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FAREY LINES DEFINING FAREY DIAGRAMS AND APPLICATION TO SOME DISCRETE STRUCTURES

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The aim of the paper is to study some of the analytical properties of Farey diagrams of order (m, n), which are associated to the (m, n)-cubes, that is the pieces of discrete planes, occuring in discrete mathematics. We give a closed formula for the number of Farey lines defining Farey diagrams. This number asymptotically behaves as $mn(m + n)/\zeta(3)$. Then we establish the relation with some discrete structures in the field of discrete geometry in particular.

1. INTRODUCTION

In [**3**], one of the strategies for the enumeration of discrete pieces of planes, was to estimate, in the most precise possible way, the number of vertices in a diagram called a Farey diagram. The upper bound for the cardinality of the pieces of discrete planes (or $(m, n) - cubes$) obtained in [**3**], is a homogeneous polynomial of degree 8 : $m^3n^3(m + n)^2$. The strategy used in this previous article to obtain the upper bound was to study the cardinality of Farey lines. It was not clear whether the degree 3 was optimal or not. We show that the degree 3 is optimal for the cardinality of Farey lines. We obtain a new closed and exact formula for computing this number of straight lines, and we derive from this an asymptotic value, which definitely establishes that the degree 3 is optimal.

2. DEFINITIONS

Farey diagrams are involved in several fields of imagery, surgery, recognition, and in Computer Science, but some problems related to Farey diagrams remain.

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Tomás proves in [**15**] and [**14**] that there is an important link between Accelerator Physics and Farey diagrams. We notice that the Farey diagram of order (n, n) has the same degree as the resonance diagram of order n. The asymptotic behaviour of the two different structures only differs by a factor. There seem to be some similarities between (m, n)-cubes, which we redefine in this article, and threshold functions on a two-dimensional rectangular grid, for which asymptotic values have been derived in [**5**]. They are also used when we study the preimage of a discrete piece of the plane in discrete mathematics. The Farey diagram for discrete segments was also studied by McIlroy in [**10**]. In [**10**], it was shown that the number of straight lines with some particular conditions (which we call Farey lines) is $\mathcal{O}(mn(m+n))$ In this work, we derive an asymptotic value whose major term is $mn(m+n)/\zeta(3)$.

Let $[\![-m, m]\!]$ denote the set of integers between $-m$ and m.

Definition 1 (Farey lines of order (m, n))**.** *A Farey line of order (m, n) is a line whose equation is $u\alpha + v\beta + w = 0$ with $(u, v, w) \in [\![-m, m]\!] \times [\![-n, n]\!] \times \mathbb{Z}$, and which has at least 2 intersection points with the frontier of $[0, 1]^2$. We denote the set of Farey lines of order (m, n) by $DF(m, n)$.*

The term Farey sets was used by Rémy and Thiel in [**13**] to talk about some subsets of $\mathbb{Q}^2$ of irreductible points $(y/x, z/x)$ between 0 and 1 whose numerators and denominators do not exceed n. It is interesting to notice that this definition which was introduced in the Ph.D. thesis of Thiel makes us think of the x-intercept and y-intercept of Farey lines, of coordinates $(-w/u, -w/v)$. These lines play an essential role in discrete geometry, when we consider discrete planes. It explains why they are so important in theoretical computer sciences and discrete mathematics.

Definition 2 (Farey vertex)**.** *A Farey vertex of order (m, n) is the intersection of two Farey lines.*

We will denote the set of Farey vertices of order (m, n), obtained as intersection points of Farey lines of order (m, n), by $SF(m, n)$.

Definition 3 (Farey diagrams for the discrete pieces of planes of order (m, n) (or (m, n)-cubes))**.** *The Farey diagram for the (m, n)-cubes of order (m, n) is the diagram composed of Farey connected component of order (m, n).*

We use $\lfloor\ \rfloor$ and $\langle\ \rangle$ to (respectively) denote the integer part and the fractional part of a real number. If a and b are two integers, $a \wedge b$ denotes the greatest common divisor of a and b. We let φ denote Euler's totient function.

