A GENERALIZATION OF FIBONACCI NUMBERS

V .C. HARRIS and CAROLYN C. STYLES
San Diego State College and San Diego Mesa College,
San Diego, California

1. INTRODUCTION

Presented here is a generalization of Fibonacci numbers which is intimately connected with the arithmetic triangle. It at once goes beyond and falls short of other generalizations. In section 2 the numbers are defined and denoted by $u(n;\ p,\ q)$ where p is a non-negative integer and q is a positive integer. The characteristic equation is shown to be

$$x^p(x - 1)^q - 1 = 0. \tag{1.1}$$

The numbers are represented in the usual manner in terms of powers of roots of the equation and certain initial conditions. In section 3 certain sums and properties involving sums are developed and in section 4 there is made a beginning in the study of divisibility properties.

The generalization made here may be compared with characteristic equations obtained in other generalizations:

by Dickinson [2], $x^c - x^a - 1 = 0$ (a, c integers)

by Miles [4], $x^k - x^{k-1} - \ldots - x - 1 = 0$ (k integral, ≥ 2)

by Raab [5], $x^{r+1} - ax^r - b = 0$ (a, b real; r integral, ≥ 1)

by Feinberg [8], $x^{nu+1} - \sum_{i=0}^{n} x^{ui} = 0$, various positive integral values of u, n.

Generalizations by Basin [1] and Horadam [3] involve altering only the initial conditions of the Fibonacci sequence.

The numbers studied here are special cases of sums defined in Netto [6] and Dickinson [2] and their definition and relation to the arithmetic triangle appear in Hochster [7].

2. THE NUMBERS u(n; p, q)

Let p and q be integers with $p \geqq 0$ and $q > 0$. Then by definition the n-th generalized Fibonacci number of step $p,\ q$ is

$$(2.1)\qquad u(n;\,p,\,q) = \sum_{i=0}^{\left[\frac{n}{p+q}\right]} \binom{n-ip}{iq}, \quad n \geq 1, \qquad u(0;\,p,\,q) = 1$$

Here $[x]$ denotes the greatest integer $\leq x$. In particular,

$$u(n-1;\,1,\,1) = f_n \text{ (the } n\text{-th Fibonacci number)}, \; n = 1,\,2,\,\ldots$$

$$u(n;\,0,\,1) = 2^n$$

When the definition is related to the arithmetic triangle one sees that $u(n;\,p,q)$ is the sum of the term in the first column and the n-th row (counting the top row as the zero-th row) and the terms obtained starting from this term by taking steps p, q -- that is, p units up and q units to the right.

It follows that

$$u(0;\,p,\,q) = u(1;\,p,\,q) = \ldots = u(p+q-1;\,p,q) = 1, \; u(p+q;\,p,\,q) = 2$$

If ∇ is the backward difference operator, so that

$$\nabla f(x) = f(x)-f(x-1),$$

then

$$(2.2)\qquad \nabla^q u(n;\,p,q) = u(n-p-q;\,p,\,q), \qquad n \geq p+q\,.$$

From properties of binomial coefficients and

$$\nabla^q u(n;\,p,\,q) = \nabla^{q-1}\nabla u(n;\,p,\,q)$$

it follows that

$$\nabla^q u(n;\,p,\,q) = \sum_{i=0}^{\left[\frac{n-p-q}{p+q}\right]} \binom{n-p-q-ip}{iq}$$

$$= u(n-p-q;\,p,\,q) \qquad , \; n \geq p+q\,.$$

This proves (2.2). In terms of forward differences this is

$$\nabla^q u(n-q;\ p,\ q) = u(n-p-q;\ p,\ q) \quad , \quad n \geq p+q \ .$$

The characteristic equation and initial conditions consequently are

$$x^p(x-1)^q - 1 = 0 \tag{2.3}$$

$$u(n;\ p,\ q) = 1, \qquad n = 0,\ 1,\ \ldots,\ p+q-1.$$

Let

$$u(n;\ p,\ q) = \sum_{i=1}^{p+q} c_i\, x_i^{n+1}$$

where x_i, $i = 1, 2, \ldots, (p+q)$ are the roots of (2.3).
The derivative of

$$f(x) = x^p\,(x-1)^q - 1 \quad \text{is} \quad f'(x) = px^{p-1}(x-1)^q + q\,x^p\,(x-1)^{q-1}$$
$$= x^{p-1}(x-1)^{q-1}((p+q)\,x-p) \ .$$

