REMARKS ON COMPLEMENTARY SEQUENCES

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Abstract. The aim of this paper is to find the general term of the complementary sequence of a given strictly increasing sequence of non-negative integers. Finally, some applications are given.

1. Introduction

We are motivated by the large number of papers in connection with the problem of computing or estimating the so called modified harmonic series, i.e., the series obtained by omitting some terms of the harmonic series.

One of the first preoccupations in this sense refers to the series formed by the inverses of all positive integers whose denominator expressed in base 10 does not contain the digit 9. Kempner [3] proved that this series is convergent and that the limit is smaller than 80. This bound is rough though. Irwin [2] reduced the bound to 24. Baillie and Schmelzer [1, 4] developed an algorithm for the numerical approximation of such limits.

Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of non-negative integers. Then the set of all non-negative integers which are not terms of the sequence $(a_n)_{n\in\mathbb{N}}$, being assumed an infinite set, can be ordered as a strictly increasing sequence, which will be called in the sequel *the complementary sequence* of $(a_n)_{n\in\mathbb{N}}$.

As a simple example, the complementary sequence of the sequence of the even non-negative integers is the sequence of the odd non-negative integers.

Note that if $(b_n)_{n\in\mathbb{N}}$ is the complementary sequence of the sequence $(a_n)_{n\in\mathbb{N}}$, then $(a_n)_{n\in\mathbb{N}}$ is the complementary sequence of the sequence $(b_n)_{n\in\mathbb{N}}$.

2. The Main Results

In this section we consider the strictly increasing sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}} \subset \mathbb{N}$, being complementary each one to the other. We start with the following theorem.

Theorem 2.1. *For every n, $k \in \mathbb{N}$, the following implication holds true:*

$$n \in \mathbb{N} \cap [a_k - k, a_{k+1} - k - 1) \Rightarrow b_n = n + k + 1.$$

Proof. The sequence $(b_n)_{n\in\mathbb{N}}$ takes all the values between two consecutive terms $a_k < a_{k+1}$, so for every $m \in \mathbb{N}$, the following implication holds true:

$$a_k < b_m < b_{m+1} < a_{k+1} \Rightarrow b_{m+1} - b_m = 1. \tag{2.1}$$

Now, for a given index k with $a_{k+1} - a_k \geq 2$, let m be the minimal rank such that $a_k < b_m$. Then $b_m = a_k + 1$.

There are $k + 1$ terms $a_0, a_1, \ldots, a_k$ of the sequence $(a_n)_{n\in\mathbb{N}}$ which are less than b_m. The others $a_k + 1 - (k + 1) = a_k - k$ terms are terms of the sequence $(b_n)_{n\in\mathbb{N}}$, in fact they are $b_0, b_1, \ldots, b_{a_k-k-1} < b_m$.

Now we can deduce that $m = a_k - k$ and so $b_{a_k-k} = a_k + 1$, which is $b_m = m + k + 1$.

Finally, if we assume that $b_m, b_{m+1}, \ldots, b_{m+s}$ are all the (consecutive) terms between a_k and a_{k+1}, then the conclusion follows from (2.1). $\square$

Now we can conclude by giving the result which provides the formula of the general term of the complementary sequence.

Theorem 2.2. *Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of non-negative integers, with $a_0 = 0$ such that the sequence $(a_n - n)_{n\in\mathbb{N}}$ is unbounded from above. Then the complementary sequence $(b_n)_{n\in\mathbb{N}}$ of the sequence $(a_n)_{n\in\mathbb{N}}$ is given by the formula $b_n = n + u_n + 1$, where $u_n = \max\{k \in \mathbb{N} \mid a_k - k \le n\}$.*

Proof. Note that for every non-negative integer n, there exists a rank k depending on n, say $k = u_n$, such that

$$a_k - k \le n < a_{k+1} - (k+1). \tag{2.2}$$

By Theorem 2.1, we have $b_n = n + k + 1$, or equivalent $b_n = n + u_n + 1$. $\square$

In consequence, we can see that the problem of finding the general term of the complementary sequence is in fact the problem of solving the double inequality (2.2).

If we look carefully at relation (2.2), we can see that it is more convenient to denote the given sequence as in the following.