Definition 4 (Farey sequences of order n [**4**])**.** *The Farey sequence of order n is*

$$F_n = \{0\} \cup \{p/q \mid 1 \le p \le q \le n, p \wedge q = 1\}.$$

We mention [**4**] as a forthcoming modern reference work on Farey sequences. Several standard variants of the notion of Farey diagram are mentioned there, so we use in this paper a non-standard definition.

Definition 5 (Generalized Farey sequences [**1**])**.** *Let m and n be two integers such that $m \leq n$. The generalized Farey sequence is $F_{m,n} = \{i/j \in F_n \mid i \leq m\}$.*

Properties of generalized Farey sequences are also studied in [**6**]. A more precise study is given in [**9**], and formulas for cardinality of generalized Farey sequences are derived. An application of generalized Farey sequences to X-rays and Radon transform can be found in [**12**].

Definition 6 (Farey edge)**.** *A Farey edge of order (m, n) is an edge of the Farey diagram of order (m, n). We denote the set of Farey edges by $EF(m, n)$.*

Definition 7 (Farey graph)**.** *The Farey graph $GF(m, n)$ of order (m, n) is the graph $(SF(m, n), EF(m, n))$.*

For positive integers m, n, we let $\mathcal{F}_{m,n}$ denote the set $[\![0, m-1]\!] \times [\![0, n-1]\!]$.

Definition 8 ((m, n)-pattern [**3**])**.** *Let m and n be two positive integers. A (m, n)-pattern is a map $w : \mathcal{F}_{m,n} \longrightarrow \mathbb{Z}$. The size of the (m, n)-pattern w is denoted as $m \times n$. The set of the (m, n)-patterns will be denoted by $\mathcal{M}_{m,n}$.*

Definition 9 ((m, n)-cube [**3**], see Figure 1)**.** *The (m, n)-pattern $w_{i,j}(\alpha, \beta, \gamma)$ at the position (i, j) of a discrete plane $\mathcal{P}_{\alpha,\beta,\gamma}$ is the (m, n)-pattern w defined by:*

$$\begin{aligned} w(i', j') &= p_{\alpha,\beta,\gamma}(i + i', j + j') - p_{\alpha,\beta,\gamma}(i, j) \\ &= \lfloor \alpha(i + i') + \beta(j + j') + \gamma \rfloor - \lfloor \alpha i + \beta j + \gamma \rfloor, \end{aligned}$$

for $(i', j') \in \mathcal{F}_{m,n}$.

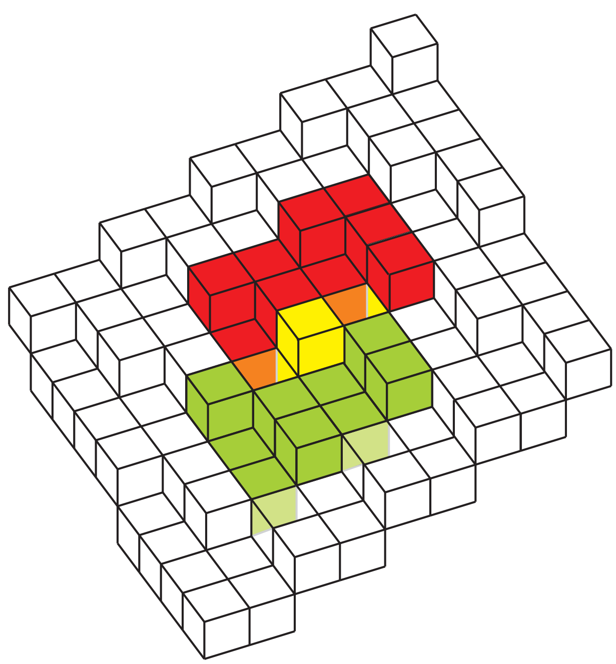

Figure 1. Example of two $(4, 3)$-cubes (red and green)

