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A LINEAR BINOMIAL RECURRENCE AND THE BELL NUMBERS AND POLYNOMIALS

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Let $B(n)$ denote the n^{th} Bell number. It is well known that $B(n)$ obeys the recurrence relation

$$B(n+1)=\sum_{k=0}^{n}\binom{n}{k}B(k),\qquad n\geq 0. \tag{0.1}$$

The goal of this paper is to study arbitrary functions $f(n)$ that obey (0.1), namely,

$$f(n+1)=\sum_{k=0}^{n}\binom{n}{k}f(k),\qquad n\geq 1. \tag{0.2}$$

By iterating (0.2), $f(n+r)$ can be written as a linear combination of binomial coefficients with polynomial coefficients $A_j^r(n)$, $0\leq j\leq r-1$. The polynomials $A_j^r(n)$ have various interesting properties. This paper provides a sampling of these properties, including two new ways to represent $B(n)$ in terms of $A_j^r(n)$.

1. INTRODUCTION

Our Bell polynomials $\phi_n(t)$ are defined [**1**] by the generating function

$$e^{t(e^x-1)}=\sum_{n=0}^{\infty}\phi_n(t)\,\frac{x^n}{n!} \tag{1.1}$$

and the Bell (or exponential) numbers are given by $B(n)=\phi_n(1)$. These numbers 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, ..., enumerate the number of possible partitions of a set of n objects.

It is well known that

$$\phi_n(t)=\sum_{k=0}^{n}S(n,k)t^k,\qquad n\geq 0, \tag{1.2}$$

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where $S(n,k)$ are the Stirling numbers of the second kind. Moreover,

$$\phi_{n+1}(t) = t\sum_{k=0}^{n}\binom{n}{k}\phi_k(t), \qquad n \geq 0. \tag{1.3}$$

In particular, the Bell numbers satisfy the binomial recurrence

$$B(n+1) = \sum_{k=0}^{n}\binom{n}{k}B(k), \qquad n \geq 0. \tag{1.4}$$

Our work here is motivated by the desire to iterate this expansion and determine the coefficients $K(n,k,r)$ so that

$$B(n+r) = \sum_{k=0}^{n}K(n,k,r)B(k), \qquad n \geq 0, \qquad r \geq 0. \tag{1.5}$$

We will find that a more interesting set of results occurs when we relax the condition $n \geq 0$ in (1.3)–(1.4) and study the general binomial recurrence

$$f(n+1) = \sum_{k=0}^{n}\binom{n}{k}f(k), \qquad n \geq 1, \tag{1.6}$$

where $f(0)$ and $f(1)$ are arbitrary initial values so that $f(n)$ is uniquely determined for $n \geq 1$ by (1.6).

2. GENERAL EXTENSION OF RECURRENCE (1.6)

From (1.6), we have, by successive substitutions

$$\begin{aligned}
f(n+2) &= \sum_{k=0}^{n+1}\binom{n+1}{k}f(k) = f(n+1) + \sum_{k=0}^{n}\binom{n+1}{k}f(k)\\
&= \sum_{k=0}^{n}\binom{n}{k}f(k) + \sum_{k=0}^{n}\binom{n+1}{k}f(k) = \sum_{k=0}^{n}\left(\binom{n}{k}+\binom{n+1}{k}\right)f(k);\\
f(n+3) &= \sum_{k=0}^{n+2}\binom{n+2}{k}f(k) = f(n+2) + (n+2)f(n+1) + \sum_{k=0}^{n}\binom{n+2}{k}f(k)\\
&= \sum_{k=0}^{n}\left((n+3)\binom{n}{k}+\binom{n+1}{k}+\binom{n+2}{k}\right)f(k);
\end{aligned}$$

$$\begin{aligned}
f(n+4) = \sum_{k=0}^{n}&\left(\frac{(n+3)(n+6)}{2}\binom{n}{k}+(n+4)\binom{n+1}{k}\right)f(k)\\
&+\sum_{k=0}^{n}\left(\binom{n+2}{k}+\binom{n+3}{k}\right)f(k);
\end{aligned}$$

and in general, we surmise that $f(n+r)$ may be written using a linear combination of the binomial coefficients $\binom{n+j}{k}$, $j = 0, 1, 2, \ldots, r-1$, with coefficients being polynomials in n. This we summarize in the following theorem.

