Integer Programming
Tutorial 2
Answers

The Cutting Plane Algorithm Let:

$$a_1 + x_1 + a_2x_2 + ... + a_nx_n = b, \qquad (1)$$

be an equation which is to be satisfied for integers $x_1, x_2, ..., x_n \leq 0$ and let S be a set of possible solutions.

Now let $a_j = [a_j] + f_j$ and $b = [b] + f$ so (1) becomes:

$$\begin{aligned} \sum_{j=1}^{n}([a_j] + f_j)x_j &= [b] + f \Rightarrow \\ \sum_{j=1}^{n} f_jx_j - f &= [b] - \sum_{j=1}^{n}[a_j]x_j. \end{aligned} \qquad (2)$$

For $x \in S$ the right hand side of (2) is integer, so

$$\varsigma = \sum_{j=1}^{n} f_jx_j - f.$$

Also $x \geq 0, x \in S$ so $\varsigma \geq 0$ and

$$\sum_{j=1}^{n} f_jx_j \geq f.$$

If we solved the continuous problem in step 1 and the solution is not an integer. Then there exists a basic variable x_i such that:

$$x_i + \sum_{j \notin I} b_{ij}x_j = b_{i0},$$

where b_{i0} is not an integer. Putting $f_j = b_{ij} - [b_{ij}]$ and $f = b_{i0} - [b_{i0}]$ we deduce that:

$$\sum_{j \notin I} f_j x_j \geq f. \tag{3}$$

Since b_{i0} is not an integer $\Rightarrow f > 0$, then (3) is not satisfied by the current solution so it is a new cut.

Exercise 1 *Solve:*

$$\begin{array}{rl} \max & x_1 + 4x_2 \\ s.t. & 2x_1 + 4x_2 \leq 7 \\ & 10x_1 + 3x_2 \leq 14 \\ & x_1, x_2 \geq 0, integers \end{array} \tag{4}$$

Solution :

$$\begin{array}{rl} \max & x_1 + 4x_2 \\ s.t. & 2x_1 + 4x_2 + x_3 = 7 \\ & 10x_1 + 3x_2 + x_4 = 14 \\ & x_1, x_2, x_3, x_4 \geq 0, integers \end{array} \tag{5}$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	-1	-4					0
x_3	2	*4	1				7
x_4	10	3		1			14
x_0	1		1				7
x_2	$\frac{1}{2}$	1	$\frac{1}{4}$				$*\frac{7}{4}$
x_4	$\frac{17}{2}$		$-\frac{3}{4}$	1			$\frac{35}{4}$

There is a non–integer solution, so we add cut, from the second row:

$$\frac{1}{2}x_1 + \frac{1}{4}x_3 \geq \frac{3}{4}.$$

Adding artificial ς and a slack variable x_5 the following cut is obtained:

$$\frac{1}{2}x_1 + \frac{1}{4}x_3 - x_5 + \varsigma = \frac{3}{4}.$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	1		1				7
x_2	$\frac{1}{2}$	1	$\frac{1}{4}$				$\frac{7}{4}$
x_4	$\frac{17}{2}$		$-\frac{3}{4}$	1			$\frac{35}{4}$
ς	$\frac{1}{2}$		$*\frac{1}{4}$		-1		$\frac{3}{4}$
x_0	-1				4		4
x_2		1			1		1
x_4	*10			1	-3		11
x_3	2		1	0	-4		3
x_0				$\frac{1}{10}$	$\frac{37}{10}$		$\frac{51}{10}$
x_2		1			1		1
x_1	1			$\frac{1}{10}$	-$\frac{3}{10}$		$\frac{11}{10}$
x_3			1	-$\frac{1}{5}$	-$\frac{17}{5}$		$\frac{4}{5}$

A new cut is added now:

$$\frac{1}{10}x_4 + \frac{7}{10}x_5 \geq \frac{3}{4},$$

or, after adding a slack variable:

$$\frac{1}{10}x_4 + \frac{7}{10}x_5 - x_6 + \varsigma = \frac{3}{4}.$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0				$\frac{1}{10}$	$\frac{37}{10}$		$\frac{51}{10}$
x_2		1			1		1
x_1	1			$\frac{1}{10}$	-$\frac{3}{10}$		$\frac{11}{10}$
x_3			1	-$\frac{1}{5}$	-$\frac{17}{5}$		$\frac{4}{5}$
ς				$*\frac{1}{10}$	$\frac{7}{10}$	-1	$\frac{1}{10}$
x_0					3		5
x_2		1			1		1
x_1	1				-1	1	1
x_3			1		-1	- 2	1
x_4				1	7	-10	1

In the lecture notes it is shown graphically how the cuts are added.

