ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by July 1, 2008. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1031 **Proposed by Andrew Cusumano, Great Neck, NY**

For any positive integer a, find an integer x such that

$$\lim_{n\to\infty}\left(\frac{F_{n+a}}{F_n}\right) = \frac{(2+\sqrt{5})F_a + F_{a-x}}{2}.$$

B-1032 **Proposed by Andrew Cusumano, Great Neck, NY**

Find a positive integer x such that

$$\frac{F_{n+3}(F_{n+1}F_{n+2}+F_n^2)(F_{n+2}F_{n+3}+F_{n+1}^2) - F_{n+1}^x}{F_{n+2}} - F_{n+1}F_{n+2}F_{n+3}(2F_{n+2}+F_n) = 1.$$

B-1033 **Proposed by Jyoti P. Shiwalkar and M.N. Despande, Nagpur, India**

Let i be a positive integer and $j = 1, 2, \cdots, i$. Define a triangular array $a(i,j)$ satisfying the following three conditions.

(1) $a(i,1) = F_i$ for all i
(2) $a(i,i) = i$ for all i
(3) $a(i,j) = a(i-1,j) + a(i-2,j) + a(i-1,j-1) - a(i-2,j-1)$.
For instance, if $i = 8$, then the array will be as follows:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & & & & & \\
 & & & & & 2 & & 2 & & 3 & & & & & \\
 & & & & 3 & & 5 & & 3 & & 4 & & & & \\
 & & & 5 & & 8 & & 9 & & 4 & & 5 & & & \\
 & & 8 & & 15 & & 15 & & 14 & & 5 & & 6 & & \\
 & 13 & & 26 & & 31 & & 24 & & 20 & & 6 & & 7 & \\
21 & & 46 & & 57 & & 54 & & 35 & & 27 & & 7 & & 8
\end{array}$$

Find a closed form for $\sum_{j=1}^{i} a(i,j)$.

B-1034 **Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA**

(a) Find a positive integer k and a polynomial $f(x)$ with rational coefficients such that

$$F_{kn} = f(L_n + F_n)$$

is an identity or prove that no identity of this form exists.

(b) Repeat part (a) with $f(L_n + F_n)$ replaced with $f(L_n - F_n)$.

(c) Repeat part (a) with $f(L_n + F_n)$ replaced with $f(L_n \times F_n)$.

B-1035 **Proposed by Hiroshi Matsui, Naoki Saita, Kazuki Kawata, Yusuke Sakurama, Toshiyuki Yamauchi, and Ryohei Miyadera, Kwansei Gakuin University, Nishinomiya, Japan**

Define $\{a_n\}$ by $a_1 = a_2 = 1$ and

$$a_n = a_{n-1} + a_{n-2} + \begin{cases} 1 & \text{if } n \equiv 1 \pmod 6 \\ 0 & \text{if } n \not\equiv 1 \pmod 6 \end{cases}$$

for $n \geq 3$. Express a_n in terms of F_n.

SOLUTIONS

Another Recursive Relation and Fibonacci

B-1019 **Proposed by Hiroshi Matsui, Naoki Saita, Kazuki Kawata, Yusuke Sakurama, and Ryohei Miyadera, Kwansei Gakuin University, Nishinomiya, Japan**
(Vol. 44, no. 3, August 2006)

(a) Define $\{a_n\}$ by $a_1 = a_2 = 1$ and

$$a_n = a_{n-1} + a_{n-2} + \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ 0 & \text{if } n \not\equiv 1 \pmod 4 \end{cases}$$

for $n \geq 3$. Express a_n in terms of F_n.
(b) Prove that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \alpha$.