3. PROPERTIES

Now we give some important lemmas:

Proposition 1 ([**8**]). *Let $d \in \mathbb{N}^*$. If $d \mid n$, then the number of integers w such that $1 \leq w \leq n$ and $w \wedge d = 1$ is $(n/d)\varphi(d)$.*

Proof. Indeed, as $d \mid n$, we have $n = dk$. If $1 \leq w \leq d$, then we have $\varphi(d)$ integers for w for which $w \wedge d = 1$. If $d+1 \leq w \leq 2d$, then $1 \leq w - d \leq d$, so this gives again $\varphi(d)$ integers for which $w \wedge d = 1$. In fact, for each $j \in [\![1, k]\!]$, if $(j-1)d + 1 \leq w \leq jd = n$, then we have $\varphi(d)$ such integers w for which for which $w \wedge d = 1$, because $1 \leq w - (j-1)d \leq d$, and because $[w - (j-1)d] \wedge d = 1$. Thus, altogether, we have $(n/d)\varphi(d)$ such w's.

Proposition 2. *Let $u \leq -1$ and $v \geq 1$ be two integers such that $u \wedge v = d$. The number of integers w such that $w \wedge d = 1$ and such that $u \leq w \leq v$ is :*

$$v - u + 1 \text{ if } d = 1,$$

or

$$((v-u)/d)\varphi(d) \text{ if } d > 1.$$

Proof.

1. If $d = 1$, then all the integers w such that $u \leq w \leq v$ have $w \wedge d = 1$. There are $v - u + 1$ such integers.
2. If $d > 1$, then $0 \wedge d \neq 1$. By Proposition 1, if we consider $u \leq w \leq v$ with $w \neq 0$, then since $u \wedge v \mid u$ and $u \wedge v \mid v$, we have $(-u/d)\varphi(d)$, and $(v/d)\varphi(d)$ suitable integers, so overall, we have $((v-u)/d)\varphi(d)$ such integers.

4. AN EXACT AND CLOSED FORMULA FOR THE ENUMERATION OF FAREY LINES

Now, we seek to estimate the number of Farey lines. We consider that a strictly increasing (respectively, decreasing) line has the form : $\beta(\alpha) = a\alpha + b$, where $a > 0$ (respectively, $a < 0$). In the other cases, the line is such that β (respectively, α) is constant, that is, a horizontal (respectively, vertical) line.

Theorem 1. *The number of lines defining the Farey diagram for the (m, n)-cubes of order (m, n) is*

$$|DF(m,n)| = |F_m| + |F_n| + 2 \sum_{\substack{1 \leq u \leq m \\ 1 \leq v \leq n \\ u \wedge v = d \geq 1}} \frac{u+v}{d}\,\varphi(d) - 2A_{m,n},$$

where $|F_m|$ denotes the cardinality of the Farey sequence of order m, and $A_{m,n}$ denotes the number of coprime pairs $(u, v) \in \mathbb{Z}^2$ with $1 \leq u \leq m$ and $1 \leq v \leq n$.

Proof. A Farey line has an equation of the form $u\alpha + v\beta + w = 0$ with $(u, v, w) \in [\![-m, m]\!] \times [\![-n, n]\!] \times \mathbb{Z}$. We can suppose $u \wedge v \wedge w = 1$ (if it is not equal to 1, we divide by $u \wedge v \wedge w$ to avoid the redundancies).

Furthermore, we can suppose $v \geq 0$ (even consider the equation $(-u)\alpha + (-v)\beta + (-w) = 0$). Then we have two cases: either $v = 0$ or $v > 0$. If $v = 0$, then necessarily $u \neq 0$ and we have $1 + \sum_{1 \leq q \leq m} \varphi(q) = |F_m|$ possible lines, and we are done in this case. Thus, we restrict attention to the case $v > 0$ throughout the rest of the proof. Then we can write :

$$\beta = -\frac{u}{v}\alpha - \frac{w}{v}.$$

Then we consider the slope of this real line : $-u/v$. Now we have two cases to consider for u. If $u = 0$, then we have $1 + \sum_{1 \leq q \leq n} \varphi(q) = |F_n|$ possible straight lines. Summarizing from the beginning, we have $|F_m| + |F_n|$ possible lines, where the 4 lines of the frontier of the square are taken into account, and we are done in this case. Therefore, we can now restrict attention to the case $u \neq 0$ throughout the rest of the proof.