Exercise 2 *Solve:*

$$\begin{array}{ll} \max & 3x_1 + 4x_2 \\ s.t. & \frac{2}{5}x_1 + x_2 \leq 3 \\ & \frac{2}{5}x_1 - \frac{2}{5}x_2 \leq 1 \\ & x_1, x_2 \geq 0, integers \end{array} \tag{6}$$

Solution : To ensure that the slacks are also integer variables we eliminate the non–integer coefficients:

$$\begin{array}{ll} \max & 3x_1 + 4x_2 \\ s.t. & 2x_1 + 5x_2 + x_3 = 15 \\ & 2x_1 - x_2 + x_4 = 5 \\ & x_1, x_2, x_3, x_4 \geq 0, integers \end{array} \tag{7}$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0	-3	-4						0
x_3	2	*5	1					15
x_4	2	-2		1				5
x_0	-$\frac{7}{5}$		$\frac{4}{5}$					12
x_2	$\frac{2}{5}$	1	$\frac{1}{5}$					3
x_4	*$\frac{14}{5}$		$\frac{2}{5}$	1				11
x_0			1	$\frac{1}{2}$				$\frac{35}{2}$
x_2		1	$\frac{1}{7}$	-$\frac{1}{7}$				$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$				$\frac{55}{14}$

There is a non–integer solution, so we add cut, from the x_1 row:

$$\frac{1}{7}x_3 + \frac{5}{14}x_4 \geq \frac{13}{14}.$$

Adding artificial ς and a slack variable x_5 the following cut is obtained:

$$\frac{1}{7}x_3 + \frac{5}{14}x_4 - x_5 + \varsigma = \frac{13}{14}.$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0			1	$\frac{1}{2}$				$\frac{35}{2}$
x_2		1	$\frac{1}{7}$	-$\frac{1}{7}$				$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$				$\frac{55}{14}$
ς			$\frac{1}{7}$	*$\frac{5}{14}$	-1			$\frac{13}{14}$
x_0			$\frac{4}{5}$		$\frac{7}{5}$			$\frac{81}{5}$
x_2		1	$\frac{1}{5}$		-$\frac{2}{5}$			$\frac{9}{5}$
x_1	1				1			3
x_4			$\frac{2}{5}$	1	-$\frac{14}{5}$			$\frac{13}{5}$

A new cut is added, based on x_2 row:

$$\frac{1}{5}x_3 + \frac{3}{5}x_5 \geq \frac{4}{5},$$

or, after adding a slack variable:

$$\frac{1}{5}x_3 + \frac{3}{5}x_5 - x_6 + \varsigma = \frac{4}{5}.$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0			$\frac{4}{5}$		$\frac{7}{5}$			$\frac{81}{5}$
x_2		1	$\frac{1}{5}$		-$\frac{2}{5}$			$\frac{9}{5}$
x_1	1				1			3
x_4			$\frac{2}{5}$	1	-$\frac{14}{5}$			$\frac{13}{5}$
ς			$\frac{1}{5}$		*$\frac{3}{5}$	-1		$\frac{4}{5}$
x_0			$\frac{1}{3}$			$\frac{7}{3}$		$\frac{43}{3}$
x_2		1	$\frac{1}{3}$			-$\frac{2}{3}$		$\frac{7}{3}$
x_1	1		-$\frac{1}{3}$			$\frac{5}{3}$		$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		-$\frac{14}{3}$		$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	-$\frac{5}{3}$		$\frac{4}{3}$

Third and final cut is based on x_2 row:

$$\frac{1}{3}x_3 + \frac{1}{3}x_6 \geq \frac{1}{3},$$

or, after adding a slack variable:

$$\frac{1}{3}x_3 + \frac{1}{3}x_6 - x_7 + \varsigma = \frac{1}{3}.$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0			$\frac{1}{3}$			$\frac{7}{3}$		$\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$		$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$		$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$		$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$		$\frac{4}{3}$
ς			$*\frac{1}{3}$			$\frac{1}{3}$	-1	$\frac{1}{3}$
x_0						2	1	14
x_2		1				-1	1	2
x_1	1					2	-1	2
x_4				1		-6	4	5
x_5					1	-2	1	1
x_3			1			1	-3	1

$x^* = (2, 2)$.

Exercise 3 *Solve the following IP problem:*

$$\begin{array}{rl} \max & 5x_1 + 6x_2 \\ s.t. & 0.2x_1 + 0.3x_2 \leq 1.8 \\ & 0.2x_1 + 0.1x_2 \leq 1.2 \\ & 0.3x_1 + 0.3x_2 \leq 2.4 \\ & x_1, x_2 \geq 0, integers \end{array} \tag{8}$$

Solution : $x^* = (3, 4)$.