Theorem 2.1. (*Extension of the general binomial recurrence* (1.6).) *There exist functions $A_j^r(n)$, which are polynomials in n of degree $r-2-j$ when $0 \le j \le r-2$ and $A_{r-1}^r(n) = 1$, such that*

$$f(n+r) = \sum_{k=0}^{n} \sum_{j=0}^{r-1} \left(A_j^r(n) \binom{n+j}{k} \right) f(k), \qquad r \ge 1,\ n \ge 1 \tag{2.1}$$

where the A's satisfy the recurrence relation

$$A_j^{r+1}(n) = \sum_{i=0}^{r-j-1} \binom{n+r}{i} A_j^{r-i}(n), \qquad 0 \le j \le r-1 \tag{2.2}$$

and with $A_r^{r+1}(n) = 1$. We assume $A_j^r(n) = 0$ for $j < 0$ or $j > r-1$.

Proof. We use induction. By successive substitutions, we have

$$\begin{aligned} f(n+r+1) = f(n+r) &+ \binom{n+r}{1} f(n+r-1) + \binom{n+r}{2} f(n+r-2) \\ &+ \cdots + \binom{n+r}{r-1} f(n+1) + \sum_{k=0}^{n} \binom{n+r}{k} f(k). \end{aligned}$$

By applying (2.1) to each term, we find

$$f(n+r+1) = \sum_{k=0}^{n} \left(\sum_{j=0}^{r-1} \sum_{i=0}^{r-j-1} \binom{n+r}{i} A_j^{r-i}(n) \binom{n+j}{k} \right) f(k) + \sum_{k=0}^{n} \binom{n+r}{k} f(k).$$

But we wish to have

$$f(n+r+1) = \sum_{k=0}^{n} \left(\sum_{j=0}^{r} \binom{n+j}{k} A_j^{r+1}(n) \right) f(k),$$

so that by equating the coefficients of $f(k)$, we find that (2.2) allows the induction on r to go through. $\square$

Table 9.1 gives some values of the $A_j^r(n)$. Inspection of this table leads one to suspect the following.

Theorem 2.2. *The $A_j^r(n)$ coefficients satisfy the relation*

$$A_{j+1}^{r+1}(n) = A_j^r(n+1), \qquad j \ge 1, \qquad r \ge 0. \tag{2.3}$$

Proof. We use (2.1) and the simple fact that $(n+1)+r = n+(r+1)$. On the one hand, we have

$$\begin{aligned} f(n+1+r) &= \sum_{k=0}^{n+1}\left(\sum_{j=0}^{r-1}\binom{n+1+j}{k}A_j^r(n+1)\right)f(k) \\ &= \sum_{k=0}^{n+1}\left(\sum_{j=0}^{r}\binom{n+j}{k}A_{j-1}^r(n+1)\right)f(k). \end{aligned}$$

On the other hand, we have

$$f(n+1+r) = \sum_{k=0}^{n}\left(\sum_{j=0}^{r}\binom{n+j}{k}A_j^{r+1}(n+1)\right)f(k),$$

so that by equating the coefficients of $f(k)$, we have $A_{j-1}^r(n+1) = A_j^{r+1}(n)$, which we restate at (2.3). $\square$

From this relation, we see that if we compute $A_0^r(n)$ using (2.2), we may then compute diagonally down the table to find all the other $A's$. This is how we found $A_1^7(n)$, since it is equal to $A_0^6(n+1)$.

We end this section by stating an explicit formula for $f(n+1)$ in terms of the two initial values of $f(0)$ and $f(1)$. This result, which we state as Corollary 2.1, is the basis for the results given in Sections 8 and 9. To prove Corollary 2.1, we simply let $n=1$ in (2.1) and rename r as n.