So we can consider the x-intercept $\alpha_o = -w/u$, together with the y-intercept $\beta_o = -w/v$. If the line is strictly increasing, then $-u/v > 0 \Rightarrow u < 0$ and the necessary condition for the line going through the square $[0, 1]^2$ is :

$$\begin{aligned} &\beta_o \leq (n-1)/n \text{ and } \alpha_0 \leq (m-1)/m \\ &\Leftrightarrow -w/v \leq (n-1)/n \text{ and } -w/u \leq (m-1)/m \\ &\Leftrightarrow -v(n-1)/n \leq w \leq -u(m-1)/m. \end{aligned}$$

It is equivalent to search for the integers w such that $-v + v/n \leq w \leq \lfloor -u + u/m \rfloor$, so $-v + v/n \leq w \leq -u - 1$ or $-v + 1 \leq w \leq -u - 1$ for $(u, v) \in [\![1, m]\!] \times [\![1, n]\!]$.

We see that if $w = 0$ (which corresponds to the origin), the condition of primality becomes $u \wedge v \wedge w = 1 \Rightarrow u \wedge v = 1$. With the condition $-m \leq u \leq -1$ and $1 \leq v \leq n$, this leads us to the following problem:

For a couple of integers $(m, n) \in \mathbb{N}^{*2}$, we have to find the number $A_{m,n}$ of integer couples $(u, v) \in [\![1, m]\!] \times [\![1, n]\!]$ such that $u \wedge v = 1$. So, we have $A_{m,n}$ other lines (increasing) i.e. if we summarize, the total number of lines is

$$|F_m| + |F_n| + A_{m,n}.$$

So, now, we search for the integers different from 0, w such that $-v + 1 \leq w \leq -u - 1$ for $(u, v) \in [\![1, m]\!] \times [\![1, n]\!]$.

If $u \wedge v = 1$, then all the nonzero w's between $-v + 1$ and $u - 1$ are suitable; there are $u + v - 2$ of them. Otherwise, the number of suitable w is equal to $((u + v)/d)\varphi(d)$.

The equation of a Farey line now has the form:

$$\beta(\alpha) = -(u/v)\alpha - w/v$$

By the same idea, using the fact that $\beta(1/n) \geq 0$ and $\beta(1) \leq (n-1)/n$ (which are the last points before the border of the Farey sequence, i.e. $-u-v+1 \leq w \leq -1$), we find that the number of decreasing Farey lines of order (m,n) is:

$$\sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=1}} (u+v-1) + \sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=d\geq 2}} \frac{u+v}{d}\varphi(d)$$

Henceforth, we can enumerate the number of lines which are strictly decreasing or strictly increasing:

$$A_{m,n} + \sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=d\geq 2}} \frac{u+v}{d}\varphi(d) + \sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=1}} (u+v-2) + \sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=1}} (u+v-1) + \sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=d\geq 2}} \frac{u+v}{d}\varphi(d),$$

which becomes (by the fact that $\sum_{u\wedge v=1} 1 = A_{m,n}$) exactly

$$2A_{m,n} + 2\sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=d\geq 2}} \frac{u+v}{d}\varphi(d) + 2\sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=1}} (u+v-2).$$

Hence, in adding the vertical and horizontal lines, we finally obtain that the number of Farey lines is:

$$|F_m| + |F_n| + 2A_{m,n} + 2\sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=d\geq 2}} \frac{u+v}{d}\varphi(d) + 2\sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=1}} (u+v-2).$$

In a more reduced form, this is equivalent to

$$|F_m| + |F_n| + 2\sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=d\geq 1}} \frac{u+v}{d}\varphi(d) - 2A_{m,n}. \tag{1}$$

This completes the proof. ☐

Let $DFD(m,n)$ denote the set of decreasing Farey lines.

