Algorithms for Optimal Decisions
Tutorial 9
Answers

Exercise 1 *Using the Goldstein–Levitin–Polyak algorithm solve the following NLP problem:*

$$\begin{aligned} \min_{x} \quad F(x) &= x_1^3 + x_2^2 - 3x_1 - 4x_2 \\ s.t. \quad h_1(x) &= x_1^2 + x_2 - 1 \le 0 \\ h_2(x) &= -x_1 \le 0 \\ h_3(x) &= -x_2 \le 0. \end{aligned} \tag{1}$$

Starting point : $x^{(0)} = (x_1^{(0)}, x_2^{(0)}) = (\frac{1}{4}, \frac{1}{4})$.

Solution : First, we need to compute the unconstrained step, for some $\overline{\alpha} \in [1, 2)$. For simplicity we choose $\overline{\alpha} = 1$, and consequently the step–size $\alpha_0 = 1$
($\alpha_0 \in [1, \overline{\alpha}]$).

Hence the unconstrained step becomes (see (2.2) in the notes):

$$\begin{aligned} \bar{x} &= x^{(0)} - \alpha_0 (\nabla^2 F(x^{(0)})^{-1} \nabla F(x^{(0)}) \\ &= \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} - 1 \cdot \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{pmatrix} -2.8125 \\ -3.5 \end{pmatrix} \\ &= \begin{pmatrix} 2.125 \\ 2 \end{pmatrix} \end{aligned} \tag{2}$$

The inverse of the Hessian matrix $\nabla^2 F(x^{(0)})$ is given by:

$$(\nabla^2 F(x^{(0)})^{-1} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}. \tag{3}$$

Now we need to define the problem which will be used to determine the projection of the unconstrained step $\bar{x}$ onto the feasible region:

$$\mathcal{R} = \{x \in R^2 \quad | \quad h_i(x) \le 0, i = 1, 2, 3\}. \tag{4}$$

$$\begin{array}{rrcl} \min\limits_x & p(x) & = & \frac{1}{2}\|x - \bar{x}\|_{\nabla^2 F(x^{(0)})} \\ s.t. & h_1(x) & = & x_1 + x_2 - 1 \le 0 \\ & h_2(x) & = & -x_1 \le 0 \\ & h_3(x) & = & -x_2 \le 0. \end{array} \tag{5}$$

The objective function $p(x)$ of problem (5) has the following analytic form:

$$\begin{aligned} p(x) &= \frac{1}{2}(x - \bar{x})^t \nabla^2 F(x^{(0)})(x - \bar{x}) = \\ &= \frac{1}{2}(x_1 - 2.125, x_2 - 2)^t \cdot \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{pmatrix} x_1 - 2.125 \\ x_2 - 2 \end{pmatrix} \\ &= \frac{3}{4}(x_1 - 2.125)^2 + (x_2 - 2)^2, \end{aligned} \tag{6}$$

and the projection problem (5) becomes:

$$\begin{array}{rrcl} \min\limits_x & p(x) & = & \frac{3}{4}(x_1 - 2.125)^2 + (x_2 - 2)^2 \\ s.t. & h_1(x) & = & x_1 + x_2 - 1 \le 0 \\ & h_2(x) & = & -x_1 \le 0 \\ & h_3(x) & = & -x_2 \le 0. \end{array} \tag{7}$$

We need to find the optimum solution of (7). The Lagrangian of the problem is given by:

$$L(x, \mu) = \tfrac{3}{4}(x_1 - 2.125)^2 + (x_2 - 2)^2 + \mu_1(x_1 + x_2 - 1) + \mu_2(-x_1) + \mu_3(-x_2),$$

and the corresponding KKT conditions are:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= \frac{3}{2}(x_1 - 2.125) + \mu_1 - \mu_2 = 0 \\ \frac{\partial L}{\partial x_2} &= 2(x_2 - 2) + \mu_1 - \mu_3 = 0 \end{aligned}$$

$$
\begin{aligned}
&\mu_1(x_1 + x_2 - 1) = 0\\
&\mu_2(-x_1) = 0\\
&\mu_3(-x_2) = 0\\
&h_i(x) \leq 0, \quad i = 1, 2, 3\\
&\mu_i \geq 0, \quad i = 1, 2, 3.
\end{aligned}
$$

As an exercise try to solve the above KKT conditions, in order to find the optimum solution of (7). Or, alternatively you can use the steepest descent method or **SUMT** to solve it. In any case the optimum solution of (7) is

$$x_p^{(0)} = (0.339286, 0.660714) \in \mathcal{R}.$$

The current point $x^{(0)}$ and the solution $x_p^{(0)}$ of the projection problem (7) define a line. Since the feasible region $\mathcal{R}$ is a convex set and both points $x^{(0)}$ and $x_p^{(0)}$ belong to $\mathcal{R}$, we have that all the points on that line also belong to the feasible region $\mathcal{R}$.

