Algorithms for Optimal Decisions
Tutorial 6
Answers

Exercise 1 *Solve the following problem by using the active set method and taking* $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0,0,1)$ *as a starting point*

$$\begin{aligned} \min_x f(x) &= x_1^2 + 2x_2^2 + 3x_3^2 \\ s.t. \quad & x_1 + x_2 + x_3 - 1 \geq 0 \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{1}$$

Solution : First, we rewrite the problem, so we have constraints which are less or equal to zero:

$$\begin{aligned} \min_x f(x) &= x_1^2 + 2x_2^2 + 3x_3^2 \\ s.t. \quad & 1 - x_1 - x_2 - x_3 \leq 0 \\ & -x_1, -x_2, -x_3 \leq 0. \end{aligned} \tag{2}$$

- The starting point $x^{(0)}$ is feasible, since $g_i(x^{(0)}) \leq 0, \quad i = 1,2,3,4.$
- Set $k = 0$, where k is the iteration counter. The set of active constraints at the point $x^{(0)}$ is $J_0 = \{1,2,3\}$.
- The direction of movement $d_0 = x - x^{(0)} = x$ will be found by solving the following equality constrained problem:

$$\begin{aligned} \min_x \quad & f(x) = x_1^2 + 2x_2^2 + 3x_3^2 \\ s.t. \quad & g_1(x) = 1 - x_1 - x_2 - x_3 = 0 \\ & g_2(x) = -x_1 = 0 \\ & g_3(x) = -x_2 = 0 \end{aligned} \tag{3}$$

- It follows from (3) that $d_0 = 0$.

- Since $d_0 = 0$ we need to compute multipliers $\mu^{(1)} = (\mu_1^{(1)}, \mu_2^{(1)}, \mu_3^{(1)})$ for problem (3).

- The Lagrangian of (3) is:

$$L(x, \mu^{(1)}) = x_1^2 + 2x_2^2 + 3x_3^2 + \mu_1^{(1)}(1 - x_1 - x_2 - x_3) + \mu_2^{(1)}(-x_1) + \mu_3^{(1)}(-x_2). \tag{4}$$

- The optimality conditions for (3) are:

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= 2x_1 - \mu_1^{(1)} - \mu_2^{(1)} = 0 \\
\frac{\partial L}{\partial x_2} &= 4x_2 - \mu_1^{(1)} - \mu_3^{(1)} = 0 \\
\frac{\partial L}{\partial x_3} &= 6x_3 - \mu_1^{(1)} = 0 \\
\frac{\partial L}{\partial \mu_1^{(1)}} &= 1 - x_1 - x_2 - x_3 = 0 \\
\frac{\partial L}{\partial \mu_2^{(1)}} &= -x_1 = 0 \\
\frac{\partial L}{\partial \mu_3^{(1)}} &= -x_2 = 0
\end{aligned}$$

Solution to the above system is
$(x_1, x_2, x_3, \mu_1^{(1)}, \mu_2^{(1)}, \mu_3^{(1)}) = (0, 0, 1, 6, -6, -6)$.

- Only one of the Lagrange multipliers are negative $\mu_2^{(1)}$.

- From step 3 of the algorithm (see your notes) we can drop the constraint $g_2(x) = -x_1 \leq 0$ from the active set J_0. Thus the new active set is $J_1 = \{1, 3\}$.

- Now we need to solve the following equality constrained quadratic problem:

$$\begin{aligned}
\min_x \quad f(x) &= x_1^2 + 2x_2^2 + 3x_3^2 \\
s.t. \quad g_1(x) &= 1 - x_1 - x_2 - x_3 = 0 \\
g_3(x) &= -x_2 = 0.
\end{aligned} \tag{5}$$

- The Lagrangian of (5) is:

$$L(x, \mu^{(2)}) = x_1^2 + 2x_2^2 + 3x_3^2 + \mu_1^{(2)}(1 - x_1 - x_2 - x_3) + \mu_2^{(2)}(-x_2). \quad (6)$$

- The optimality conditions for (5) are:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 2x_1 - \mu_1^{(2)} = 0 \\ \frac{\partial L}{\partial x_2} &= 4x_2 - \mu_1^{(2)} - \mu_2^{(2)} = 0 \\ \frac{\partial L}{\partial x_3} &= 6x_3 - \mu_1^{(2)} = 0 \\ \frac{\partial L}{\partial \mu_1^{(2)}} &= 1 - x_1 - x_2 - x_3 = 0 \\ \frac{\partial L}{\partial \mu_2^{(2)}} &= -x_2 = 0. \end{aligned} \quad (7)$$

  Solution to the above system is
  $(x_1, x_2, x_3, \mu_1^{(2)}, \mu_1^{(2)}) = (\frac{3}{4}, 0, \frac{1}{4}, \frac{3}{2}, -\frac{3}{2})$.

- One of the Lagrange multipliers of problem (5) is negative, so constraint $g_3(x)$ is dropped.

- The direction d_1 is then the vector from point $x^{(1)} = (\frac{3}{4}, 0, \frac{1}{4})$ to the solution of the following constrained quadratic problem:

$$\begin{aligned} \min_x \quad f(x) &= x_1^2 + 2x_2^2 + 3x_3^2 \\ s.t. \quad & 1 - x_1 - x_2 - x_3 = 0. \end{aligned} \quad (8)$$

- Optimality conditions of (8):

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 2x_1 - \mu_1^{(3)} = 0 \\ \frac{\partial L}{\partial x_2} &= 4x_2 - \mu_1^{(3)} = 0 \\ \frac{\partial L}{\partial x_3} &= 6x_3 - \mu_1^{(3)} = 0 \\ \frac{\partial L}{\partial \mu_1^{(2)}} &= 1 - x_1 - x_2 - x_3 = 0. \end{aligned} \tag{9}$$

- The point $(x_1^*, x_2^*, x_3^*, \mu_1^{(1)}) = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11}, \frac{12}{11})$.
- New point is feasible, so we can take that point as a new point. That means that $\tau = 1$. Also the Lagrange multiplier is positive, so point $x^* = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11})$ is the solution to our problem.