Lemma 1. *For all pairs* $(m,n) \in {\mathbb{N}^*}^2$, *we have*

$$|DFD(m,n)| = \sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=1}} (u+v-1) + \sum_{\substack{1\leq u\leq m\\ 1\leq v\leq n\\ u\wedge v=d\geq 2}} \frac{u+v}{d}\varphi(d).$$

Proof. If $u \wedge v = 1$, there are $u+v-1$ integers w such that $-u-v+1 \leq w \leq -1$. Otherwise, if $u \wedge v \geq 2$, there are $(u+v)\varphi(d)/d$ integers w with $d = u \wedge v$.

Remark 1. If we let $DFC(m,n)$ denote the set of increasing Farey lines, then we notice that $|DFC(m,n)| = |DFD(m,n)|$.

5. NUMERICAL RESULTS

For the case $(m,n) = (4,3)$, we numerically and theoretically find the number 88, and in the case $(m,n) = (3,3)$, we find 60 lines.

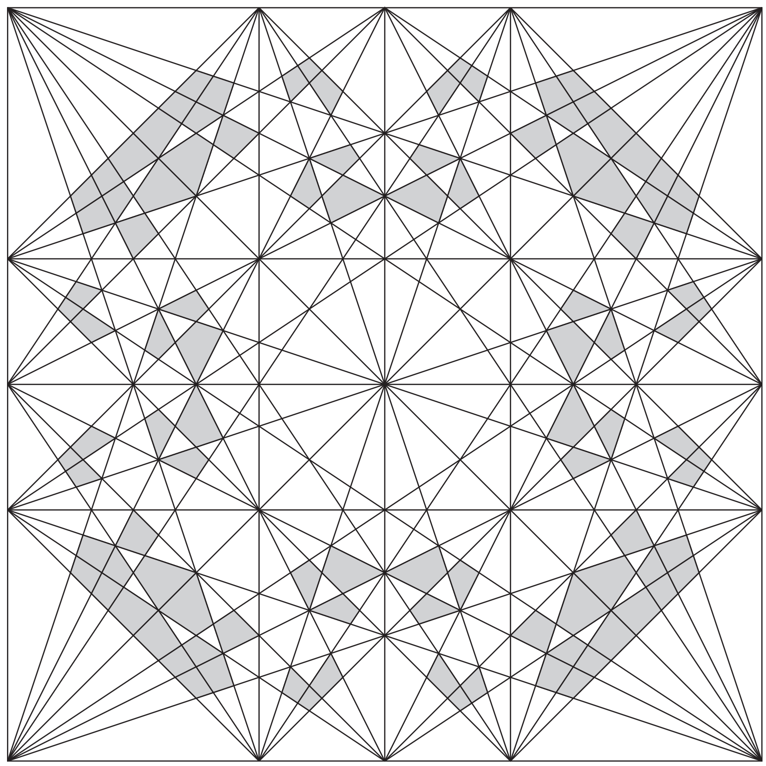

Figure 2. Farey lines of order $(3,3)$

6. ASYMPTOTIC VALUE FOR THE NUMBER OF FAREY LINES OF ORDER (m,n)

In order to give a more precise estimation (like an asymptotic behaviour of the number of Farey lines), we need to use the definition of the Möbius function that we find, for example, in [**2**] and [**16**]:

Definition 10. *The Möbius function* $\mu(n)$ *is defined as follows*: $\mu(1) = 1$; *if* n *has a square factor, then* $\mu(n) = 0$; *if all the prime numbers* $p_1, p_2, \dots, p_k$ *are distinct, then* $\mu\Big(\prod_{j=1}^{k} p_i\Big) = (-1)^k$.

