Game Theory
Tutorial 2
Answers

Exercise 1 (Minimax problem) *Three linear functions y_1, y_2 and y_3 are defined as follows:*

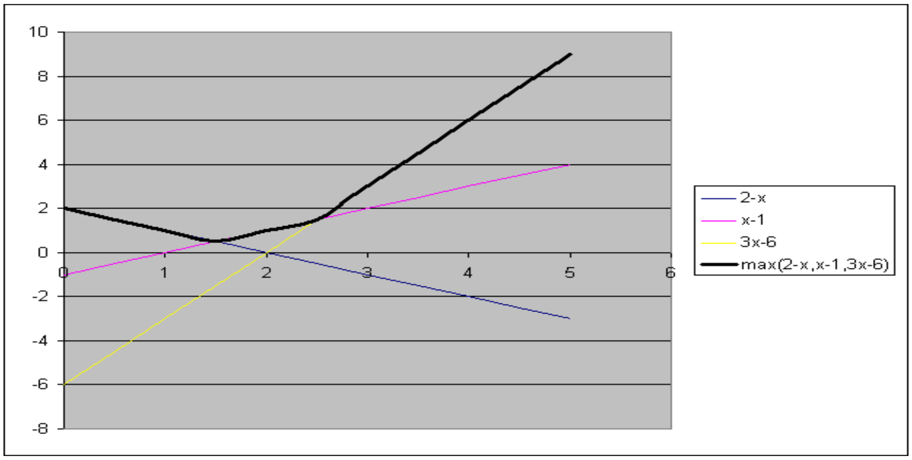
$$\begin{aligned} y_1 &= 2 - x_1, \\ y_2 &= x_1 - 1, \\ y_3 &= 2x_1 - 6. \end{aligned}$$

Find

$$\min_{x} \max_{i=1,2,3} \{y_i\}. \tag{1}$$

Solution : A way to solve problem (1) is to introduce an auxiliary variable, say x_0 and then to solve the equivalent problem:

$$\begin{aligned} min \quad & x_0 \\ s.t. \quad & 2 - x_1 \leq x_0, \\ & x_1 - 1 \leq x_0, \\ & 2x_1 - 6 \leq x_0. \end{aligned}$$



Graph of **max** function.

The solution to this problem, as can be seen from the above figure is

$$x_1^* = 1.5, \quad x_0^* = 0.5.$$

What would the graph of the following function:

$$\max_x \min_i y_i$$

look like and what is the solution to that problem?

Exercise 2 (Minimax problem again) *Find x_1, x_2 satisfying*

$$\begin{array}{rcl} x_1 + x_2 & \leq & 2, \\ x_1, x_2 & \geq & 0, \end{array} \tag{2}$$

and having the maximum of

$$\begin{array}{rr} 3x_1 - & x_2 \\ -x_1 + & x_2 \end{array} \tag{3}$$

as small as possible.

Solution : The problem can be transformed into the following LP problem:

$$\begin{aligned} \min \quad & w \\ s.t. \quad & 3x_1 - x_2 \le w \\ & -x_1 + 2x_2 \le w \\ & x_1 + x_2 \le 2, \\ & x_1, x_2 \ge 0. \end{aligned}$$

To use the simplex algorithm we need all variables to be ≥ 0. However, w could be any real value as it is free variable, so we define $w = x_3 - x_4$, where $x_3, x_4 \ge 0$.

So we are left with the following problem to solve:

$$\begin{aligned} \min \quad & x_3 - x_4 \\ s.t. \quad & 3x_1 - x_2 - x_3 + x_4 \le 0 \\ & -x_1 + 2x_2 - x_3 + x_4 \le 0 \\ & x_1 + x_2 \le 2, \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned}$$

Now, introduce slack variables x_5, x_6, x_7:

$$\begin{array}{lllllll} \min & x_0 = x_3 - x_4 & & & & \\ s.t. & 3x_1 - x_2 - x_3 + x_4 & +x_5 & & & = 0 \\ & -x_1 + 2x_2 - x_3 + x_4 & & +x_6 & & = 0 \\ & x_1 + x_2 & & & +x_7 & = 2 \\ & x_1, x_2, x_3, x_4 \ge 0. & & & & \end{array} \tag{4}$$

The solution is (**check!!!**) $(x_0, x_1, x_2) = (0, 0, 0)$.

