COMPOSITIONS WITH PAIRWISE RELATIVELY PRIME SUMMANDS WITHIN A RESTRICTED SETTING

Temba Shonhiwa
School of Mathematics, University of the Witwatersrand, P. Bag X3, Wits 2050, JHB, South Africa
e-mail: Temba@maths.wits.ac.za

(Submitted October 2004-Final Revision June 2005)

ABSTRACT

The paper studies the counting function

$$R_2(n,k) = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ (a_i,a_j)=1 \\ i\neq j}} 1, a_i \geq 1, k \geq 2$$

with a_i, n and k positive integers and establishes a relationship between $R_2(n,k)$ and $P_2(n,k)$ where

$$P_2(n,k) = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ 1\leq a_1\leq a_2\leq\cdots\leq a_k\leq n \\ (a_i,a_j)=1 \\ i\neq j}} 1, a_i \geq 1, k \geq 2$$

with a_i, n, k positive integers.

1. PRELIMINARIES

Gould [4], studied the number-theoretic function

$$R(n,k) = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ (a_1,a_2,\cdots,a_k)=1}} 1, a_i \geq 1, k \geq 2 \tag{1}$$

and showed amongst other results that:

Theorem 1:

$$R(n,k) = \sum_{d|n} \binom{d-1}{k-1} \mu(\frac{n}{d})$$

and

Theorem 2:

$$\sum_{j=1}^{\infty} R(j,k) \frac{x^j}{1-x^j} = \frac{x^k}{(1-x)^k}.$$

Motivated by definition 1, we now define

$$R_2(n,k) = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ (a_i,a_j)=1 \\ i\neq j}} 1, a_i \geq 1, k \geq 2. \tag{2}$$

From theorem 1, it follows that:

$$R(n,2) = R_2(n,2) = \sum_{d|n} \binom{d-1}{1} \mu(\frac{n}{d})$$

$$= n \sum_{d|n} \frac{\mu(d)}{d} - \sum_{d|n} \mu(d).$$

And hence

Corollary 1: *$R_2(n,2) = \phi(n)$, for all $n > 1$, where ϕ is Euler's totient function.*

Catalan [1], [2], [3, Vol. 2, 114, 126] proved in 1838 that the equation:

$$a_1 + a_2 + \cdots + a_k = n, (a_i \geq 0)$$

has $\binom{n+k-1}{k-1}$ solutions. Further, he then noted in 1868 that

$$C(n,k) = \binom{n-1}{k-1} = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ a_i \geq 1}} 1. \tag{3}$$

Consequently, with $\binom{n}{k} = \sum_{j=k}^{n} \binom{j-1}{k-1}$, it now follows that

$$\binom{n}{2} = \sum_{j=2}^{n} C(n,2) = \sum_{j=2}^{n} \sum_{\substack{a_1+a_2=j \\ a_i \geq 1}} 1$$

$$= \sum_{j=2}^{n} \left[\frac{n}{j}\right] \sum_{\substack{a_1+a_2=j \\ (a_1,a_2)=1}} 1 = \sum_{j=2}^{n} \left[\frac{n}{j}\right] \phi(j).$$

Hence, using the results

$$\sum_{n=j}^{\infty} \left[\frac{n}{j}\right] x^{n-j} = \frac{1}{(1-x)(1-x^j)} \text{ and } \sum_{n=j}^{\infty} \binom{n}{j} x^{n-j} = (1-x)^{-j-1}\, ; |x| < 1,$$

we may set

$$\sum_{n=2}^{\infty} \binom{n}{j} x^n = \sum_{n=2}^{\infty} \sum_{j=2}^{n} \left[\frac{n}{j}\right] \phi(j) x^n$$

$$= \sum_{j=2}^{\infty} \sum_{n=j}^{\infty} \left[\frac{n}{j}\right] \phi(j) x^n$$

which simplifies to give the result

Theorem 3: *The Lambert series for Euler's totient function is*

$$\sum_{j=1}^{\infty} \frac{\phi(j) x^j}{(1-x^j)} = \frac{x}{(1-x)^2}.$$

This is a known result, see Sivaramakrishnan [7, page 71] albeit our approach here is quite different.

2. MAIN RESULTS AND PROOFS

The following lemma will be used in our subsequent investigations.