Theorem 2.3. *Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of non-negative integers, with $a_0 = 0$, given in the form $a_n = n + f(n)$, where the function $f : [0,\infty) \to [0,\infty)$ is increasing, invertible and assume that f transforms integers into integers. Then the general term of the complementary sequence $(b_n)_{n\in\mathbb{N}}$ of the sequence $(a_n)_{n\in\mathbb{N}}$ is given by the formula*

$$b_n = n + 1 + [f^{-1}(n)]. \tag{2.3}$$

($[\cdot]$ means rounding towards 0).

Proof. The sequence $(a_n - n)_{n\in\mathbb{N}}$ is unbounded above, so formula (2.2) can be written equivalently as $f(k) \le n < f(k+1)$ and by the monotonicity of the functions f and f^{-1}, we obtain $k \le f^{-1}(n) < k+1$.

It follows that $k = [f^{-1}(n)]$, so we are done. $\square$

3. Examples of Complementary Sequences

Corollary 3.1. *The general term of the sequence of the positive integers which are not perfect squares is given by the formula*

$$z_n = n + \left[\sqrt{n} + \frac{1}{2}\right], \qquad n \in \mathbb{N} \setminus \{0\}.$$

Proof. The sequence $(b_n)_{n\in\mathbb{N}}$ is the complementary sequence of the sequence $a_n = n^2$. In order to use Theorem 2.3, we write $a_n = n + f(n)$, where the function $f : [1,\infty) \to [0,\infty)$ is $f(x) = x^2 - x$, with $f^{-1}(x) = \frac{1+\sqrt{4x+1}}{2}$.

Directly using formula (2.3), we deduce that

$$b_n = n + \left[\frac{3 + \sqrt{4n+1}}{2}\right], \quad n \in \mathbb{N}.$$

To finish the proof, we have to show that $b_n = z_{n+1}$, for every $n \in \mathbb{N}$, which is

$$\left[\sqrt{n+\frac{1}{4}}+\frac{1}{2}\right] = \left[\sqrt{n+1}+\frac{1}{2}\right].$$

Indeed, by Hermite's formula $[x+\frac{1}{2}] = [2x]-[x]$, we get

$$\left[\sqrt{n+\frac{1}{4}}+\frac{1}{2}\right] = [\sqrt{4n+1}] - \left[\sqrt{n+\frac{1}{4}}\right] = [\sqrt{4n+4}] - [\sqrt{n+1}] = \left[\sqrt{n+1}+\frac{1}{2}\right].$$

The penultimate equality follows from the fact that

$$[\sqrt{4n+4}] - [\sqrt{4n+1}] = [\sqrt{n+1}] - \left[\sqrt{n+\frac{1}{4}}\right] \in \{0,1\},$$

where the value 1 is taken if and only if $n+1$ is a perfect square. □

Using the same method, we can find the sequence $b_0 = 2$ and

$$b_n = n+1+\left[\sqrt[3]{\frac{n}{2}+\sqrt{\frac{n^2}{4}-\frac{1}{27}}}+\sqrt[3]{\frac{n}{2}-\sqrt{\frac{n^2}{4}-\frac{1}{27}}}\right], \quad n \geq 1$$

of the positive integers which are not perfect cubes.

Similarly, the complementary sequence of the sequence of triangular numbers $T_n = 1+2+\cdots+n$, $n \geq 1$ is

$$b_n = n + \left[\frac{3+\sqrt{8n+1}}{2}\right], \quad n \in \mathbb{N},$$

while the complementary sequence of the sequence of tetrahedral numbers $\tau_n = T_1+T_2+\cdots+T_n$, $n \geq 1$ is $c_0 = 2$, $c_1 = 3$, $c_2 = 5$, with

$$c_{n+1} = n+1+\left[\sqrt[3]{3n+\sqrt{9n^2-\frac{343}{27}}}+\sqrt[3]{3n-\sqrt{9n^2-\frac{343}{27}}}\right], \quad n \geq 2.$$

4. Sums of Complementary Sequences

These results can be used to compute some interesting sums involving the values of the inverse of a given function. More precisely, let us take the example of the sum of all integers less than or equal to n, which are not squares. This sum can be obtained by retracting from the sum $1+2+\cdots+n$ the sum of all squares which are less than or equal to n. The advantage of this method is that we can compute the sum of the first few squares involved, so the other sum can be also calculated. We illustrate this idea in the following theoretical background.

