THE PASCAL MATRIX

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The $n \times n$ matrix P or $P(n)$ whose coefficients are the elements of Pascal's triangle has been suggested as a test datum for matrix inversion programs, on the grounds that both itself and its inverse have integer coefficients. For example, if $n = 4$

$$(1) \qquad P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 4 & -6 & 4 & -1 \\ -6 & 14 & -11 & 3 \\ 4 & -11 & 10 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 6 & 4 & 1 \\ 6 & 14 & 11 & 3 \\ 4 & 11 & 10 & 3 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix},$$

$$|P - \lambda I| = \lambda^4 - 29\lambda^3 + 72\lambda^2 - 29\lambda + 1.$$

It occurred to us to take a closer look at this entertaining object. We shall require a couple of binomial coefficient identities, both of which are easily proved by induction from the fundamental relation

$$\binom{i}{j} = \binom{i-1}{j-1} + \binom{i-1}{j} = (i+j)!/i!j!,$$

or 0 unless $0 \leqslant j \leqslant i$.

$$(2) \qquad \sum_k \binom{s}{k+u}\binom{t}{k} = \binom{s+t}{s-u}.$$

$$(3) \qquad \sum_k \binom{s+k}{u}\binom{t}{k}(-)^k = \binom{s}{u-t}(-)^t.$$

(Here and subsequently all summations over i, j, k, etc., are implicitly over the values 0 to $n - 1$. Notice that our matrix subscripts are also taken over this domain.) P is defined by

$$P_{ij} = \binom{i+j}{i}.$$

First notice that the determinant of P is unity. For subtracting from each row the row above, and similarly differencing the columns, we find

$$(4) \qquad P(n) = \begin{vmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & P(n-1) \\ 0 & \end{vmatrix} = |P(n-1)| = |P(0)| = 1$$

It follows that P^{-1} has integer coefficients, since they are signed minors of P divided by $|P|$. As it happens, there is a nice explicit formula for them:

(5) $$(P^{-1})_{ij} = (-)^{i+j} \sum_k \binom{k}{i}\binom{k}{j}.$$

Proof of (5). Let the RHS temporarily define a matrix Q. Then

$$(PQ)_{ij} = \sum_p \binom{i+p}{i}\left[(-)^{p+j}\sum_k \binom{k}{p}\binom{k}{j}\right]$$

$$= \sum_k \binom{k}{j}(-)^j\left[\sum_p \binom{i+p}{i}\binom{k}{p}(-)^p\right]$$

$$= \sum_k \binom{k}{j}\binom{i}{k}(-)^{i+j} \quad \text{by (3)}$$

$$= \binom{0}{i-j}(-)^{i+j} = \delta_{ij} \quad \text{by (3) again.}$$

That is, $PQ = I$ and $Q = P^{-1}$.

The decomposition of P into lower- and upper-triangular factors is simply

(6) $$P = LU, \text{ where } L_{ij} = \binom{i}{j},\quad U_{ij} = \binom{j}{i};$$

since $(LU)_{ij}$ is immediately reducible to P_{ij} via (2). And from (5) it is immediate that

(7) $$(UL)_{ij} = |(P^{-1})_{ij}|,$$

or the coefficients of UL are the moduli of those of P^{-1}.

Turning to the characteristic polynomial of P, we need the following method of computing

$$|A - \lambda I| = \sum_m c_m \lambda^{n-m}$$

for any matrix A :—

(8) $$\text{Let } d_k = \operatorname{trace}(A^k) = \sum_i (A^k)_{ii} \text{ for } k > 0.$$

$$d_0 = m \quad (\text{instead of } n),$$

$$c_0 = 1,$$

then

$$\sum_k c_{m-k} d_k = 0.$$

This relation enables us to compute the c's in terms of the d's or vice versa, e.g.,

$$c_0 = 1$$
$$c_1 = -d_1$$
$$c_2 = -\tfrac{1}{2}(c_1 d_1 + c_0 d_2) = \tfrac{1}{2}(d_1^2 - d_2)$$
$$c_3 = \tfrac{1}{3}(c_2 d_1 + c_1 d_2 + c_0 d_3) = \tfrac{1}{6}(-d_1^3 + 3d_1 d_2 - 2d_3).$$

Proof of (8). The eigenvalues of A^k are just the k^{th} powers of the eigenvalues of A and our relation is simply a special case of Newton's identity, which relates the coefficients of a polynomial to the sums of k^{th} powers of its roots, etc. (In numerical computation this formula suffers from heavy cancellation.)

Notice also that, by the definition of matrix multiplication, for $m > 0$

$$d_m = \sum_i \sum_j \sum_k \cdots \sum_q \sum_r A_{ij}A_{jk} \cdots A_{ri} \tag{9}$$

(over m summations and factors).