Exercise 3 (Duality Theory 1) *Given the primal L.P. problem:*

$$\begin{aligned} \max_{x} \quad & c^t x \\ s.t. \quad & Ax \le b \\ & x \ge 0, \end{aligned} \tag{5}$$

and its dual pair:

$$\begin{aligned} \min_y \quad & b^t y \\ s.t. \quad & A^t y \geq c \\ & y \geq 0, \end{aligned} \tag{6}$$

show that the dual of (6) is (5).

Solution : We start from the original dual problem (6):

$$\begin{aligned} \min_y \quad & b^t y \\ s.t. \quad & A^t y \geq c \\ & y \geq 0, \end{aligned}$$

which is equivalent to :

$$\left.\begin{array}{rr} \max_y & (-b^t)y \\ s.t. & A^t y \geq c \\ & y \geq 0 \end{array}\right\} \rightarrow \left.\begin{array}{rr} \max_y & (-b^t)y \\ s.t. & -A^t y \leq -c \\ & y \geq 0 \end{array}\right\} \tag{7}$$

Thus problem (6) is equivalent to the problem (7). We will find the dual of (6) applying the rule onto the equivalent problem (7):

$$\left.\begin{array}{rr} \min_x & (-c^t)x \\ s.t. & (-A^t)^t x \geq -b \\ & x \geq 0 \end{array}\right\} \rightarrow \left.\begin{array}{rr} \min_x & (-c^t)x \\ s.t. & -Ax \geq -b \\ & x \geq 0 \end{array}\right\} \rightarrow \tag{8}$$

$$\rightarrow \left.\begin{array}{rr} \max_x & (c^t)x \\ s.t. & -Ax \geq -b \\ & x \geq 0 \end{array}\right\} \rightarrow \left.\begin{array}{rr} \max_y & c^t x \\ s.t. & Ax \leq b \\ & x \geq 0 \end{array}\right\} \rightarrow \tag{9}$$

Hence the dual of the dual problem (6) of the initial problem (5) is (9) which is equivalent to (5).

Exercise 4 (Duality Theory 2) *Find the dual problem of the following*

L.P. problem:

$$\begin{array}{lll} \max & x_0 = & 3x_1 + 2x_2 \\ s.t. & & 5x_1 + 2x_2 \le 0 \\ & & 4x_1 + 6x_2 \le 24 \\ & & x_1 + x_2 \ge 1 \\ & & x_1 + 3x_2 = 1 \\ & & x_1 \ge 0. \end{array} \tag{10}$$

Solution : We are going to use rules (1),(2) and (3) from your notes to find the dual of (10).

$\min y_0$	$x_1 \ge 0$	x_2 free	
$y_1 \ge 0$	5	2	≤ 0
$y_2 \ge 0$	4	6	≤ 24
$y_3?$	1	1	≥ 1
$y_4?$	1	3	$= 1$
	≥ 3	?2	

- Since the *3rd* primal constraint is $\ge$ inequality, then *3rd* dual variable y_3 must satisfy $y_3 \le 0$;
- Since the *4th* primal constraint is an equality constraint then *4th* dual variable y_4 must be free – unrestricted in sign;
- Since *2nd* primal variable x_2 is free then *2nd* dual constraint will be an equality.

The new table becomes:

$\min y_0$	$x_1 \ge 0$	x_2 free	
$y_1 \ge 0$	5	2	≤ 0
$y_2 \ge 0$	4	6	≤ 24
$y_3 \le 0$	1	1	≥ 1
$y_4$free	1	3	$= 1$
	≥ 3	=2	

Hence, the dual problem of (10) is:

$$\begin{array}{llrl}
\min & y_0 = & 0y_1 + 24y_2 + y_3 + y_4 & \\
s.t. & & 5y_1 + 4y_2 + y_3 + y_4 & \geq 3 \\
& & 2y_1 + 6y_2 + y_3 + 3y_4 & = 2 \\
& & y_1, y_2 \geq 0 & y_3 \leq 0
\end{array} \tag{11}$$