Since no root of $f'(x)$ is a root of $f(x)$, it follows that $f(x)$ has no multiple root. Hence the determinant of the coefficients of

$$\sum_{i=1}^{p+q} c_i\, x_i^{n+1} = u(n;\ p,\ q) = 1,\ n = 0,\ \ldots,\ p+q-1$$

is different from zero. The system can be solved by Cramer's rule with Vandermondians (as in several of the references). It results that

$$c_i = 1/((p+q)x_i - p)$$

and

$$u(n;\ p,\ q) = \sum_{i=1}^{p+q} \frac{x_i^{n+1}}{(p+q)\,x_i - p}\ , \quad n = 0,\ 1,\ 2,\ \ldots \tag{2.4}$$

There is a positive real root $x_1 > 1$. This follows from $f(1) < 0$ and $f(2) \geqq 0$. Since $f'(x) \neq 0$ for $x > 1$ there is no other real root > 1. Also $|x_1|$ exceeds the absolute value of each other root. For if $x_2 \neq x_1$ is a root and $|x_2| \geq x_1$ then

$$|x_2^p(x_2-1)^q| = |x_2|^p\,|x_2-1|^q > |x_1|^p\,|x_1-1|^q > 1$$

so that (2.2) cannot be satisfied, a contradiction. From this it follows

$$\lim_{n\to\infty} \frac{u(n+1;\, p, q)}{u(n;\, p, q)} = x_1 \tag{2.5}$$

To show this, merely note

$$\lim_{n\to\infty} \frac{u(n+1;\, p, q)}{u(n;\, p, q)} = \lim_{n\to\infty} \frac{u(n+1;\, p, q)/x_1^{n+2}}{u(n;\, p, q)/x_1^{n+2}} = x_1 \;.$$

We remark that if we choose initial conditions $u(0;\, p, q) = u(1;\, p, q) = \ldots = u(p+q-2;\, p, q) = 1$, $u(p+q-1\,;\, p, q) = p+q+1$, then we have a sequence $(w(n;\, p, q))$, where

$$w(n;\, p,\, q) = \sum_{i=1}^{p+q} x_i^{n+1} \quad , \quad n = 0,\, 1,\, 2,\, \ldots$$

Moreover, a convenient form for expressing $u(n,\, p, q)$ arises from writing the difference equation as

$$u(n;\, p,\, q) = \binom{q}{1} u(n-1;\, p,\, q) - \binom{q}{2} u(n-2;\, p,\, q) + - \cdots + (-1)^{q-1} u(n-q;\, p, q) + u(n-p-q;\, p,\, q), \quad n \geq p + q \;. \tag{2.6}$$

3. SUMS

Theorem 3.1. The relation

$$\sum_{i=0}^{n} u(i;\, p,\, q) = \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} u(n+p+q-i;\, p,\, q) - \delta_{1q} \tag{3.1}$$

holds, where δ_{1q} is Kroneeker's δ and $\binom{q-1}{i} = 1$ in the case

$$q = 1, \quad i = 0 \;.$$

If (3.1) holds for n, for $q \geq 2$, then

$$\sum_{i=0}^{n+1} u(i;\, p,\, q) = u(n+1;\, p,\, q) + \sum_{i=0}^{n} u(i;\, p,\, q)$$

$$= \sum_{i=0}^{q} (-1)^i \binom{q}{i} u(n+1+p+q-i;\, p,\, q)$$

$$+ \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} u(n+p+q-i;\, p,\, q) - \delta_{1q}$$

$$= \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} u(n+1+p+q-i;\, p,\, q) - \delta_{1q}$$