Corollary 2.1. *Let $f(n)$ be defined by recursion* (1.6). *Then*

$$f(n+1) = f(0)\sum_{j=0}^{n-1}A_j^n(1) + f(1)\sum_{j=0}^{n-1}(j+1)A_j^n(1), \qquad r \geq 1 \tag{2.4}$$

Remark 2.1. In view of (2.3), we can rewrite (2.4) as

$$f(n+1) = f(0)\sum_{j=0}^{n-1}A_{j+1}^{n+1}(0) + f(1)\sum_{j=0}^{n-1}(j+1)A_{j+1}^{n+1}(0), \qquad r \geq 1$$

3. GENERALIZING THE NUMBER OF INITIAL CONDITIONS

We have assumed that (1.6) held for all $n \geq 1$. For the Bell numbers, it holds for all $n \geq 0$. It is natural to ask what we can say when we only assume that (1.6) holds for all $n \geq a$, where a is a non-negative integer, and $f(0), f(1), \ldots, f(a)$ are prescribed values. It is easy to see that we have the following result, which is a generalization of Theorem 2.1 and Corollary 2.2.

Theorem 3.1. *Let $f(0), f(1), \ldots, f(a)$ be prescribed values and let*

$$f(n+1) = \sum_{k=0}^{n}\binom{n}{k}f(k), \qquad n \geq a, \tag{3.1}$$

where $a \geq 0$ is any given integer. Then,

$$f(n+r) = \sum_{k=0}^{n} \left(\sum_{j=0}^{r-1} A_j^r(n) \binom{n+j}{k} \right) f(k), \qquad r \geq 1,\, n \geq a \tag{3.2}$$

where the A coefficients are the same as in Theorem 2.1. Moreover, by letting $n = a$ in (3.2) and then renaming r as n, we note

$$f(n+a) = \sum_{k=0}^{a} \left(\sum_{j=0}^{n-1} A_j^n(a) \binom{a+j}{k} \right) f(k), \qquad n \geq 1. \tag{3.3}$$

Example. With $a = 2$, Equation (3.3) implies that $f(n+2) = C(n)f(0) + D(n)f(1) + E(n)f(2)$, where for $n \geq 1$,

$$C(n) = \sum_{j=0}^{n-1} A_j^n(2),\;\; D(n) = \sum_{j=0}^{n-1} (2+j) A_j^n(2),\;\; E(n) = \sum_{j=0}^{n-1} \frac{(2+j)(1+j)}{2} A_j^n(2).$$

4. BELL NUMBERS

If we take $f(n) = B(n)$, then, as we noted in the introduction, relation (1.6) is valid for all $n \geq 0$. Thus, (2.1) becomes

$$B(n+r) = \sum_{k=0}^{n} \left(\sum_{j=0}^{r-1} A_j^r(n) \binom{n+j}{k} \right) B(k), \qquad r \geq 1,\, n \geq 0. \tag{4.1}$$

Setting $n = 0$ in (4.1), we get the explicit formula,

$$B(r) = \sum_{j=0}^{r-1} A_j^r(0), \qquad r \geq 1. \tag{4.2}$$

This affords a new representation of the Bell numbers using the array of A coefficients. Thus, we can use the row sums Table 9.2 to obtain the Bell Numbers, since the n^{th} row sum is the n^{th} Bell number. We have been unable to find this result in the literature and should note that rows and diagonals of the array of $A_j^r(0)$ numbers are conspicuously absent from the Online Encyclopedia of Integer Sequences (OEIS). We have entered the sequence $A_0^r(0)$, denoted $A(r)$, as A040027 in OEIS, where one can now find references to other manifestations of this sequence.

A useful simplification of (2.4) by way of (4.2) is as follows:

Corollary 4.1. *Let $f(n)$ satisfy (1.6). Then for all $r \geq 1$,*

$$f(r) = f(0)(B(r) - A(r)) + f(1)A(r). \tag{4.3}$$

In fact, Equation (2.4) holds for all $r \geq 0$ if we define $A(0) = 0$.

Proof. We have from (2.4)

$$
\begin{aligned}
f(r+1) &= f(0)\sum_{j=0}^{r-1} A_j^r(1) + f(1)\sum_{j=0}^{r-1}(j+1)A_j^r(1) \\
&= f(0)\sum_{j=0}^{r-1} A_{j+1}^{r+1}(0) + f(1)\sum_{j=0}^{r-1}(j+1)A_{j+1}^{r+1}(0), \text{ by (2.3)} \\
&= f(0)\sum_{j=1}^{r} A_j^{r+1}(0) + f(1)\sum_{j=0}^{r}(j)A_j^{r+1}(0) \\
&= f(0)(B(r+1)-A(r+1)) + f(1)\sum_{j=0}^{r}(j)A_j^{r+1}(0), \text{ by (4.2) } (*)
\end{aligned}
$$