We recall the very important lemma due to Vinogradov:

Lemma 2 (Vinogradov)**.** *Let m and n be two integers greater or equal than* 1 *and let f be a function defined on the integers. Then we have*

$$\sum_{\substack{k=1\\k\wedge m=1}}^{n} f(k)=\sum_{d|m}\mu(d)\sum_{k\leq\frac{n}{d}}f(kd)$$

With this lemma, we can make precise the major term in the closed formula giving the number of Farey lines of order (m,n) :

$$\sum_{\substack{1\leq u\leq m\\1\leq v\leq n\\u\wedge v=d\geq 1}}\frac{u+v}{d}\varphi(d)=\sum_{1\leq d\leq\min(m,n)}\varphi(d)\sum_{\substack{1\leq u'\leq\frac{m}{d}\\1\leq v'\leq\frac{n}{d}\\u'\wedge v'=1}}u'+v'$$

Thanks to Vinogradov's lemma, we have

$$\begin{aligned}\sum_{\substack{1\leq u'\leq\frac{m}{d}\\1\leq v'\leq\frac{n}{d}\\u'\wedge v'=1}}u'+v' &=\sum_{1\leq u'\leq\frac{m}{d}}u'\sum_{e|u'}\mu(e)\sum_{k\leq\frac{n}{de}}1+\sum_{1\leq v'\leq\frac{n}{d}}v'\sum_{e|v'}\mu(e)\sum_{k\leq\frac{m}{de}}1\\&=\sum_{1\leq u'\leq\frac{m}{d}}u'\sum_{e|u'}\mu(e)\left\lfloor\frac{n}{de}\right\rfloor+\sum_{1\leq v'\leq\frac{n}{d}}v'\sum_{e|v'}\mu(e)\left\lfloor\frac{m}{de}\right\rfloor\\&=\sum_{1\leq e\leq\frac{m}{d}}e\mu(e)\left\lfloor\frac{n}{de}\right\rfloor\sum_{f\leq\frac{m}{ed}}f+\sum_{1\leq e\leq\frac{n}{d}}e\mu(e)\left\lfloor\frac{m}{de}\right\rfloor\sum_{f\leq\frac{n}{ed}}f\end{aligned}$$

The value of the last term is:

$$\frac{1}{2}\sum_{1\leq e\leq\frac{m}{d}}e\mu(e)\left\lfloor\frac{n}{de}\right\rfloor\left(\left\lfloor\frac{m}{de}\right\rfloor^2+\left\lfloor\frac{m}{de}\right\rfloor\right)+\frac{1}{2}\sum_{1\leq e\leq\frac{n}{d}}e\mu(e)\left\lfloor\frac{m}{de}\right\rfloor\left(\left\lfloor\frac{n}{de}\right\rfloor^2+\left\lfloor\frac{n}{de}\right\rfloor\right)\tag{2}$$

Now, we can derive the asymptotic value of this expression. Let us do it for the first term in equation (2).

Proposition 3. *We have*

$$\sum_{1\leq e\leq\frac{m}{d}}e\mu(e)\left\lfloor\frac{n}{de}\right\rfloor\left(\left\lfloor\frac{m}{de}\right\rfloor^2+\left\lfloor\frac{m}{de}\right\rfloor\right)=\frac{nm^2}{d^3}\sum_{1\leq e\leq\frac{m}{d}}\frac{\mu(e)}{e^2}+\mathcal{O}\left(\frac{m\log(m)(m+n)}{d^2}\right).$$