Now suppose that $A = P(n)$, and denote by C_m and D_k the values of $(-)^m c_m$ and d_k; the former are tabulated for a few small n at the end. The first thing to strike the eye is their symmetry:—

$$C_m = C_{n-m}\ . \tag{10}$$

To prove this, it is by (8) enough to show that $D_m = D_{n-m}$. Also since the eigenvalues of P^{-1} are the reciprocals of those of P, and the determinant of P is unity (4), the characteristic polynomial of P^{-1} is just the reverse of that of P. So it is enough to show that $D_m = d_m(P^{-1})$. But by (9) and (5)

$$\begin{aligned} d_m(P^{-1}) &= \sum_{i,j,k,\ell} \left[\sum_p \binom{p}{i}\binom{p}{j}\right]\left[\sum_q \binom{q}{j}\binom{q}{k}\right]\left[\sum_r \binom{r}{k}\binom{r}{\ell}\right]\cdots \\ &= \sum_{p,q,r} \left[\sum_j \binom{p}{j}\binom{q}{j}\right]\left[\sum_k \binom{q}{k}\binom{r}{k}\right]\cdots \\ &= \sum_{p,q,r} \binom{p+q}{p}\binom{q+r}{q} \cdots \quad \text{by (2)} \\ &= \sum_{p,q,r} P_{pq}P_{qr} \cdots = D_m \quad \text{by (6) and (9), QED.} \end{aligned}$$

Incidentally, setting $m = 2$ shows that the sums of squares of coefficients (the d_2) are the same for P and P^{-1}.

The next striking feature is

$$C_m > 0. \tag{11}$$

If the characteristic polynomial of some A is expanded explicitly in the form $|A - \lambda I|$, it is easily seen that $(-)^m c_m$ is the sum of all principal $m \times m$ minors of A. So (11) is a consequence of the more general result

(12) Every minor of P is positive.

We denote by $M = M(i,k,\cdots,o,q;j,\ell,\cdots,p,r)$ the $m \times m$ minor of P

$$\begin{vmatrix} P_{ij} & & & & \\ & P_{k\ell} & & & \\ & & \ddots & & \\ & & & P_{op} & \\ & & & & P_{qr} \end{vmatrix}$$

and define the "type" of M to be the triple (m,q,r). One triple is said to be "less than or equal to" another if this relation holds between corresponding pairs of elements. With this ordering we prove (12) by induction on the type.

The result is clearly true for $m = 0$, since any 0×0 determinant has value 1. Suppose then that $m > 0$ and the result is true for all types less than (m,p,r). According to the fundamental relation $P_{qr} = P_{q-1,r} + P_{q,r-1}$ etc., so we can decompose the final row of M to obtain $M =$

$$M(i, \cdots, q-1; j, \cdots, r) + \begin{vmatrix} P_{ij} & & \\ & \ddots & \\ & & P_{op} \\ P_{q,j-1} & \cdots & P_{q,r-1} \end{vmatrix},$$

where the final row of the latter determinant has been shifted one place to the left. Repeating the decomposition on the new minor, we eventually reach a zero minor when the final row coincides with row o, and so

$$M = \sum_{q'=o+1}^{q} \begin{vmatrix} P_{ij} & & \\ & \ddots & \\ & & P_{op} \\ P_{q',j-1} & \cdots & P_{q',r-1} \end{vmatrix}.$$

Decomposing all the other rows of the summand in turn, we finally get them lined up again to form a respectable minor, thus

(13) $$M = \sum_{i',k',\cdots,o',q'} M(i',k',\cdots,o',q'; j-1, \ell-1, \cdots, p-1, r-1),$$

where $-1 < i' \leqslant i < k' \leqslant k < \cdots < o' \leqslant o < q' \leqslant q$.

If $j > 0$, each summand is of type at most $(m,q,r-1)$. If $j = 0$, we need to introduce another row and column for P, defined by $P_{-1,k} = P_{k,-1} = \delta_{0k}$, to preserve the sense of (13): we need then only consider the case $i' = 0$, and (13) becomes

$$M = \sum_{k',\cdots,o',q'} M(k', \cdots o',q'; \ell-1, \cdots, p-1, r-1),$$

in which each summand is of type at most $(m-1, q, r-1)$. In either case M is a sum of minors of lesser type and therefore is positive, QED.

We can squeeze more than (11) out of (12): since $C_m(n+1)$ includes all the minors in $C_m(n)$, it follows that

(14) $C_m(n)$ is an increasing function of n.

A squint at the data suggests the tougher conjecture

(15) $C_m(n)$ is an increasing function of m for $m < \frac{1}{2}n$?

Concerning P in general, some further questions suggest themselves. The maximum element of P is clearly $P_{nn} \sim 4^n/\sqrt{(\frac{1}{2}\pi n)}$ by Stirling's approximation; but what about that of P^{-1}?

How are the eigenvalues of P distributed? By (10) they occur in inverse pairs, with 1 an eigenvalue for all odd n; how big is the largest? Since $P = LL'$, it is positive definite and they are all positive.

1							
1	1						
1	3	1					
1	9	9	1				
1	29	72	29	1			
1	99	626	626	99	1		
1	351	6084	13869	6084	351	1	
1	1275	64974	347020	347020	64974	1275	1

Coefficients of $|P(n) + \lambda I|$, n (descending) = 0(1)7.

REFERENCES

1. J. Riordan, *Combinatorial Identities,* Wiley (1968).
2. A. Aitken, *Determinants and Matrices,* Oliver & Boyd (1962).
3. I. N. Herstein, *Topics in Algebra,* Blaisdell (1964).
4. Whittaker & Watson, *Modern Analysis,* C. U. P. (1958).

Riordan discusses binomial coefficients, Aitken elementary matrix properties, Herstein mentions Newton's identity, Whittaker and Watson Stirling's approximation.

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