Since for the Bell numbers $f(0) = f(1) = 1$ and $f(r+1) = B(r+1)$, we must have

$$\sum_{j=1}^{r} jA_j^{r+1}(0) = A(r+1). \tag{4.4}$$

Substituting (4.4) into $(*)$ proves the desired result. □

Remark. Another interesting formula, whose proof is an adaptation of the methodology used for Corollary (4.1), is

$$\sum_{j=1}^{r-1} j^2 A_j^r(0) = A(r) + 2A_1^r(0).$$

In a later paper we will discuss the series $\sum_{j=1}^{r-1} j^p A_j^r(0)$.

We are next able to offer another remarkable formula for the Bell numbers. This formula is given in Corollary 4.2.

Corollary 4.2. *Let $B(n)$ be the n^{th} Bell number. Then*

$$B(n) = A_0^{n+1}(-1), \qquad n \geq 0. \tag{4.5}$$

Proof. In (2.2), set $j = 0, n = -1$, and then replace r by $r+1$. We get

$$A_0^{r+2}(-1) = \sum_{i=0}^{r}\binom{r}{i}A_0^{r+1-i}(-1) = \sum_{i=0}^{r}\binom{r}{r-i}A_0^{i+1}(-1),$$

so that we have

$$A_0^{r+2}(-1) = \sum_{i=0}^{r}\binom{r}{i}A_0^{i+1}(-1), \text{ with } A_0^1(-1) = 1. \tag{4.6}$$

We know from (1.4) that the Bell numbers $B(n)$ satisfy

$$B(r+1) = \sum_{i=0}^{r} \binom{r}{i} B(i), \qquad n \geq 0,$$

being the unique solution of this with $B(0) = 1$, whence $A_0^{i+1}(-1)$ must be identical to $B(i)$. □

Remark 4.2. By applying (2.3) to (4.5) we obtain

$$B(n-j) = A_{j-1}^{n}(-1), \qquad n \geq j \geq 1.$$

We propose calling the $A_j^r(n)$ Binomial Recurrence Coefficients. Undoubtedly, they possess other remarkable properties, intimately associated with the Bell Numbers. The remaining sections of this paper expound on the various properties of the Binomial Recurrence Coefficients.

5. BELL POLYNOMIALS

We wish to next indicate how we can extend the concept of the Binomial Recurrence Coefficients to expansions of the Bell polynomials. If we suppose that

$$\phi_{n+1}(t) = t \sum_{k=0}^{n} \binom{n}{k} \phi_k(t), \qquad n \geq 0, \tag{5.1}$$

and iterate the relation, we find

Theorem 5.1. *There exist coefficients $A_j^r(n,t)$ such that*

$$\phi_{n+r}(t) = \sum_{k=0}^{n} \left(\sum_{j=0}^{r-1} A_j^r(n,t) \binom{n+j}{k} \right) \phi_k(t), \qquad n \geq 0,\ r \geq 1, \tag{5.2}$$

with a recurrence relation parallel to (2.2), *which is*

$$A_j^{r+1}(n,t) = t \sum_{i=0}^{r-j-1} \binom{n+r}{i} A_j^{r-i}(n,t), \qquad 0 \leq j \leq r-1 \tag{5.3}$$

and with $A_r^{r+1}(n,t) = 1$. Also we assume $A_j^r(n,t) = 0$ for $j < 0$ or $j > r-1$.

The formula corresponding to (2.3) is

$$A_{j+1}^{r+1}(n,t) = A_j^r(n+1,t), \qquad j \geq 0,\ r \geq 0. \tag{5.4}$$

Equation (4.2) extends to

$$\phi_r(t) = \sum_{j=0}^{r-1} A_j^r(0,t), \qquad r \geq 1. \tag{5.5}$$

Here are examples of (5.2) for $r = 2, 3$, and 4:

$$\phi_{n+2}(t) = \sum_{k=0}^{n} \phi_k(t)\left(t^2\binom{n}{k} + t\binom{n+1}{k}\right),$$

