcal{F} = \cap_i L_i$ is an intersection of convex sets, therefore also convex.

(b) To show that the objective function $f(x)$ is convex we need to show that its Hessian matrix

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \tag{17}$$

is positive semi–definite.

Matrix H is positive semi–definite because for any $v^t = (v_1, v_2) \in R^2$ we have:

$$v^t H v = (v_1, v_2) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2v_1^2 + 2v_2^2 \geq 0. \tag{18}$$