Exercise 2 *Solve the following problem using the interior point method:*

$$\begin{aligned} \min_x \quad f(x) &= x_1 + x_2 \\ s.t. \quad g_1(x) &= -x_1^2 + x_2 \geq 0 \\ g_2(x) &= x_1 \geq 0. \end{aligned} \tag{10}$$

Solution : We shall use the logarithmic barrier function to solve the problem (10). Thus problem (10) is approximated by a sequence of unconstrained problems:

$$\min_x f(x) - \eta_k \sum_{i=1}^{2} \log(g_i(x)), \tag{11}$$

where the values of the parameter η_k decrease and approach zero. We are going to solve a number of problems (11) for a decreasing sequence of values of the barrier parameter η_k, such that

$$\lim_{k\to\infty} \eta_k = 0.$$

First, we find the optimality conditions of the unconstrained problem (11) where the value of the barrier parameter is fixed:

$$\begin{aligned} \tfrac{\partial}{\partial x_1}(x_1 + x_2 - \eta_k(\log(-x_1^2 + x_2) + \log(x_1))) = 0 \\ \tfrac{\partial}{\partial x_2}(x_1 + x_2 - \eta_k(\log(-x_1^2 + x_2) + \log(x_1))) = 0 \end{aligned} \quad \Rightarrow \tag{12}$$

$$\Rightarrow \begin{aligned} 1 - \eta_k \tfrac{1}{-x_1^2+x_2} \cdot (-2x_1) - \tfrac{\eta_k}{x_1} = 0 \\ 1 - \eta_k(\tfrac{1}{-x_1^2+x_2}) = 0 \end{aligned} \tag{13}$$

Solving (13) we have:

$$-\frac{\eta_k}{-x_1^2 + x_2} = -1, \tag{14}$$

and

$$1 - (-2x_1) - \frac{\eta_k}{x_1} = 0 \Rightarrow 1 + 2x_1 - \frac{\eta_k}{x_1} = 0 \Rightarrow$$
$$2x_1^2 + x_1 - \eta_k = 0. \tag{15}$$

The solution of (15) is given by the formula:

$$x_1 = \frac{-1 \pm \sqrt{1 + 8\eta_k}}{4}. \tag{16}$$

Since x_1 must be positive, only the root

$$x_1 = \frac{-1 + \sqrt{1 + 8\eta_k}}{4} \tag{17}$$

is of interest. Substituting (17) into (14) yields:

$$\frac{-\eta_k}{-(\frac{-1+\sqrt{1+8\eta_k}}{4})^2 + x_2} = -1 \Rightarrow \ldots \Rightarrow$$
$$\Rightarrow \quad x_2 = \frac{(-1 + \sqrt{1 + 8\eta_k})^2}{16} + \eta_k. \tag{18}$$

Formulae (17) and (18) give the optimum of the unconstrained problem (10) where the value of the barrier parameter η_k is fixed. For example, if $\eta_k = 1$ then the point

$$(x_1^{(1)}, x_2^{(1)}) = (\frac{-1 + \sqrt{1 + 8}}{4}, \frac{(-1 + \sqrt{1 + 8})^2}{16} + 1) = (0.5, 1.25) \tag{19}$$

is the optimum solution of the following unconstrained problem:

$$\min_x f(x) - 1 \cdot \sum_{i=1}^{2} \log(g_i(x)). \tag{20}$$

Now, if η_k is fixed to a smaller value, say $\eta_k = \frac{1}{2}$ then the point

$$(x_1^{(2)}, x_2^{(2)}) = (\frac{-1 + \sqrt{1 + 8\frac{1}{2}}}{4}, \frac{(-1 + \sqrt{1 + 8\frac{1}{2}})^2}{16} + 1) = (0.309, 0.595) \tag{21}$$

is the optimum solution of the following unconstrained problem:

$$\min_x f(x) - \frac{1}{2} \cdot \sum_{i=1}^{2} \log(g_i(x)). \tag{22}$$

The following table shows the computed value of the points $(x_1^{(k)}, x_2^{(k)})$ for different values of η_k.

k	η_k	$x_1^{(k)}$	$x_2^{(k)}$
1	$\eta_1 = 1$	0.5	1.25
2	$\eta_2 = \frac{1}{2}$	0.309	0.595
3	$\eta_2 = \frac{1}{4}$	0.183	0.283
4	$\eta_2 = \frac{1}{10}$	0.085	0.107
	$\downarrow$	$\downarrow$	$\downarrow$
	0	0	0

In the limit (i.e. $\lim_{k\to\infty} \eta_k = 0$) the minimizing points $(x_1^{(k)}, x_2^{(k)})$ approach the solution $(x_1^*, x_2^*) = (0, 0)$ of the original constrained problem (10).

In this problem there is only one unconstrained local minimum for each value of η_k. The problem happens to have the unique solution. It turns out that in problems with many local optima there is a sequence of local unconstrained minima converging to each set of constrained local minima. This is illustrated in the next example.