$$\phi_{n+3}(t) = \sum_{k=0}^{n} \phi_k(t)\left(\left(t^3 + (n+2)t^2\right)\binom{n}{k} + t^2\binom{n+1}{k} + t\binom{n+2}{k}\right),$$

$$\phi_{n+4}(t) = \sum_{k=0}^{n} \phi_k(t)\left(\left(t^4 + (2n+5)t^3 + \frac{(n+2)(n+3)}{2}t^2\right)\binom{n}{k}\right)$$
$$+ \sum_{k=0}^{n} \phi_k(t)\left(\left(t^3 + (n+3)t^2\right)\binom{n+1}{k} + t^2\binom{n+2}{k} + t\binom{n+3}{k}\right).$$

Evidently one may prove also that $A_j^r(n,t)$ is a polynomial of degree r in t. We shall name them Binomial Recurrence Polynomials.

6. INVERSE SERIES RELATIONS

We can learn more about the array of coefficients $A_j^r(n)$ by looking at inverse series relations. We announce the following result whose proof we omit:

Theorem 6.1.

$$f(r) = \sum_{j=0}^{r-1} A_j^r(0) g(j), \qquad r \geq 1 \tag{6.1}$$

and only if

$$g(r) = f(r+1) - \sum_{j=1}^{r} \binom{r}{j} f(j), \quad \textit{with} \quad g(0) = f(1). \tag{6.1}$$

From this point of view, the array $A_j^r(0)$ arises as the inverse of the simple binomial matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & ... \\ -1 & 1 & 0 & 0 & 0 & 0 & ... \\ -2 & -2 & 1 & 0 & 0 & 0 & ... \\ -3 & -3 & -1 & 1 & 0 & 0 & ... \\ -4 & -6 & -4 & -1 & 1 & 0 & ... \\ -5 & -10 & -10 & -5 & -1 & 1 & ... \\ ... & ... & ... & ... & ... & ... & ... \end{bmatrix}.$$

Theorem 6.1 reveals how the array $A_j^r(0)$ comes from the inversion of the Bell number recurrence (1.4). Indeed, set $g(j) = 1$ identically. Then (6.1) becomes our earlier Bell number formula (4.2), where now $f(r) = B(r)$; by (6.2), the inverse is

$$1 = B(r+1) - \sum_{j=1}^{r} \binom{r}{j} B(j).$$

In as much as $B(0) = 1$, this just becomes

$$B(r+1) = \sum_{j=1}^{r} \binom{r}{j} B(j),$$

which is exactly (1.4). Thus, we may think of (4.2) as an inverse of the classical (1.4).

Here is another instructive example of Theorem 6.1. We choose $f(j) = (-1)^j$. Then, Theorem 6.1 yields

$$A_0^r(0) = (-1)^{r-1} + 2 \sum_{1 \le 2j-1 \le r-1} A_{2j-1}^r(0), \tag{6.3}$$

and since $A_{r-1}^r(0) = A_{r-2}^r(0) = 1$, we may rewrite this as

$$A_0^r(0) = 1 + 2 \sum_{1 \le 2j-1 < r-1} A_{2j-1}^r(0). \tag{6.4}$$

Equation (6.4) provides a useful check of rows in Table 9.1.

Relation (6.3), or (6.4), gives immediate proof that $A(r)$ is odd for all $r \ge 1$.

Combining (6.3) with (4.2), we find the Bell number formula

$$2B(r) = A(r) + (-1)^r + 2 \sum_{0 \le 2j \le r-1} A_{2j}^r(0), \tag{6.5}$$

which we may rewrite as

$$2B(r) = 3A(r) + (-1)^r + 2 \sum_{1 \le j \le (r-1)/2} A_{2j}^r(0). \tag{6.6}$$

From (6.5) and (4.2) we find the formula

$$\sum_{j=0}^{r-1} (-1)^j A_j^r(0) = B(r) - A(r) - (-1)^r, \qquad r \ge 1. \tag{6.7}$$

We close this section by noting an unusual binomial identity that flows from the inverse series relation pair (6.1)-(6.2). Choosing $f(r) = \binom{x}{r-1}$. Relations (6.1) and (6.2) show that $g(r) = \binom{x}{r} - \binom{x+r}{r-1}$. We then obtain the identity

$$\sum_{j=0}^{r-1} A_j^r(0) \binom{x}{j} = \binom{x}{r-1} + \sum_{j=0}^{r-1} A_j^r(0) \binom{x+j}{j-1}, \qquad r \ge 1. \tag{6.8}$$