Theorem 4.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of integers with $a_0 = 0$, such that $(a_n - n)_{n\in\mathbb{N}}$ is unbounded from above and let $(b_n)_{n\in\mathbb{N}}$ be its complementary sequence. Then for every positive integer s and any injective function $\varphi : \mathbb{N} \to \mathbb{R}$, we have*

$$\sum_{j=0}^{s} \varphi(b_j) = \sum_{k=0}^{b_s} \varphi(k) - \sum_{i=0}^{b_s-s-1} \varphi(a_i). \tag{4.1}$$

In particular, the following inequality holds true:

$$\sum_{j=0}^{s} b_j = \frac{b_s(b_s+1)}{2} - \sum_{i=0}^{b_s-s-1} a_i. \tag{4.2}$$

Proof. The particular case (4.2) is obtained from (4.1) with $\varphi(x) = x$. Let us consider the disjoint union of sets:

$$\{\varphi(b_0), \varphi(b_1), \ldots, \varphi(b_s)\} \cup \{\varphi(a_0), \varphi(a_1), \ldots, \varphi(a_k)\} = \{\varphi(0), \varphi(1), \ldots, \varphi(b_s-1), \varphi(b_s)\}, \tag{4.3}$$

where k is the greatest integer such that $a_k < b_s$. Indeed, from $(s+1)+(k+1) = b_s+1$, it results that $k = b_s - s - 1$. Moreover, by writing the equality of the sums of the elements from the terms of (4.3), we obtain

$$\sum_{j=0}^{s} \varphi(b_j) + \sum_{i=0}^{b_s-s-1} \varphi(a_i) = \sum_{k=0}^{b_s} \varphi(k),$$

which is the conclusion. □

If we use the representations from Theorem 2.3 for the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, then we obtain the following interesting form.

Theorem 4.2. *Let $f : [0,\infty) \to [0,\infty)$ be an increasing, invertible function which transforms integers into integers. For every positive integer s, we have:*

$$\sum_{j=1}^{s} \left[f^{-1}(j)\right] = (s+1) \cdot \left[f^{-1}(s)\right] - \sum_{i=1}^{[f^{-1}(s)]} f(i). \tag{4.4}$$

Proof. Let us define the increasing sequence $a_n = n + f(n)$ and let $b_n = n + 1 + \left[f^{-1}(n)\right]$ be its complementary sequence by Theorem 2.3. Then (4.2) becomes

$$\sum_{j=1}^{s} (j+1+[f^{-1}(j)]) = \frac{b_s(b_s+1)}{2} - \sum_{i=1}^{b_s-s-1} (i+f(i)),$$

or

$$\frac{(s+1)(s+2)}{2} + \sum_{j=1}^{s} [f^{-1}(j)] = \frac{b_s(b_s+1)}{2} - \frac{(b_s-s-1)(b_s-s)}{2} - \sum_{i=1}^{b_s-s-1} f(i).$$

Now the conclusion follows by replacing b_s and after some easy calculations. □

If we take $f(x) = x^2$, then by (4.4), we deduce

$$\sum_{j=1}^{s} \left[\sqrt{j}\right] = (s+1) \cdot \left[\sqrt{s}\right] - \frac{[\sqrt{s}]([\sqrt{s}]+1)(2[\sqrt{s}]+1)}{6},$$

while for $f(x) = x^3$, we get

$$\sum_{j=1}^{s} \left[\sqrt[3]{j}\right] = (s+1) \cdot \left[\sqrt[3]{s}\right] - \left(\frac{[\sqrt[3]{s}]\left([\sqrt[3]{s}]+1\right)}{2}\right)^2.$$

Further, by applying (4.4) to the function $f(x) = x^2 - x$ and from Corollary 3.1, we obtain the formula

$$\sum_{j=1}^{s} \left[\frac{1+\sqrt{4j+1}}{2}\right] = (s+1)v_s - \frac{v_s(v_s^2-1)}{3},$$

where $v_s = \left[\frac{1+\sqrt{4s+1}}{2}\right]$.

Similarly we can obtain the following results about modified harmonic series. By taking $\varphi(x) = x^{-1}$ in (4.1), the sum of the inverses of the first s integers which are not triangular numbers is equal to

$$\sum_{j=0}^{s} \frac{1}{j+\left[\frac{3+\sqrt{8j+1}}{2}\right]} = \sum_{k=1}^{s+\left[\frac{3+\sqrt{8s+1}}{2}\right]} \frac{1}{k} + \frac{2}{\left[\frac{3+\sqrt{8s+1}}{2}\right]} - 2.$$

We are convinced that our summation method can be successfully used to compute or to evaluate other interesting partial sums of modified harmonic series.

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