Lemma 4: *Let $P(n, n; r_1, r_2, \cdots, r_k)$ denote the number of $n-$ permutations which may be formed from the multiset $S = \{r_1 a_1, r_2 a_2, \cdots, r_k a_k\}$. Then*

$$P(n, n; r_1, r_2, \cdots, r_k) = \frac{n!}{r_1! r_2! \cdots r_k!}.$$

Using this result, we give a selection of results and proofs as follows:

Example 1:

1. $R_2(k+1, k) = k$
2. $R_2(k+2, k) = k$
3. $R_2(k+3, k) = k + k(k-1)$
4. $R_2(k+7, k) = k + 2k(k-1) + k(k-1)(k-2)$
5. $R_2(k+9, k) = k + 4k(k-1) + 2k(k-1)(k-2)$
6. $R_2(k+11, k) = k + 5k(k-1) + 2k(k-1)(k-2)$.

Proof:

$$R_2(k+1, k) = \sum_{\substack{a_1+a_2+\cdots+a_k=k+1 \\ (a_i, a_j)=1 \\ i\neq j}} 1 = \frac{k!}{(k-1)!} = k.$$

Here the only possible solutions arise from permutations of the sum: $1_1 + 1_2 + 1_3 \cdots + 2_k = k+1$.

Similarly for $R_2(k+2, k)$ where the only possible solutions arise from permutations of the sum $1_1 + 1_2 + 1_3 \cdots + 3_k = k+2$.

But

$$R_2(k+3, k) = \sum_{\substack{a_1+a_2+\cdots+a_k=k+3 \\ (a_i, a_j)=1 \\ i\neq j}} 1 = \frac{k!}{(k-1)!} + \frac{k!}{(k-2)!} = k^2.$$

Here the sum $1_1 + 1_2 + 1_3 \cdots + 1_k + 3$ gives rise to precisely two possible compositions: $1_1 + 1_1 + \cdots + 4_k$ and $1_1 + 1_1 + \cdots + 2_{k-1} + 3_k$.

Further,

$$R_2(k+7, k) = \sum_{\substack{a_1+a_2+\cdots+a_k=k+7 \\ (a_i, a_j)=1 \\ i\neq j}} 1 = \frac{k!}{(k-1)!} + \frac{k!}{(k-2)!} + \frac{k!}{(k-2)!} + \frac{k!}{(k-3)!}.$$

Here the sum $1_1 + 1_2 + 1_3 \cdots + 1_k + 7$ gives rise to precisely four possible compositions:

$$\begin{gathered} 1_1 + 1_2 + \cdots + 1_{k-1} + 8_k, \\ 1_1 + 1_2 + \cdots + 2_{k-1} + 7_k, \\ 1_1 + 1_2 + \cdots + 4_{k-1} + 5_k, \\ 1_1 + 1_2 + \cdots + 2_{k-2} + 3_{k-1} + 5_k. \end{gathered}$$

The enumeration is now effected as follows: first we define the function

$$P_2(n,k) = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ 1\le a_1\le a_2\le\cdots\le a_k\le n \\ (a_i,a_j)=1 \\ i\neq j}} 1. \tag{4}$$

We then compute $P_2(9,2)$ and $P_2(10,3)$ and apply lemma 4.

The arguments above can be generalised to give the following result.

Lemma 5:

$$R_2(n+k,k) = \sum_{j=1}^{k} \frac{k!}{(k-j)!} \left(P_2(n+j,j) - P_2(n+j-1,j-1)\right)$$

with conditions $P_2(n,1)=1, P_2(n,0)=0$ *in order to initialize the counting process. Further, for consistency we shall also require the condition* $R_2(n,1)=1$ *for all* $n\ge 1$.

Proof: The process proceeds in stages as follows: first we start off by filling each of the k positions with a 1. We then count how many times the last two positions can be filled in, such that $a_{k-1}+a_k=n+2, a_{k-1}<a_k\le n+1, (a_{k-1},a_k)=1$. This is precisely $P_2(n+2,2)$ where we note that the case when $a_{k-1}=1$ and $a_k=n+1$ counts $\frac{k!}{(k-1)!}$ times, and the other cases count $\frac{k!}{(k-2)!}$ times each. Next, we count how many times the last three positions can be filled in, such that $a_{k-2}+a_{k-1}+a_k = n+3, a_{k-2}<a_{k-1}<a_k<n, (a_i,a_j) = 1, i\neq j$. This is precisely $P_2(n+3,3)$. To find the total count thus far we therefore compute $(P_2(n+3,3)-P_2(n+2,2))+P_2(n+2,2)$. The process stops when we fill in all the k positions such that $a_1+a_2+\cdots+a_k=n+k, 1\le a_1\le a_2\le\cdots\le a_{k-1}<a_k\le n+k, (a_i,a_j)=1, i\neq j$.