Solution by H.-J. Seiffert, Thorwaldsenstr. 13, Berlin, Germany

(a) Let $a_0 = 0$, so that the given recurrence remains valid for $n = 2$. We shall prove that

$$a_n = F_{2\lfloor n/4\rfloor+2}\{F_{n-2\lfloor n/4\rfloor}\} \quad \text{for all } n \geq 0, \tag{1}$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function. If b_n denotes the expression on the right hand side of (1), then

$$\begin{matrix} b_{4k} = F_{2k+2}F_{2k}, & b_{4k-1} = F_{2k}F_{2k+1}, \\ b_{4k-2} = F_{2k}^2, \quad \text{and} & b_{4k-3} = F_{2k}F_{2k-1}, \end{matrix} \quad \text{for } k \geq 1. \tag{2}$$

From the basic recurrence of the Fibonacci numbers, it immediately follows that $b_{4k} = b_{4k-1} + b_{4k-2}$, $b_{4k-1} = b_{4k-2} + b_{4k-3}$, and $b_{4k-2} = b_{4k-3} + b_{4k-4}$, for $k \geq 1$. The relation [see [1], eqn. (3.32)] $F_{2k}F_{2k-1} - F_{2k+1}F_{2k-2} = 1$ together with $F_{2k+1} = F_{2k} + F_{2k-1}$ imply $b_{4k-3} = b_{4k-4} + b_{4k-5} + 1$, $k \geq 1$, where $b_{-1} := 0$. Now, it is easily seen that $a_n = b_n$ for all $n \geq 0$, which proves (1).

(b) From the Binet form of the Fibonacci numbers, it follows that $\lim_{j\to\infty} F_{j+1}/F_j = \alpha$. Hence, by (2) and $a_n = b_n$, $n \geq 0$,

$$\lim_{k\to\infty} \frac{a_{4k}}{a_{4k-1}} = \lim_{k\to\infty} \frac{a_{4k-1}}{a_{4k-2}} = \lim_{k\to\infty} \frac{a_{4k-2}}{a_{4k-3}} = \lim_{k\to\infty} \frac{a_{4k-3}}{a_{4k-4}} = \alpha,$$

which implies the desired limit relation.

References:

1 A.F. Horadam and Bro. J. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985): 7-20.

Also solved by Paul S. Bruckman, Charles K. Cook, G. C. Greubel, Russell J. Hendel, Harris Kwong, Jaroslav Seibert, and the proposers.

Two Fibonacci Identities

B-1020 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 44, no. 3, August 2006)

(a) Let $(A_j)_{j\geq 0}$ be any sequence of numbers such that $A_j \neq 0$ and $A_{j+1} = A_j + A_{j-1}$ for $j \geq 1$. Prove that, for all positive integers n,

$$A_1 \sum_{k=1}^{n} \left[A_0^{n-k} F_k \prod_{j=k}^{n} \frac{F_j}{A_j} \right] = F_n.$$

(b) Deduce the identities

$$\sum_{k=1}^{n} \left[2^{n-k} F_k \prod_{j=k}^{n} \frac{F_j}{L_j} \right] = F_n$$

and

$$3\sum_{k=1}^{n} 2^{n-k} F_k^2 F_{k+1} F_{k+2} = F_n F_{n+1} F_{n+2} F_{n+3}.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

(a) We first use induction to prove that

$$F_n \prod_{k=1}^{n} A_k = A_1 \sum_{k=1}^{n} \left[A_0^{n-k} F_k \left(\prod_{j=1}^{k-1} A_j \right) \left(\prod_{j=k}^{n} F_j \right) \right] \tag{1}$$

for all positive integers n. The identity holds when $n = 1$ because it reduces to $F_1 A_1 = A_1 F_1^2$. Observe that $F_n = A_0 F_{n-1} + A_1 F_n$, using which we can complete the inductive step as follows:

$$\begin{aligned}
F_n \prod_{k=1}^{n} A_k &= F_n \left(\prod_{k=1}^{n-1} A_k \right) A_n \\
&= F_n \left(\prod_{k=1}^{n-1} A_k \right) (A_0 F_{n-1} + A_1 F_n) \\
&= A_0 F_{n-1} \left(\prod_{k=1}^{n-1} A_k \right) F_n + A_1 F_n \left(\prod_{k=1}^{n-1} A_k \right) F_n \\
&= A_0 A_1 \sum_{k=1}^{n-1} \left[A_0^{n-1-k} F_k \left(\prod_{j=1}^{k-1} A_j \right) \left(\prod_{j=k}^{n-1} F_j \right) \right] F_n + A_1 F_n \left(\prod_{k=1}^{n-1} A_k \right) F_n \\
&= A_1 \sum_{k=1}^{n} \left[A_0^{n-k} F_k \left(\prod_{j=1}^{k-1} A_j \right) \left(\prod_{j=k}^{n} F_j \right) \right].
\end{aligned}$$

The desired result is obtained by dividing both sides of (1) by $\prod_{k=1}^{n} A_k$.