Hence (3.1) holds for $n + 1$. When $n = 0$, with $q \geq 2$, then (3.1) becomes

$$\sum_{i=0}^{0} u(i;\, p,\, q) = \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} u(p+q-i;\, p,\, q) - \delta_{1q}$$

$$= u(p+q;\, p,\, q) + \sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i} = 1 = u(0;\, p,\, q)$$

To complete the proof, we consider $q = 1$. Then

$$u(i;\, p,\, 1) = u(p+1+i;\, p,\, 1) - u(p+i;\, p,\, 1)$$

Hence

$$\sum_{i=0}^{n} u(i;\, p,\, 1) = u(n+p+1;\, p,\, 1) - u(p;\, p,\, 1)$$

$$= u(n+p+1;\, p,\, 1) - \delta_{11}\ .$$

Theorem 3.2.

$$(3.2)\quad \sum_{i=0}^{m} (-1)^{m-i} u(i;p,q) = \frac{1}{1-(-1)^{p+q}2^q}\Bigg[\sum_{k=0}^{q-1}\sum_{j=0}^{k}(-1)^k\binom{q}{j} u(m+p+q-k;p,q) + (-1)^{m+p+q}2^q \sum_{i=m+1}^{m+p}(-1)^i u(i;p,q) + (-1)^{m-1}2^{q-1} + (-1)^{m+p+q-1}2^q\epsilon\Bigg],$$

where $\epsilon = 0$, $p+q$ even, and $\epsilon = 1$, $p+q$ odd.

Proof. Writing

$$(-1)^j u(m-j;\, p,q) = (-1)^{m-j} u(m+p+q-j;\, p,q) + (-1)^{m-j-1}\binom{q}{1} u(m+p+q-j-1;\, p,q) + \dots + (-1)^{m+q}\binom{q}{q} u(m+p-j;\, p,q)$$

and summing for $j = 0, 1, \dots, m$ gives for the sum S,

$$S = \sum_{k=0}^{q-1}\sum_{j=0}^{k}(-1)^k\binom{q}{j} u(m+p+q-k;\, p,q) + (-1)^q 2^q \sum_{r=0}^{m-q}(-1)^r u(m+p-r;\, p,q) + (-1)^{m-1}2^{q-1}$$

$$= \sum_{k=0}^{q-1}\sum_{j=0}^{k}(-1)^k\binom{q}{j} u(m+p+q-k;\, p,q) + (-1)^q 2^q \sum_{r=0}^{m+p}(-1)^r u(m+p-r;\, p,q) + (-1)^{m-1}2^{q-1} + (-1)^{q-1}2^q \sum_{r=m-q+1}^{m+p}(-1)^r u(m+p-r;\, p,q)$$

$$= \sum_{k=0}^{q-1}\sum_{j=0}^{k}(-1)^k\binom{q}{j} u(m+p+q-k;\, p,q) + (-1)^{p+q} 2^q \sum_{i=0}^{m+p}(-1)^{m-i} u(i;\, p,q) + (-1)^{m-1}2^{q-1} + (-1)^{m+p+q-1}2^q \sum_{i=0}^{p+q-1}(-1)^i u(i;\, p,q)$$

Solving for S, and noting

$$\sum_{i=0}^{p+q-1} (-1)^i u(i; p, q) = \begin{Bmatrix} 0 & p+q & \text{even,} \\ 1 & p+q & \text{odd} \end{Bmatrix} = \epsilon ,$$

we get the result (3.2).