Exercise 4 (Branch and Bound Method – 1) *Solve the following problem using branch and bound method:*

$$\begin{array}{rl} \max & x_1 + 2x_2 \\ s.t. & 2x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0, integers \end{array} \tag{9}$$

Solution : First solve the continuous relaxation of the given L.P. problem

$$\begin{array}{rl} \max & x_0^{P_0} = x_1 + 2x_2 \\ s.t. & 2x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0; \end{array} \tag{10}$$

Using simplex method the following solution is obtained:

$$x_*^{P_0} = (9.99, 1.33, 4.33).$$

Choose the variable x_2 to branch on. Two new L.P. problems are generated:

$$\begin{array}{rl} \max & x_0^{P_1} = x_1 + 2x_2 \\ s.t. & 2x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_2 \leq [4.33] = 4 \\ & x_1, x_2 \geq 0, \end{array} \tag{11}$$

and

$$\begin{array}{rl} \max & x_0^{P_2} = x_1 + 2x_2 \\ s.t. & 2x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_2 \geq [4.33] + 1 = 5 \\ & x_1, x_2 \geq 0; \end{array} \tag{12}$$

Optimum solutions for both problems are:

$$x_*^{P_1} = (9.5, 1.5, 4),$$

and there is no optimum solution for (12) because it is infeasible. So fathom all this branch. As the optimum solution $x_*^{P_1}$ contains non–integer values we expand (11).

$$\begin{array}{rl} \max & x_0^{P_3} = x_1 + 2x_2 \\ s.t. & 2x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_2 \leq 4 \\ & x_1 \leq [1.5] = 1 \\ & x_1, x_2 \geq 0, \end{array} \tag{13}$$

and

$$\begin{array}{rl} \max & x_0^{P_4} = x_1 + 2x_2 \\ s.t. & 2x_1 + x_2 \leq 7 \\ & -x_1 + x_2 \leq 3 \\ & x_2 \leq 4 \\ & x_1 \geq [1.5] + 1 = 2 \\ & x_1, x_2 \geq 0, \end{array} \tag{14}$$

Solving (13) we obtain $x_*^{P_3} = (9, 1, 4)$. This is the incumbent solution, i.e. the best integer solution found so far. Solving (14) we obtain $x_*^{P_4} = (8, 2, 3)$. As $8 < 9$ and this is a maximisation problem the incumbent solution does not change.

Since both problems (13) and (14) have integer solutions none of them can be expanded. Hence the B&B process has been terminated since there are no more unsolved problems. The optimum solution of the IP problem is the current solution $x_*^{P_3} = (9, 1, 4)$.

Exercise 5 (Branch and Bound – 2) *Consider the following problem:*

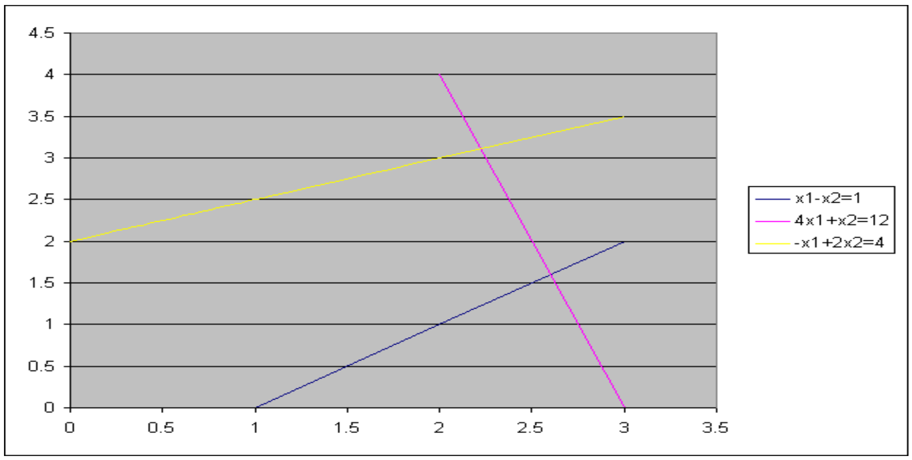
$$\begin{array}{rl} \max & x_0 = 5x_1 + x_2 \\ s.t. & -x_1 + 2x_2 \leq 4 \\ & x_1 - x_2 \leq 1 \\ & 4x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0, integers; \end{array} \tag{15}$$

- *Solve this problem graphically;*
- *Solve LP relaxation. Round this solution to the nearest integer solution and check whether it is feasible. Then enumerate all the rounded solutions, check them for feasibility and calculate* x_0 *for those that are*

feasible. Are any of these feasible rounded solutions optimal for the IP problem?

Solution :

- The feasible region of the IP problem is shown in the next figure.



Graph of the feasible region.

It can be seen that the following pairs of integers are in the feasible region:

$$\begin{matrix} (0,0) & (0,1) & (0,2) \\ (1,0) & (1,1) & (1,2) \\ (2,1) & (2,2) & (2,3), \end{matrix} \tag{16}$$

and the optimal solution is $x_* = (13, 2, 3)$.

- Solving the LP relaxation we obtain $x_* = (14.6, 2.6, 1.6)$.

  Rounding the optimal solution of the LP relaxation 4 pairs of integers are obtained:

$$\begin{matrix} (3,2) & (3,1) \\ (2,2) & (2,1). \end{matrix} \tag{17}$$

  For each of the four pairs we whether they are feasible, and if yes, the objective function value:

rounded solutions	Constraints violated	x_0
(3,2)	3rd	–
(3,1)	2nd, 3rd	–
(2,2)	none	12
(2,1)	none	11

It can be seen that none of the rounded solutions are optimal for the IP problem.