Integer Programming
Tutorial 1
Answers

Exercise 1 *Olympic Airways Wants to load* n *containers on one of its cargo air planes. Container* j *weighs* a_j *tons and its value is* c_j *dollars. The maximum capacity of the air plane is* b *tons. The airline wants to load the air plane in such a way that the value of its cargo is as large as possible. Formulate the problem as an integer programming problem.*

Solution : Let us define the following binary variable:

$$x_j = \begin{cases} 1, & j \in airplane \\ 0, & otherwise \end{cases} \quad \forall j = 1, 2, ..., n. \tag{1}$$

The loading problem can then be modelled as the following I.P. problem:

$$\begin{array}{rl} \max_x & \sum_{j=1}^{n} c_j x_j \\ s.t. & \sum_{j=1}^{n} a_j x_j \leq b \\ & x \in \{0,1\}^n. \end{array} \tag{2}$$

Exercise 2 *The owner of a big motor company wants to build* $k = 10$ *new factories in different areas. All factories make the same product. The owner has* $n = 15$ *customers. Customer* i *demands* d_i *units of the product. The operating cost of the factory* j *is* $f_j \geq 0$ *and the maximum number of units it can make is* M_j*. The cost of delivering 1 unit from factory* i *to customer* j *is* $c_{i,j}$*.*

Where should the owner build his new factories in order to minimise the delivery cost? Formulate the above problem as an I.P. programming problem.

Solution : Let us define the following variables:

- $y_{i,j}$ denotes the quantity of the product that goes from the factory i to customer j.
- Let x_i be the binary variable with:

$$x_i = \begin{cases} 1, & i \quad is \quad used \\ 0, & otherwise \end{cases} \quad \forall i = 1, 2, ..., k. \tag{3}$$

Then the above problem can be modelled as the following I.P. problem:

$$\begin{array}{rl} \min_x & \{\sum_{i=1}^k \sum_{j=1}^n c_{i,j} y_{i,j} + \sum_{i=1}^k x_i f_i\} \\ s.t. & \sum_{j=1}^n y_{i,j} = d_j \\ & \sum_{j=1}^n y_{i,j} \leq M_i x_i \\ & y_{i,j} \geq 0, \forall i, j \\ & x \in \{0,1\}^k. \end{array} \tag{4}$$

Exercise 3 *Reformulate as IP problem the following problem:*

$$\begin{array}{rl} \min_{x_1,x_2} & 2x_1 - 7x_2 \\ s.t. & 0 \leq x_1 \leq 10 \\ & 0 \leq x_2 \leq 10, \end{array} \tag{5}$$

and at least one of the following holds:

$$-2x_1 + 3x_2 \geq 0$$
$$5x_1 - 4x_2 \geq 0.$$

Solution : We note that the following holds:

$$\max\{2x_1 - 3x_2 | 0 \leq x_1, x_2 \leq 10\} = 20,$$

and

$$\max\{-5x_1 + 4x_2 | 0 \leq x_1, x_2 \leq 10\} = 40.$$

So the problem can be reformulated as the following IP problem:

$$\begin{array}{rl} \min_{x,\delta} & 2x_1 - 7x_2 \\ s.t. & 2x_1 - 3x_2 - 20\delta_1 \leq 0 \\ & -5x_1 + 4x_2 - 40\delta_2 \leq 0 \\ & 0 \leq x_1, x_2 \leq 10 \\ & \delta_1 + \delta_2 \leq 1 \\ & \delta_1, \delta_2 \in \{0,1\}. \end{array} \tag{6}$$

Exercise 4 *Solve the following problem:*

$$\begin{array}{rl} \min_x & c^t x \\ s.t. & Ax = b \\ & x \geq 0 \\ & x_1 \in \{r_1, r_2, ..., r_q\}. \end{array} \tag{7}$$

Solution : Write x_1 as:

$$x_1 = \delta_1 r_1 + \delta_2 r_2 + ... + \delta_q r_q = \sum_{j=1}^{q} \delta_j r_j.$$

δ_j is a binary variable and exactly one $\delta_j = 1$, so

$$\sum_{j=1}^{q} \delta_j = 1.$$

The problem defined as a IP problem is:

$$\begin{array}{rl} \min_x & c^t x \\ s.t. & Ax = b \\ & x \geq 0 \\ & x_1 = \sum_{j=1}^{q} \delta_j r_j \\ & \sum_{j=1}^{q} \delta_j = 1, \delta_j \in \{0, 1\}. \end{array} \tag{8}$$

Exercise 5 *Formulate the following model as a mixed integer programming problem:*

$$\begin{array}{rl} \min_x & \sum_{j=1}^{n} \mathcal{C}_j(x_j) \\ s.t. & Ax \leq b \\ & x \geq 0 \\ & \mathcal{C}_j(x_j) = \left\{ \begin{array}{cc} 0 & x_j = 0 \\ k_j + c_j x_j & x_j > 0 \end{array} \right. \end{array} \tag{9}$$

where $c_j, k_j > 0$ *and* k_j *are called fixed changes.*

Solution : First of all we need to reformulate the objective function. We do so by introducing binary variables $\delta \in \{0, 1\}^n$, so the objective function becomes:

$$\sum_{j=1}^{n} \mathcal{C}_j(x_j) = \sum_{j=1}^{n} (c_j x_j + \delta_j k_j).$$

We also need to add a notional upper bound, so we set M as a large number so that the following constraint is satisfied – $x_j \leq M\delta_j$. The reformulated problem is:

$$\begin{array}{rl} \min_x & \sum_{j=1}^{n}(c_j x_j + \delta_j k_j) \\ s.t. & Ax \leq b \\ & x \geq 0 \\ & x \leq M\delta \\ & \delta \in \{0,1\}^n \end{array} \tag{10}$$