(b) Setting $A_n = L_n$ yields $A_0 = L_0 = 2$ and $A_1 = L_1 = 1$, which reduces (1) to

$$\sum_{k=1}^{n} \left[2^{n-k} F_k \prod_{j=k}^{n} \frac{F_j}{L_j} \right] = F_n.$$

Setting $A_n = F_{n+3}$ yields $A_0 = F_3 = 2$ and $A_1 = F_4 = 3$, which reduces (1) to

$$3\sum_{k=1}^{n} 2^{n-k} F_k \cdot \frac{F_k F_{k+1} F_{k+2}}{F_{n+1} F_{n+2} F_{n+3}} = F_n,$$

which is equivalent to

$$3\sum_{k=1}^{n} 2^{n-k} F_k^2 F_{k+1} F_{k+2} = F_n F_{n+1} F_{n+2} F_{n+3}.$$

Also solved by Paul S. Bruckman, Jaroslav Seibert, and the proposer.

A Sum as a Product

B-1021 Proposed by Harris Kwong, SUNY, Fredonia, Fredonia, NY (Vol. 44, no. 4, November 2006)

Prove that, for any integers m and N,

$$N^4 - 2L_mN^3 + 4F_{m+1}F_{m-1}N^2 + 2F_m^2L_mN - F_{2m}^2$$

is of the form $X^2(Y^2 - Z^2)$ for some integers X, Y and Z.

Solution by Steve Edwards, Southern Polytechnic State University, Marietta, GA

We start by expanding the expression $(N - L_m)^2(N^2 - F_m^2)$:

$$(N - L_m)^2(N^2 - F_m^2) = N^4 - 2L_mN^3 + (L_m^2 - F_m^2)N^2 + 2F_m^2L_mN - L_m^2F_m^2.$$

It is well-known that $L_m + F_m = 2F_{m+1}$, $L_m - F_m = 2F_{m-1}$, and $L_mF_m = F_{2m}$. This shows that $X = N - L_m$, $Y = N$, and $Z = F_m$.

We note that the formula holds for any number N, whether or not N is an integer.

Also solved by Paul S. Bruckman, Charles K. Cook, H.-J. Seiffert, and the proposer.

A Quartic as a Sum of Two Squares

B-1022 Proposed by Paul S. Bruckman, Sointula, Canada (Vol. 44, no. 4, November 2006)

If $u_n = F_{n+1}F_n$, and $x_n = \{3u_n + (-1)^n\}\{u_n - (-1)^n\}$, prove the following idenitity, for all integers n:

$$(x_n)^2 + (4u_{n+1}u_{n-1})^2 = (F_{2n+1})^4.$$

Solution by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

First

$$\begin{aligned} F_{2n+1}^4 &= (F_{n+1}^2 + F_n^2)^4 = (F_{n+1}^4 + 2u_n^2 + F_n^4)^2 \\ &= F_{n+1}^8 + F_n^8 + 4u_n^2(F_{n+1}^4 + F_n^4) + 6u_n^4. \end{aligned}$$

Next

$$\begin{aligned} u_{n+1} = F_{n+1}^2 + u_n \quad &\text{and} \quad u_{n-1} = u_n - F_n^2, \text{ so that} \\ u_{n+1}u_{n-1} = u_n^2 + (F_{n+1}^2 - F_n^2)u_n - u_n^2 &= (F_{n+1}^2 - F_n^2)u_n. \end{aligned}$$