Proof. By reversing the roles of m and n, we will deduce the asymptotic value of the other term. We compute

$$\begin{aligned}
&\sum_{1\le e\le \frac{m}{d}} e\mu(e)\left\lfloor\frac{n}{de}\right\rfloor\left(\left\lfloor\frac{m}{de}\right\rfloor^2+\left\lfloor\frac{m}{de}\right\rfloor\right)\\
&=\sum_{1\le e\le \frac{m}{d}} e\mu(e)\Big(\frac{n}{de}-\Big\langle\frac{n}{de}\Big\rangle\Big)\Big(\Big(\frac{m}{de}-\Big\langle\frac{m}{de}\Big\rangle\Big)^2+\frac{m}{de}-\Big\langle\frac{m}{de}\Big\rangle\Big)\\
&=\frac{nm^2}{d^3}\sum_{1\le e\le \frac{m}{d}}\frac{\mu(e)}{e^2}+\frac{mn}{d^2}\sum_{1\le e\le \frac{m}{d}}\frac{\mu(e)}{e}+\mathcal{O}\Big(\frac{nm}{d^2}\log(m)\Big)+\mathcal{O}\Big(\frac{m^2}{d^2}\log(m)\Big)
\end{aligned}$$

So we conclude

$$\sum_{1\le e\le \frac{m}{d}} e\mu(e)\left\lfloor\frac{n}{de}\right\rfloor\left(\left\lfloor\frac{m}{de}\right\rfloor^2+\left\lfloor\frac{m}{de}\right\rfloor\right)=\frac{nm^2}{d^3}\sum_{1\le e\le \frac{m}{d}}\frac{\mu(e)}{e^2}+\mathcal{O}\left(\frac{m\log(m)(m+n)}{d^2}\right)\square$$

It remains to study the quantity:

$$\sum_{1\le d\le \min(m,n)}\varphi(d)\Big(\frac{nm^2}{d^3}\sum_{1\le e\le \frac{m}{d}}\frac{\mu(e)}{e^2}+\mathcal{O}\Big(\frac{m\log(m)(m+n)}{d^2}\Big)\Big).$$

Towards this goal, we first have the following proposition:

Proposition 4 ([**2**]). *For $x\ge 2$, and $\alpha>1$ and $\alpha\ne 2$, we have*

$$\sum_{n\le x}\frac{\varphi(n)}{n^\alpha}=\frac{x^{2-\alpha}}{2-\alpha}\frac{1}{\zeta(2)}+\frac{\zeta(\alpha-1)}{\zeta(\alpha)}+\mathcal{O}(x^{1-\alpha}\log x).$$

Now we apply this result in the case $\alpha=3$ for the computation of the sum, which leads us to:

Proposition 5. *For $x\ge 2$, we have*

$$\sum_{n\le x}\frac{\varphi(n)}{n^3}=-\frac{1}{x}\frac{1}{\zeta(2)}+\frac{\zeta(2)}{\zeta(3)}+\mathcal{O}(x^{-2}\log x).$$

Finally, by [**2**, page 71], we have:

Theorem 2. *If $x\ge 2$, then*

$$\sum_{n\le x}\frac{\varphi(n)}{n^2}=\frac{1}{\zeta(2)}\log x+\frac{C}{\zeta(2)}-A+\mathcal{O}\Big(\frac{\log x}{x}\Big),$$

where C is the Euler's constant and $A=\sum_{n=1}^{\infty}\frac{\mu(n)\log n}{n^2}$.

Proposition 6. *We have*

$$\sum_{1\le d\le \min(m,n)} \varphi(d)\Big(\frac{nm^2}{d^3}\sum_{1\le e\le \frac{m}{d}}\frac{\mu(e)}{e^2}+\mathcal{O}\Big(\frac{m(m+n)\log(m)}{d^2}\Big)\Big) = \frac{nm^2}{\zeta(3)}+\mathcal{O}\left(m(m+n)\log(m)\log(\min(m,n))\right) \tag{3}$$

Proof. We first observe

$$\sum_{1\le d\le \min(m,n)} \varphi(d)\Big(\frac{nm^2}{d^3}\sum_{1\le e\le \frac{m}{d}}\frac{\mu(e)}{e^2}+\mathcal{O}\Big(\frac{m(m+n)\log(m)}{d^2}\Big)\Big)$$
$$= nm^2\sum_{1\le d\le \min(m,n)}\frac{\varphi(d)}{d^3}\sum_{1\le e\le \frac{m}{d}}\frac{\mu(e)}{e^2}+\mathcal{O}\Bigg(m(m+n)\log(m)\sum_{1\le d\le \min(m,n)}\frac{\varphi(d)}{d^2}\Bigg).$$