Below is table 1 of values of $R_2(n,k)$.

$\frac{n}{k}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8
3	0	0	1	3	3	9	3	15	9	21	9	39	9	45	21	45
4	0	0	0	1	4	4	16	4	28	16	52	16	100	16	100	68
5	0	0	0	0	1	5	5	25	5	45	25	105	25	205	25	225
6	0	0	0	0	0	1	6	6	36	6	66	36	186	36	366	36
7	0	0	0	0	0	0	1	7	7	49	7	91	49	301	49	595
8	0	0	0	0	0	0	0	1	8	8	64	8	120	64	456	64
9	0	0	0	0	0	0	0	0	1	9	9	81	9	153	81	657
10	0	0	0	0	0	0	0	0	0	1	10	10	100	10	190	100
11	0	0	0	0	0	0	0	0	0	0	1	11	11	121	11	231
12	0	0	0	0	0	0	0	0	0	0	0	1	12	12	144	12
13	0	0	0	0	0	0	0	0	0	0	0	0	1	13	13	169
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	14	14
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	15
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 1: Values of $R_2(n,k)$

Remark: Since $P_2(k+1,k) = P_2(k,k-1) = 1$ we immediately retrieve the result $R_2(k+1,k) = k$ from theorem 4.

The main difficulty however is in the computation of the function $P_2(n,k)$. Does there exist a closed form solution to this function? It would appear that this is a very difficult problem and a search of the literature has not yielded any positive results in this regard. The case $k = 2$ obviously gives $P_2(n,2) = \frac{\phi(n)}{2}$ for all $n \geq 2$.

A somewhat similar function,

$$P_r(n,k) = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ 1\leq a_1\leq a_2\leq\cdots\leq a_k\leq n \\ (a_1,a_2,\cdots,a_k)=1 \\ i\neq j}} 1. \tag{5}$$

was studied by the author in [6], where it was shown that: $P_{n,k} = \sum_{d|n} P_r(d,k)$, where $P(n,k)$ is the partition of n into exactly k parts.

However, it is easily shown that $P_2(k+2,k) = 1, P_2(k+3,k) = 2, P_2(k+4,k) = 1, P_2(k+5,k) = 3, P_2(k+6,k) = 2, P_2(k+7,k) = 4$, etcetera. Further, in table 2, we give some values for $P_2(n,k)$ for $1 \leq n,k \leq 13$.

$\frac{n}{k}$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	1	2	1	3	2	3	2	5	2	6
3	0	0	1	1	1	2	1	3	2	4	2	7	2
4	0	0	0	1	1	1	2	1	3	2	4	2	7
5	0	0	0	0	1	1	1	2	1	3	2	4	2
6	0	0	0	0	0	1	1	1	2	1	3	2	4
7	0	0	0	0	0	0	1	1	1	2	1	3	2
8	0	0	0	0	0	0	0	1	1	1	2	1	3
9	0	0	0	0	0	0	0	0	1	1	1	2	1
10	0	0	0	0	0	0	0	0	0	1	1	1	2
11	0	0	0	0	0	0	0	0	0	0	1	1	1
12	0	0	0	0	0	0	0	0	0	0	0	1	1
13	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 2: Values of $P_2(n,k)$

We now let $B(n,j) = (P_2(n+j,j) - P_2(n+j-1,j-1))$ and $\mathcal{R}_2(n,x) = \sum_{k=1}^{\infty} R_2(n+k,k)x^k$ be the generating function for $R_2(n+k,k)$.

Then,

$$\mathcal{R}_2(n,x) = \sum_{k=1}^{\infty}\sum_{j=1}^{k} \frac{k!}{(k-j)!} B(n,j)x^k$$

$$= \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{k!}{(k-j)!} B(n,j) x^k$$

$$= \sum_{j=1}^{\infty} B(n,j) j! \sum_{k=j}^{\infty} \binom{k}{j} x^k$$

$$= \sum_{j=1}^{\infty} B(n,j) j! \frac{x^j}{(1-x)^{j+1}}.$$

Thus,

Theorem 6: *The generating function for* $R_2(n+k,k)$ *is*

$$= \sum_{j=1}^{\infty} B(n,j) j! \frac{x^j}{(1-x)^{j+1}}.$$

It now follows from theorem 2 that

$$\sum_{j=1}^{\infty} B(n,j) j! \frac{x^j}{(1-x)^{j+1}} = \sum_{j=1}^{\infty} B(n,j) j! \sum_{i=j}^{\infty} R(i,j) \frac{x^i}{(1-x)(1-x^i)}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{i} B(n,j) j! R(i,j) \frac{x^i}{(1-x)(1-x^i)}$$

and hence,

$$(1-x) \sum_{i=1}^{\infty} R_2(n+i,i) x^i = \sum_{i=1}^{\infty} \frac{x^i}{(1-x^i)} \sum_{j=1}^{i} B(n,j) j! R(i,j).$$