Hence

$$(4u_{n+1}u_{n-1})^2 = 16u_n^2(F_{n+1}^4 + F_n^4 - 2u_n^2).$$

The identity $F_{m-1}F_m = F_m^2 - F_{m-1}^2 + (-1)^m$ attributed to A. Struyk by Koshy on page 76 in [1] can be written as $(-1)^n = F_{n+1}^2 - F_n^2 - u_n$. Thus

$$x_n = \{2u_n + (F_{n+1}^2 - F_n^2)\}\{2u_n - (F_{n+1}^2 - F_n^2)\} = 4u_n^2 - (F_{n+1}^2 - F_n^2)^2$$

so that

$$x_n^2 = 16u_n^4 - 8u_n^2(F_{n+1}^4 + F_n^4 - 2u_n^2) + (F_{n+1}^4 + F_n^4 - 2u_n^2)^2.$$

Finally

$$\begin{aligned} x_n^2 + (4n_{n+1}u_{n-1})^2 &= 16u_n^4 + 8u_n^2(F_{n+1}^4 + F_n^4 - 2u_n^2) \\ &\quad + F_{n+1}^8 + F_n^8 + 4u_n^4 + 2u_n^4 - 4u_n^2\{F_{n+1}^4 + F_n^4\} \\ &= F_{n+1}^8 + F_n^8 + 4u_n^2\{F_{n+1}^4 + F_n^4\} + 6u_n^4 = F_{2n+1}^4 \end{aligned}$$

as requested.

References:

1 Thomas Koshy, *Fibonacci and Lucas Numbers With Applications*, New York. John Wiley & Sons, Inc., 2001.

Also solved by H.-J. Seiffert, and the proposer.

And a Cubic as a Sum of Two Squares

B-1023 Proposed by Paul S. Bruckman, Sointula, Canada (Vol. 44, no. 4, November 2006)

Prove the following identity, for all integers n:

$$(F_n u_{n+2})^2 + (F_{n+1}u_{n-1})^2 = (F_{2n+1})^3,$$

where $u_n = (F_n)^2 + 2(-1)^n$.

Solution by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

Using the identities $(-1)^n = F_{n-1}F_{n+1} - F_n^2$, $F_{2n+1} = F_{n+1}^2 + F_n^2$, and $F_{n+3} = 2F_{n+1} + F_n$, as needed, it follows that

$$\begin{aligned} u_n &= F_n^2 + 2(F_{n-1}F_{n+1} - F_n^2) = 2F_{n+1}^2 - 2F_nF_{n+1} - F_n^2, \\ u_{n+2} &= 2F_{n+3}^2 - 2F_{n+2}F_{n+3} - F_{n+2}^2 = 2(2F_{n+1} + F_n)^2 - 2(F_{n+1} + F_n)(2F_{n+1} + F_n) \\ &\quad - (F_{n+1} + F_n)^2 = 3F_{n+1}^2 - F_n^2, \quad \text{and} \\ u_{n-1} &= 2F_n^2 - 2F_{n-1}F_n - F_{n-1}^2 = 2F_n^2 - 2(F_{n+1} - F_n)F_n - (F_{n+1} - F_n)^2 \\ &= 3F_n^2 - F_{n+1}^2. \end{aligned}$$

Thus

$$\begin{aligned} (F_nu_{n+2})^2 + (F_{n+1}u_{n-1})^2 &= F_n^2(9F_{n+1}^4 - 6F_{n+1}^2F_n^2 + F_n^4) \\ &\quad + F_{n+1}^2(9F_n^4 - 6F_{n+1}^2F_n^2 + F_{n+1}^4) = 3F_{n+1}^2F_n^2(F_{n+1}^2 + F_n^2) + F_{n+1}^6 + F_n^6 \\ &= F_{n+1}^6 + 3F_{n+1}^4F_n^2 + 3F_{n+1}^2F_n^4 + F_n^6 = (F_{n+1}^2 + F_n^2)^3 \\ &= F_{2n+1}^3 \end{aligned}$$

as required.

Also solved by H.-J. Seiffert, and the proposer.

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