The next iterate $x^{(1)}$ will be one of the points on that line, and it is computed as follows:

$$x^{(1)} = x^{(0)} + \tau^{(0)}(x_p^{(0)} - x^{(0)})$$

where the step–size $\tau^{(0)}$ is defined as $\tau^{(0)} = \bar{\tau}^j$, where $\bar{\tau}$ is any number in the interval $(0, 1)$, say $\bar{\tau} = \frac{1}{2}$ and j is the smallest non–negative integer such that the following inequality (Armijo rule) is satisfied:

$$F(x^{(1)}) - F(x^{(0)}) \leq \bar{\tau}^j \rho \nabla F(x^{(0)})^t (x_p^{(0)} - x^{(0)}) \tag{8}$$

or equivalently:

$$F(x^{(0)} + \bar{\tau}^j (x_p^{(0)} - x^{(0)})) - F(x^{(0)}) \leq \bar{\tau}^j \rho \nabla F(x^{(0)})^t (x_p^{(0)} - x^{(0)}) \tag{9}$$

We choose $\bar{\tau} = \frac{1}{2}$ and $\rho = 10^{-4}$, but any other pair of values from the interval $(0, 1)$ can be chosen, and try to find the smallest non–negative integer j such that the above Armijo's rule is satisfied.

- $j = 0$: then $x^{(1)} = x^{(0)} + (\frac{1}{2})^0 (x_p^{(0)} - x^{(0)}) = x_p^{(0)}$. Let us find the value of the objective function F of the initial problem (1) at $x_p^{(0)}$ and $x^{(0)}$:

$$
\begin{aligned}
F(x_p^{(0)}) &= 0.339286^3 + 0.660714^2 - 3 \cdot 0.339286 - 4 \cdot 0.660714\\
&= -3.1851141.\\
F(x^{(0)}) &= -1.671875
\end{aligned}
$$

Next we need to calculate the following:

$$\begin{aligned}\nabla F(x^{(0)})^t(x_p^{(0)} - x^{(0)}) &= (-2.8125, -3.5) \cdot \begin{pmatrix} 0.339286 - \frac{1}{4} \\ 0.660714 - \frac{1}{4} \end{pmatrix} \\ &= -1.67606 < 0. \end{aligned} \tag{10}$$

Vector $x_p^{(0)} - x^{(0)}$ defines a descent direction for the objective function F at $x^{(0)}$, since $F(x^{(0)})^t(x_p^{(0)} - x^{(0)}) < 0$.

Substituting these computed values into (8) and setting $\bar{\tau} = \frac{1}{2}$, $j = 0$ and $\rho = 10^{-4}$ we have:

$$-1.5132391 \leq -0.000167606.$$

Hence, $j = 0$ is the smallest non–negative integer value that satisfies the Armijo rule (8).

Thus the next iterate $x^{(1)}$ is obtained:

$$x^{(1)} = x^{(0)} + \bar{\tau}^0(x_p^{(0)} - x^{(0)}) = x_p^{(0)} = (0.339286, 0.660714), \tag{11}$$

and the value of the objective function is $F(x^{(1)}) = -3.1851141$.

Recall that we started from the point $x^{(0)} = (\frac{1}{4}, \frac{1}{4})$ where $F(x^{(0)}) = -1.671875$ and applying one step of the Goldstein–Levitin–Polyak algorithm we found the point $x^{(1)} = (0.339286, 0.660714)$ where $F(x^{(1)}) = -3.1851141$. Hence the value of the objective function has decreased. At this point the first iteration of the G–L–P algorithm has been completed. As an exercise you can carry on until you find the optimum solution or you may write a computer program which will find the solution.

Exercise 2 *Using the Goldstein–Levitin–Polyak algorithm solve the following QP problem:*

$$\begin{aligned} \min_x \quad F(x) &= 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 \\ s.t. \quad h_1(x) &= x_1 + x_2 - 4 \leq 0 \\ h_2(x) &= -x_1 \leq 0 \\ h_3(x) &= -x_2 \leq 0. \end{aligned} \tag{12}$$

Starting point : $x^{(0)} = (x_1^{(0)}, x_2^{(0)}) = (1, 1)$.

Solution :