From (3.1) and (3.2) we can obtain expressions yielding

$$\sum_{i=0}^{n} u(2i; p, q) \quad \text{and} \quad \sum_{i=0}^{n} u(2i + 1; p, q) .$$

In the simpler case where $q = 1$, we find

$$(3.3) \quad \sum_{i=0}^{n} u(2i + 1; p, 1) = \frac{1}{2} (u(2n+p+2; p, 1) - 1 + \sum_{i=0}^{\frac{2n-p-\eta}{2}} u(2i+\eta ; p, 1)$$

and

$$(3.4) \quad \sum_{i=0}^{n} u(2i; p, 1) = \frac{1}{2} \left[u(2n+p+2; p, 1) - 1 - \sum_{i=0}^{\frac{2n-p-\eta}{2}} u(2i+\eta; p, 1)\right]$$

where $\eta = 0$ when p is even and $\eta = 1$ when p is odd. In this case it is simpler to start with

$$u(2i+1; p, 1) = u(2i; p, 1) + u(2i-p; p, 1) , \quad 2i \geq p$$

$$= u(2i; p, 1) , \quad 0 \leq 2i < p$$

and sum. We obtain in this way

$$(3.5) \quad \sum_{i=0}^{n} u(2i+1; p, 1) = \sum_{i=0}^{n} u(2i; p, 1) + \sum_{i=0}^{\frac{2n-p-\eta}{2}} u(2i+\eta; p, 1) .$$

Since we also can write

$$\sum_{i=0}^{2n+1} u(i;\, p, 1)$$

as

$$(3.6) \qquad \sum_{i=0}^{n} u(2i+1;\, p, 1) + \sum_{i=0}^{n} u(2i;\, p, 1) = u(2n+p+2;\, p, 1) - 1$$

by (3.1), the results (3.3) and (3.4) follow by addition and subtraction and solving for the sum.

For $p = 1$ these results reduce to the well-known relations of Fibonacci numbers:

$$(3.1') \qquad \sum_{i=1}^{n} f_i = f_{n+2} - 1$$

$$(3.2') \qquad \sum_{i=1}^{n} (-1)^{n-i} f_i = f_{n-1} + (-1)^{n-1}$$

$$(3.3') \qquad \sum_{i=1}^{n} f_{2i} = f_{2n+1} - 1$$

$$(3.4') \qquad \sum_{i=1}^{n} f_{2i-1} = f_{2n} \quad .$$

Theorem 3.3. Let $q = 1$ and define $u(i;\, p, 1) = 0$ for i a negative integer. Then

$$(3.7) \quad u(n+m;\, p, 1) = u(n;\, p, 1)u(m;\, p, 1) + \sum_{i=0}^{p-1} u(n-1-i;\, p, 1)u(m-p+i;\, p, 1) \quad ,$$

where n, m are any positive integers or zero. To prove this we note first that this is true for n any positive integer or zero and $m = 0$. For n any positive integer or zero and $0 < m = k \leq p$ we have

$$u(n;\ p,1)u(k;\ p,1) + \sum_{i=0}^{p-1} u(n-1-i;\ p,1)u(k-p+i;\ p,1)$$

$$= u(n;\ p,1) + \sum_{i=p-k}^{p-1} u(n-1-i;\ p,1)$$

$$= u(n;\ p,1) + \sum_{j=n-p}^{n+k-p-1} u(j;\ p,1)$$

$$= u(n;\ p,1) + \sum_{j=0}^{n+k-p-1} u(j;\ p,1) - \sum_{j=0}^{n-p-1} u(j;\ p,1)$$

$$= u(n;\ p,1) + u(n+k;\ p,1) - u(n;\ p,1)$$

$$= u(n+k;\ p,1)$$

where the sums have been evaluated using (3.1). Hence (3.7) is true for n any positive integer or zero and $m = 0, 1, \ldots, p$. For $m = p+1$ we get

$$u(n;\ p,1)u(p+1;\ p,1) + \sum_{i=0}^{p-1} u(n-1-i;\ p,1)u(p+1-p+i;\ p,1)$$

$$= 2\,u(n;\ p,1) + \sum_{i=0}^{p-1} u(n-1-i;\ p,1)$$

$$= u(n;\ p,1) + \sum_{j=n-p}^{n} u(j;\ p,1)$$

$$= u(n+p+1;\ p,1)\quad .$$

Assume now, finally, that (3.7) is true for n any positive integer or zero and $m = 0, 1, \ldots, p, \ldots, k$ where $k \geq p+1$. Then

$$u(n+k-p;\, p, 1) = u(n;\, p, 1)u(k-p;\, p, 1) + \sum_{i=0}^{p-1} u(n-1-i;\, p, 1)u(k-2p+i;\, p, 1)$$

$$u(n+k;\, p, 1) \quad = u(n;\, p, 1)u(k;\, p, 1) + \sum_{i=0}^{p-1} u(n-1-i;\, p, 1)u(k-p+i;\, p, 1)$$