7. GENERATING FUNCTIONS

For the general case governed by (1.6), we develop the exponential generating function for $f(n)$. We have

$$\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} f(n)\,\frac{x^n}{n!} = f(0) + f(1)x + \sum_{n=1}^{\infty} f(n+1)\,\frac{x^{n+1}}{(n+1)!} \\
&= f(0) + (f(1) - f(0))x + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} f(k)\,\frac{x^{n+1}}{(n+1)!} \\
&= f(0) + \big(f(1) - f(0)\big)x + \int_0^x \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} f(k)\,\frac{t^n}{n!}\,\mathrm{d}t \\
&= f(0) + \big(f(1) - f(0)\big)x + \int_0^x \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^n}{n!}\, f(k)\,\frac{t^k}{k!}\,\mathrm{d}t \\
&= f(0) + \big(f(1) - f(0)\big)x + \int_0^x e^t F(t)\,\mathrm{d}t,
\end{aligned}$$

whence, we have the differential equation

$$F'(x) = f(1) - f(0) + e^x F(x), \tag{7.1}$$

which is easily solved using the integrating factor $\exp(-e^x + 1)$, so that, we obtain the general solution

$$F(x) = \big(f(1) - f(0)\big)e^{e^x-1}\int_0^x e^{-e^t+1}\,\mathrm{d}t + Ce^{e^x-1}.$$

Letting $x = 0$, we find $C = f(0)$, and obtain the specific solution

$$F(x) = \big(f(1) - f(0)\big)e^{e^x-1}\int_0^x e^{1-e^t}\,\mathrm{d}t + f(0)e^{e^x-1}. \tag{7.2}$$

By (4.3) we have

$$f(n) = f(0)B(n) + \big(f(1) - f(0)\big)A(n), \qquad n \geq 0,$$

so that

$$\sum_{n=0}^{\infty} f(n)\,\frac{x^n}{n!} = f(0)\sum_{n=0}^{\infty} B(n)\,\frac{x^n}{n!} + \big(f(1) - f(0)\big)\sum_{n=0}^{\infty} A(n)\,\frac{x^n}{n!}\,,$$

which is to say

$$F(x) = f(0)e^{e^x-1} + \big(f(1) - f(0)\big)\sum_{n=0}^{\infty} A(n)\,\frac{x^n}{n!},$$

and comparing this with (7.2), we find that

$$\sum_{n=0}^{\infty} A(n)\,\frac{x^n}{n!} = e^{e^x-1}\int_0^x e^{1-e^t}\,\mathrm{d}t, \tag{7.3}$$

which gives an integral generating function for the sequence $A(n)$. This may be expanded for as many terms as one likes, since the coefficients in the expansion of $\exp(e^x-1)$ use the BELL numbers, and the coefficients in the expansion of $\exp(1-e^x)$ are well-known. For the former see sequence M1484 in SLOANE [**4**] and for the latter see sequence M1913 in [**4**]. Further detailed references may be found in GOULD [**2**]. To show some details, write

$$e^{1-e^x} = \sum_{n=0}^{\infty} D(n)\,\frac{x^n}{n!}. \tag{7.4}$$

The first seventeen $D's$ (as tabulated in [**5**]) are as follows: 1, -1, 0 1, 1, -2, -9 $-$ 9, 50, 267, 413, -2180, -17731, -50533, 110176, 1966797, 9938669. SLOANE's printed book [**4**] only gives the absolute values. A known infinite series formula for these is

$$D(n) = e\sum_{k=0}^{\infty}(-1)^k\,\frac{k^n}{k!}, \tag{7.5}$$

just as

$$B(n) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!} \tag{7.6}$$

which is known as DOBINSKI's formula.

The $D's$ satisfy a recursion similar to (1.4), but with a minus sign inserted in front. In fact, from (5.1) with $t=-1$,

$$D(n+1) = -\sum_{j=0}^{n}\binom{n}{j}D(j), \tag{7.7}$$

which allows for easy computation of them. Using the $D's$ and carrying out the integration in (7.3), and multiplying by (1.1) with $t=1$ there, one can easily multiply the resulting series and recover as many terms of the sequence $A(n)$ as desired. For $n=0,1,2,\dots$ this yields $0,1,1,3,9,31,121,523,2469,\dots$ as expected.