For $s\ge 2$, we have:

$$\sum_{n\le x}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)}+\mathcal{O}\Big(\frac{1}{x^{s-1}}\Big).$$

So we conclude

$$\sum_{1\le d\le \min(m,n)} \varphi(d)\Big(\frac{nm^2}{d^3}\sum_{1\le e\le \frac{m}{d}}\frac{\mu(e)}{e^2}+\mathcal{O}\Big(\frac{m\log(m)(m+n)}{d^2}\Big)\Big)$$
$$= nm^2\sum_{1\le d\le \min(m,n)}\frac{\varphi(d)}{d^3}\Big(\frac{1}{\zeta(2)}+\mathcal{O}\Big(\frac{d}{m}\Big)\Big)$$
$$+\mathcal{O}\Bigg(m(m+n)\log(m)\sum_{1\le d\le \min(m,n)}\frac{\varphi(d)}{d^2}\Bigg)$$
$$=\frac{nm^2}{\zeta(3)}+\mathcal{O}(nm\log(\min(m,n)))+\mathcal{O}(m(m+n)\log(m)\log(\min(m,n)))$$
$$=\frac{nm^2}{\zeta(3)}+\mathcal{O}(m(m+n)\log(m)\log(\min(m,n))).$$

In a same way, we obtain an asymptotic value for the second part of equation (2), by exchanging the roles of m and n. Now we conclude with the following theorem.

Theorem 3. *The number of Farey lines, when m and n tend to infinity, has the following behavior*

$$|DF(m,n)|\sim mn(m+n)/\zeta(3).$$

Proof. By using (1), and Proposition 3, the total number of Farey lines of order

(m, n) has the asymptotic value :

$$\begin{aligned} |DF(m,n)| &= (2)(1/2)(nm^2/\zeta(3) + mn^2/\zeta(3)) \\ &\quad + \mathcal{O}\left(m(m+n)\log(m)\log(\min(m,n))\right) \\ &\quad + \mathcal{O}\left(n(m+n)\log(n)\log(\min(m,n))\right) \\ &= (2)(1/2)(nm)(n+m)/\zeta(3) \\ &\quad + \mathcal{O}((m+n)(m\log(m) + n\log(n))\log(\min(m,n))). \qquad \square \end{aligned}$$

So, we obtained an equivalent of the cardinality of the set of Farey lines of order (m, n), whereas until now we only had a $\mathcal{O}$ upper bound.

7. COMBINATORIAL ANALYSIS OF FAREY LINES FOR DISCRETE GEOMETRY

Before this paper, we were not sure whether the order of Farey lines cardinality of order (m, n) is 3 or less than 3. Now we know that this order is exactly 3. Moreover, we can compute exactly this number. And we know the behaviour of this quantity when m and n tend to infinity. Since we know that, if we have any set of n lines, then the number of vertices which are constructed from these lines is at most $n(n-1)/2$, then we can deduce (as in [**3**]) that the order of Farey vertices is at most 6. So if we want to study more deeply the cardinality of the set of Farey vertices, this argument is not sufficient. This work allows us to say that the number of Farey vertices which is known, is impossible to improve, if we only use the previous argument, which is a basic argument of combinatorial geometry, as it is the case in [**3**]. In our proofs, we can see that terms of generalized Farey sequences appear. This result should also be interesting for the scientists who work on graph theory, because we worked on a special graph, that we call the Farey graph, and we studied (for example) the degree of the vertices. In order to improve the upper bound, another interesting work will be to focus on the diophantine aspects of Farey diagrams, combined with some other arguments of graph theory, to better estimate the cardinality of Farey vertices. In such a way, we should progress in the knowledge of the (m, n)-cubes combinatorics.

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