From this, after adjusting the summation variables on the left we obtain

$$\sum_{i=1}^{\infty} \left(R_2(n+i,i) - R_2(n+i-1,i-1) \right) x^i = \sum_{i=1}^{\infty} \frac{x^i}{(1-x^i)} \sum_{j=1}^{i} B(n,j) j! R(i,j).$$

Now, it is known, Hardy and Wright [5, page 257] that the Lambert series

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{(1-x^n)} = \sum_{n=1}^{\infty} b_n x^n$$

is equivalent to $b_n = \sum_{d|n} a_d$.

We may therefore summarize the above as

Theorem 7:

$$R_2(n+k,k) - R_2(n+k-1,k-1) = \sum_{d|k} \sum_{j=1}^{d} B(n,j) j! R(d,j)$$

$$= \sum_{d|k} B(n,d) d!.$$

From this we obtain through Möbius inversion the result

$$k! B(n,k) = \sum_{d|k} \mu(\frac{k}{d}) \left(R_2(n+d,d) - R_2(n+d-1,d-1) \right),$$

where we have used the condition, $R(j,k) = 0$ if $k < j$.

Using this and lemma 5, we obtain the result;

Theorem 8:

$$R_2(n+k,k) = \sum_{j=1}^{k} \binom{k}{j} \sum_{d|j} \mu(\frac{j}{d}) \left(R_2(n+d,d) - R_2(n+d-1,d-1) \right)$$

and

$$P_2(n+j,j) = P_2(n+j-1,j-1) + \frac{1}{j!} \sum_{d|j} \mu(\frac{j}{d}) \left(R_2(n+d,d) - R_2(n+d-1,d-1) \right).$$

Example 2:

$$P_2(n+2,2) = P_2(n+1,1) + \frac{1}{2} \sum_{d|2} \mu(\frac{2}{d}) \left(R_2(n+d,d) - R_2(n+d-1,d-1) \right)$$

$$= 1 + \frac{1}{2} \left(-R_2(n+1,1) - R_2(n+1,1) + R_2(n+2,2) \right)$$

$$= \frac{\phi(n+2)}{2}$$

as expected.

Further, on using the result: if

$$F(n) = \sum_{d|n} f(d), \text{ then } \sum_{n=1}^{N} F(n) = \sum_{n=1}^{N} f(j) \left[\frac{N}{j} \right],$$

it follows that

$$\sum_{k=1}^{N}(R_2(n+k,k)-R_2(n+k-1,k-1))=\sum_{k=1}^{N}\sum_{j=1}^{k}B(n,j)j!R(k,j)\left[\frac{N}{k}\right]$$

$$=\sum_{k=1}^{N}B(n,k)k!\left[\frac{N}{k}\right].$$

Thus the original investigation has led us into yet another problem, specifically that of the structure of $P_2(n,k)$. Investigations of this function and the implications for theorems 7 and 8 will be presented in a follow-up paper. We also note the fact that some diagonal sequences in table 1 have been studied before. Further information on these can be viewed under; *The On-Line Encyclopedia of Integer Sequences*, at www.research.att.com/∼njas/sequences/.

REFERENCES

[1] E. Catalan. "Mélanges Mathématiques." *Mém. Soc. Sci. Liége* (2) 12 (1885); orig. publ. 1868.

[2] E. Catalan. "Note sur un probléme de combinaisons." *J. Math. Pures Appl.* **3.1** (1838): 111-112.

[3] L. E. Dickson. *History of the Theory of Numbers,* Washington, Vol. 1, 1919; Vol. 2, 1920; Vol. 3, 1923.

[4] H. W. Gould. "Binomial Coefficients, the Bracket Function, and Compositions with Relatively Prime Summands." *The Fibonacci Quarterly* Vol. 2 (4), 1964.

[5] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers,* Oxford, Fifth Edition, 1979.

[6] T. Shonhiwa. "On Relatively Prime Decompositions and Related Results." *Quaestiones Mathematicae* **24** (2001): 565-573.

[7] R. Sivaramakrishnan. *Classical Theory of Arithmetic Functions*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 126, Marcel Dekker, New York, 1989.

AMS Classification Numbers: 11A25, 11P81, 11B50, 11B65, 11B75, 11Y70