Hence

$$\begin{aligned} u(n+k+1;\, p, 1) &= u(n+k;\, p, 1) + u(n+k-p;\, p, 1) \\ &= u(n;\, p, 1)\ \left[u(k;\, p, 1) + u(k-p;\, p, 1)\right] \\ &\quad + \sum_{i=0}^{p-1} u(n-1-i;\, p, 1)\ \left[u(k-p+i;\, p, 1) + u(k-2p;p, 1)\right] \\ &= u(n;p, 1)u(k+1;p, 1) + \sum_{i=0}^{p-1} u(n-1-i;p, 1) \cdot \\ &\qquad \cdot u(k+1-p+i;p, 1) \end{aligned}$$

But this is (3.7) with $m = k+1$ and the theorem is proved.

For $m = n$, equation (3.7) becomes

$$(3.8)\quad u(2n;p, 1) = u^2(n;p, 1)+u^2(n-\tfrac{p+1}{2};p, 1)+2 \sum_{i=1}^{\frac{p-1}{2}} u(n-i;p, 1)u(n-(p+1)+i;p, 1), \qquad p \text{ odd}$$

and

$$(3.9)\quad u(2n;p, 1) = u^2(n;p, 1) + 2 \sum_{i=1}^{\frac{p}{2}} u(n-i;p, 1)u(n-(p+1)+i;p, 1), \qquad p \text{ even.}$$

For $m = n+1$, equation (3.7) becomes

$$3.10)\quad u(2n+1;p, 1) = u^2(n;p, 1) + 2 \sum_{i=0}^{\frac{p-1}{2}} u(n-i;p, 1)u(n-p+i;p, 1), \qquad p \text{ odd}$$

and

(3.11) $$u(2n+1;p,1) = u^2(n;p,1) + u^2(n-\tfrac{p}{2};p,1) + 2\sum_{i=0}^{\frac{p}{2}-1} u(n-i;p,1)u(n-p+i;p,1), \quad p \text{ even}$$

When $p = 1$ equations (3.7), (3.8) and (3.10) reduce to the known relationships

(3.7') $$f_{n+m+1} = f_{n+1}f_{m+1} + f_n f_m$$

(3.8') $$f_{2n+1} = f_{n+1}^2 + f_n^2$$

(3.10') $$f_{2n} = f_n^2 + 2f_n f_{n-1}$$

4. DIVISIBILITY PROPERTIES

Theorem 4.1. Any $p + q$ consecutive terms are relatively prime.

The terms $u(0; p,q), \ldots, u(p + q - 1; p, q)$ are all unity and so relatively prime. Any $p + q$ consecutive terms containing one of these will have greatest common divisor 1. Assume $(u(n; p,q), u(n + 1; p,q), \ldots, u(n + p + q - 1; p, q)) = d$, where $n > p + q - 1$. Then because of (2.2) it follows

$$d \mid (u(n - 1; p,q), u(n; p,q), \ldots, u(n + p + q - 2; p, q)).$$

Successive applications will show

$$d \mid (u(p + q - 1; p,q), u(p + q; p, q), \ldots, u(2p + 2q - 2; p, q)) .$$

This contains $u(p + q - 1; p, q)$ so that $d = 1$ and the theorem follows.

Theorem 4.2. The least non-negative residues modulo any positive integer m of $\{u(n; p, q)\}$ are periodic with period P not exceeding m^{p+q}. There is no preperiod. Each period begins with $p + q$ terms all unity.