Another known way [**5**] to relate the B and D sequences is the formula expressing their reciprocal relationship

$$\sum_{k=0}^{n}\binom{n}{k}B(k)D(n-k) = \delta_0^n. \tag{7.8}$$

Now as a matter of fact, we can obtain an explicit formula for the A coefficients, by recalling (1.1) and (7.3). Substituting the generating functions, carrying out the integration, and simplifying, we get from (7.3) the very nice formula

$$A(n+1) = \sum_{k=0}^{n} \binom{n+1}{k+1} B(n-k)D(k), \qquad n \geq 0. \tag{7.9}$$

Since

$$\phi_n(1) = B(n) = \sum_{k=0}^{n} S(n,k)$$

and

$$\phi_n(-1) = D(n) \sum_{k=0}^{n} (-1)^k S(n,k),$$

we could then write a more complicated formula giving the $A(r)$ coefficients using Stirling numbers of the second kind and binomial coefficients.

8. OPEN QUESTION: A LIMIT CONJECTURE

We end this paper by comparing the Bell number sequence $\{B(n)\}_{n=0}^{\infty}$ to the sequence $\{f(n)\}_{n=0}^{\infty}$, where $f(n)$ obeys (1.6). This analysis leads to a conjecture, whose proof we leave as an open research question. Corollary 4.1 implies that

$$\begin{aligned} f(n) &= \big(B(n) - A(n)\big) f(0) + A(n) f(1) \\ B(n) &= \big(B(n) - A(n)\big) B(0) + A(n) B(1). \end{aligned}$$

Since both $f(n)$ and $B(n)$ have similar structure, it is natural to look at the ratio between these two terms in the form

$$\lim_{n\to\infty} \frac{f(n)}{B(n)}. \tag{8.1}$$

A particular example of (8.1) is when $f(0) = 1$, and $f(1) = 3$. In this case, $\{f(n)\}_{n=0}^{\infty}$ is 1, 3, 4,11, 33, 114, 445, 1923, 9078, 46369, 254297, 148796, ... Note that, $f(n) = B(n) + 2A(n)$ and the value of (8.1) is $2.1926\ldots$. In general, since $B(0) = 1 = B(1)$, (8.1) can be rewritten as follows:

$$\lim_{n\to\infty} \frac{f(n)}{B(n)} = f(0) + \big(f(1) - f(0)\big)L, \tag{8.2}$$

where

$$L = \lim_{n\to\infty} \frac{A(n)}{B(n)}. \tag{8.3}$$

Thus, $\lim\limits_{n\to\infty} \dfrac{f(n)}{B(n)}$ if and only if $\lim\limits_{n\to\infty} \dfrac{A(n)}{B(n)}$ exists. Empirical evidence, via a Maple computer program, suggest the following

Conjecture 8.1.

$$\lim_{n\to\infty}\frac{A(n)}{B(n)}=\int_0^\infty \frac{e^{-u}}{1+u}\,\mathrm{d}u=-eEi(-1).$$

Remark 8.1. Recall that $-eEi(-1)=0.596347362323194074341078 49...$, which is known as Gompertz constant. See A040027 in OEIS, with reference to E. Weisstein's World of Mathematics.

To gain some insight about Conjecture 8.1, note that (7.3) may be rewritten in the following form

$$\sum_{n=0}^{\infty}A(n)\frac{x^n}{n!}=\sum_{n=0}^{\infty}B(n)\,\frac{x^n}{n!}\int_0^x e^{1-e^t}\,\mathrm{d}t, \tag{8.4}$$

so that

$$\frac{\sum\limits_{n=0}^{\infty}A(n)\,\frac{x^n}{n!}}{\sum\limits_{n=0}^{\infty}B(n)\,\frac{x^n}{n!}}=\int_0^x e^{1-e^t}\,\mathrm{d}t. \tag{8.5}$$

