Algorithms for Optimal Decisions
Tutorial 2
Answers

Exercise 1 *Labor costs* 2\$/*hour and capital costs* 1\$/*unit. If* l *hours of labor and* k *units of capital are available then* $l^{2/3} \cdot k^{1/3}$ *machines can be produced. If the budget for purchasing capital and labor is* 10\$, *what is the maximum number of machines that can be produced?*

Solution : The problem can be formulated as the following equality constrained non–linear problem:

$$\begin{aligned} \max_{l,k} \quad f(l,k) &= l^{\frac{2}{3}} \cdot k^{\frac{1}{3}} \\ s.t. \quad g(l,k) &= 2l + k - 10 = 0. \end{aligned} \tag{1}$$

The Lagrangian of the problem (1) is

$$\begin{aligned} L(k,l,\lambda) &= f(l,k) + \lambda g(l,k) \\ &= l^{\frac{2}{3}} \cdot k^{\frac{1}{3}} + \lambda(2l + k - 10). \end{aligned} \tag{2}$$

A stationary point of the Lagrangian function is defined as the solution of the following system of nonlinear equations:

$$\nabla_{l,k,\lambda} L(k,l,\lambda) = \begin{bmatrix} \frac{\partial L}{\partial l} \\ \frac{\partial L}{\partial k} \\ \frac{\partial L}{\partial \lambda} \end{bmatrix} = 0. \tag{3}$$

Evaluating the partial derivatives we get:

$$\frac{\partial L}{\partial l} = \frac{2}{3} l^{-\frac{1}{3}} k^{\frac{1}{3}} + 2\lambda = \frac{2}{3}(\frac{k}{l})^{\frac{1}{3}} + 2\lambda = 0 \tag{4}$$

$$\frac{\partial L}{\partial k} = \frac{1}{3} l^{\frac{2}{3}} k^{-\frac{2}{3}} + \lambda = \frac{1}{3}(\frac{l}{k})^{\frac{2}{3}} + \lambda = 0 \tag{5}$$

$$\frac{\partial L}{\partial \lambda} = 2l + k - 10 = 0 \tag{6}$$

If we define $p = \frac{l}{k}$ then equations (4) and (5) become:

$$\frac{2}{3}(\frac{1}{p})^{\frac{1}{3}} + 2\lambda = 0 \tag{7}$$
$$\frac{1}{3}p^{\frac{2}{3}} + \lambda = 0 \tag{8}$$

Solving (7) for λ we have $\lambda = -\frac{1}{3}(\frac{1}{p})^{\frac{1}{3}}$ and substituting it into (8) we get

$$\frac{1}{3}p^{\frac{2}{3}} - \frac{1}{3p^{\frac{1}{3}}} = 0 \Rightarrow \frac{p^{\frac{2}{3}+\frac{1}{3}} - 1}{3p^{\frac{1}{3}}} = 0 \Rightarrow$$
$$\Rightarrow \quad p - 1 = 0 \Rightarrow p = 1. \tag{9}$$

Recall that $p = \frac{l}{k}$. Hence we have $\frac{l}{k} = 1 \Rightarrow l = k$ or in other words the number of labor hours and the number of capital units needed to maximize the number of machines produced must be equal.

l and k also must satisfy the constraint

$$2l + k - 10 = 0. \tag{10}$$

As $k = l$ the above becomes $3l = 10 \Rightarrow l = k = \frac{10}{3}$.

Note: Check whether the Hessian matrix of L is negative definite.

Exercise 2 *Find the optimum solution of the following constrained problem:*

$$\begin{aligned} \max_x f(x) &= x_1x_2 + x_2x_3 + x_1x_3 \\ s.t. \quad & x_1 + x_2 + x_3 = 3. \end{aligned} \tag{11}$$

Solution : The Lagrangian function of problem (11) is:

$$L(x, \lambda) = x_1x_2 + x_2x_3 + x_1x_3 + \lambda(x_1 + x_2 + x_3 - 3). \tag{12}$$

A stationary point of the Lagrangian L satisfies the following system of equations:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= x_2 + x_3 + \lambda = 0 \\ \frac{\partial L}{\partial x_2} &= x_1 + x_3 + \lambda = 0 \\ \frac{\partial L}{\partial x_3} &= x_1 + x_2 + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= x_1 + x_2 + x_3 - 3 = 0 \end{aligned} \tag{13}$$

It is easy to solve the above system of equations. It's solution is

$$x^* = (x_1^*, x_2^*, x_3^*) = (1, 1, 1), \quad \lambda = -2. \tag{14}$$

Note: Check whether the Hessian matrix of L is positive definite.

Exercise 3 *Given a fixed area of cardboard, try to find the dimensions of a cardboard box with the largest possible volume.*

Solution : Denoting the dimensions of the box by x_1, x_2, x_3 the problem can be expressed as the following equality constrained problem:

$$\begin{aligned} \max_x \quad vol(x_1, x_2, x_3) &= x_1 x_2 x_3 \\ s.t. \quad g(x_1, x_2, x_3) &= 2(x_1 x_2 + x_2 x_3 + x_1 x_3) - c = 0, \end{aligned} \tag{15}$$

where $c > 0$ is the given area of the cardboard.

The Lagrangian of the problem is:

$$L(x_1, x_2, x_3, \lambda) = x_1 x_2 x_3 + \lambda(2(x_1 x_2 + x_2 x_3 + x_1 x_3) - c). \tag{16}$$

A stationary point of L satisfies the following system:

$$\frac{\partial L}{\partial x_1} = x_2 x_3 + 2\lambda(x_2 + x_3) = 0 \tag{17}$$

$$\frac{\partial L}{\partial x_2} = x_1 x_3 + 2\lambda(x_1 + x_3) = 0 \tag{18}$$

$$\frac{\partial L}{\partial x_3} = x_1 x_2 + 2\lambda(x_1 + x_2) = 0 \tag{19}$$

$$\frac{\partial L}{\partial \lambda} = 2(x_1 x_2 + x_2 x_3 + x_1 x_3) - c = 0. \tag{20}$$

Adding equations (17),(18) and (19) we have

$$(x_1 x_2 + x_2 x_3 + x_1 x_3) + 4\lambda(x_1 + x_2 + x_3) = 0. \tag{21}$$

Using equation (20), from equation (21) we have:

$$\frac{c}{2} + 4\lambda(x_1 + x_2 + x_3) = 0. \tag{22}$$

¿From (22) it is clear that $\lambda \neq 0$, since $c > 0$. We can also show that x_1, x_2, x_3 are always $\neq 0$. This follows because $x_1 = 0$ implies $x_3 = 0$ from equation (18) and $x_2 = 0$ from equation (19).

Similarly it is easy to see that if either of the dimensions x_1, x_2, x_3 is zero, all the others must be zero which is impossible.

To solve the equations (17)–(20) we multiply (17) by x_1 and (18) by x_2 and subtract the two to obtain

$$\lambda(x_1 - x_2)x_3 = 0. \tag{23}$$

Apply similar operations on (18) and (19) to obtain

$$\lambda(x_2 - x_3)x_1 = 0. \tag{24}$$

Since no variable can be zero, it follows that $x_1 = x_2 = x_3$. Hence the box must be a cube.

To compute the dimension of the cube we note that

$$2(x_1x_2 + x_2x_3 + x_1x_3) = c \Rightarrow 6x_1^2 = c \Rightarrow x_1 = \sqrt{\frac{c}{6}}. \tag{25}$$