There are m possible least non-negative residues modulo m for each $u(n; p,q)$ and m^{p+q} possible arrangements of residues in $p + q$ consecutive terms. Since by (2.2) the residue of $u(n; p,q)$

depends upon the residues of the preceding $p + q$ terms, after m^{p+q} terms at most the residues must repeat with a period P. Suppose $u(n + p;\ p,\ q)$ is the first term such that the residues repeat and assume $n > 0$. Then

$$u(n + P + j;\ p, q) \equiv u(n + j;\ p, q) \pmod{m}, \quad j = 0,\ 1,\ \ldots,\ p + q\ .$$

In view of the recursion formula, this shows

$$u(n - 1 + P;\ p, q) \equiv u(n - 1;\ p, q) \pmod{m}\ ,$$

a contradiction to the assumption $u(n + P;\ p,\ q)$ is the first term such that the residues repeat. Thus $n = 0$ and there is no preperiod. Hence each period begins with $p + q$ terms each unity.

As an example, we have residues (mod 7) for $u(n;\ 2, 1)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
r	1	1	1	2	3	4	6	2	6	5	0	6	4	4	3	0	4	0	0	4	4	4

n	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43
r	1	5	2	3	1	3	6	0	3	2	2	5	0	2	0	0	2	2	2	4	6	1

n	44	45	46	47	48	49	50	51	52	53	54	55	56			
r	5	4	5	3	0	5	1	1	6	0	1	0	0	1	1	1

Here $P = 57$.

Theorem 4.3 Any prime divides infintely many $u(n;\ p, q)$. If the period of the residues (mod m) is P, then m divides each of

$$u(P - 1 + P\,k;\ p, q),\ u(P - 2 + P\,k;\ p, q),\ \ldots,\ u(P - p + P\,k;\ p, q),$$
$$k = 0,\ 1,\ 2, \ldots$$

Since the residues are periodic it is sufficient, to establish the first part of the theorem, to show that any prime divides one $u(n;\ p, q)$. Let m be any given prime or multiple of any given prime. Then with P the period,

$$u(P;\ p, q) \equiv u(P + 1;\ p, q) \equiv \ldots \equiv u(P + p + q - 1;\ p, q) \equiv 1 \pmod{m}.$$

From the recursion formula,

$$u(P-1;\, p, q) \equiv \sum_{i=0}^{q} (-1)^i \binom{q}{i} u(P - 1 + p + q - i;\, p,\, q)$$

$$\equiv \sum_{i=0}^{q} (-1)^i \binom{q}{i}$$

$$\equiv 0 \pmod{m}$$

Hence $m \mid u(P-1;\, p,\, q)$. Similarly for $u(P-2;\, p,\, q), \ldots, u(P-p;\, p, q)$.

In the previous example, we note $7 \mid u(56;\, 2,\, 1)$, and $7 \mid u(55;\, 2,\, 1)$. Of course, 7 also divides other terms, as the table indicates.

REFERENCES

1. Basin, S. L., Generalized Fibonacci Sequences and Squared Rectangles, American Mathematical Monthly, Vol. 70, 1963, pp. 372-379.
2. Dickinson, David, On Sums Involving Binomial Coefficients, American Mathematical Monthly, Vol. 57, 1950, pp. 82-86.
3. Horadam, A. F., A Generalized Fibonacci Sequence, American Mathematical Monthly, Vol. 68, 1961, pp. 455-459.
4. Miles, E. P., Jr., Generalized Fibonacci Numbers and Associated Matrices, American Mathematical Monthly, Vol. 67, 1960, pp. 745-752.
5. Raab, Joseph A., A Generalization of the Connection Between the Fibonacci Sequence and Pascal's Triangle, The Fibonacci Quarterly, Vol. 1, Oct., 1963, pp. 21-31.
6. Netto: Lehrbuch der Kombinatorik, Teubner, Leipzig, 1901, p. 247.
7. Hochster, Melvin, Fibonacci-type series and Pascal's triangle, Particle, Vol. IV, 1962, pp. 14-28.
8. Feinberg, Mark, New Slants, The Fibonacci Quarterly, Vol. 2, 1964, pp. 223-227.

XXXXXXXXXXXXXXX