We observe next that

$$\int_0^\infty \frac{e^{-u}}{1+u}\,\mathrm{d}u=\int_1^\infty \frac{e^{1-t}}{t}\,\mathrm{d}t=\int_0^\infty \frac{e^{1-e^x}e^x}{e^x}\,\mathrm{d}x=\int_0^\infty e^{1-e^x}\,\mathrm{d}x,$$

so that relation (7.3) leads to

$$\lim_{x\to\infty}\frac{\sum\limits_{n=0}^{\infty}A(n)\,\frac{x^n}{n!}}{\sum\limits_{n=0}^{\infty}B(n)\,\frac{x^n}{n!}}=\int_0^\infty \frac{e^{-u}}{1+u}\,\mathrm{d}u. \tag{8.6}$$

By combining Conjecture 8.1 with (8.6) we observe that

$$\lim_{x\to\infty}\frac{\sum\limits_{n=0}^{\infty}A(n)\,\frac{x^n}{n!}}{\sum\limits_{n=0}^{\infty}B(n)\,\frac{x^n}{n!}}=\lim_{n\to\infty}\frac{A(n)}{B(n)}, \tag{8.7}$$

which is a kind of extended l'Hospital Rule.

For finite degree polynomials, we know that

$$\lim_{x\to\infty}\frac{\sum\limits_{k=0}^{n}A(k)\,\frac{x^k}{k!}}{\sum\limits_{k=0}^{n}B(k)\,\frac{x^k}{k!}}=\frac{A(n)}{B(n)}. \tag{8.8}$$

However, only under special conditions could (8.8) hold when we allow $n \to \infty$. Finding necessary and sufficient conditions are what is needed to prove Conjecture 8.1. We suspect that when the functions defined by the series in (8.8) are doubly exponential in nature, like $\exp(e^t - 1)$, as in the case of the BELL numbers, then we might expect (8.8) to be true.

9. TABLES

Table 9.1: Some Values of $A_j^r(n)$ Polynomials

- $A_0^1(n) = 1, \quad A_0^2(n) = A_1^2(n) = 1, \quad A_0^3(n) = n+3, \quad A_1^3(n) = A_2^3(n) = 1,$
- $A_0^4(n) = \dfrac{(n+3)(n+6)}{2}, \quad A_1^4(n) = n+4, \quad A_2^4(n) = A_3^4(n) = 1,$
- $A_0^5(n) = \dfrac{(n+3)(n^2+18n+62)}{6}, \quad A_1^5(n) = \dfrac{(n+4)(n+7)}{2}, \quad A_2^5(n) = n+5,$
- $A_3^5(n) = A_4^5(n) = 1, A_0^6(n) = \dfrac{(n+3)(n^3+43n^2+386n+968)}{24},$
- $A_1^6(n) = \dfrac{(n+4)(n^2+20n+81)}{6}, \quad A_2^6(n) = \dfrac{(n+5)(n+8)}{2}, \quad A_3^6(n) = n+6,$
- $A_4^6(n) = A_5^6(n) = 1, \quad A_0^7(n) = \dfrac{(n+3)(n^4+97n^3+1684n^2+10328n+20920)}{120},$
- $A_1^7(n) = \dfrac{(n+4)(n^3+46n^2+475n+1398)}{24}, \quad A_2^7(n) = \dfrac{(n+5)(n^2+22n+102)}{6},$
- $A_3^7(n) = \dfrac{(n+6)(n+9)}{2}, \quad A_4^7(n) = n+7, \quad A_5^7(n) = A_6^7(n) = 1$

Table 9.2: Array of $A_j^r(0)$ Coefficients

1									
1	1								
3	1	1							
9	4	1	1						
31	14	5	1	1					
121	54	20	6	1	1				
523	233	85	27	7	1	1			
2469	1101	400	125	35	8	1	1		
12611	5625	2046	635	175	44	9	1	1	
69161	30846	11226	3488	952	236	54	10	1	1

REMARK 9.1. For this and all ensuing tables, rows correspond to $r = 1, 2, 3, \ldots$ and diagonals to $j = 0, 1, \ldots, r-1$. Note that the row sums are the Bell numbers.

Table 9.3: Array of $A_j^r(1)$ Coefficients

1						
1	1					
4	1	1				
14	5	1	1			
54	20	6	1	1		
233	85	27	7	1	1	
1101	400	125	35	8	1	1

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