JACOBSTHAL AND PELL CURVES

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1. INTRODUCTION

In an earlier paper [2], a study was made of Fibonacci and Lucas curves in the plane, and their Laser-printed graphs were exhibited. These graphs were drawn from the equations of the curves, rather than from the tabulated lists of values of the Cartesian coordinates x and y, which also served a purpose of their own. It is desirable to extend the work in [2] and so produce a more complete theory.

Here, we present basic information about the corresponding curves associated with (i) Pell and Pell-Lucas numbers, and (11) Jacobsthal and *Jacobsthal-Lucas* numbers, in our nomenclature.

Curves associated with (i) will carry the generic name of *Pell curves* while those connected with (ii) will be designated *Jacobsthal curves*. There seems to be no theory related to (i) and (ii) which corresponds to the result of Halsey [1] for Fibonacci numbers.

To avoid unnecessary duplication in our discussion, we will consider the numbers in (i) and (ii) (as well as the Fibonacci and Lucas numbers) to be special instances of a general sequence $\{w_n\}$ whose relevant properties will be investigated.

Thus, the Pell and Jacobsthal curves, as well as the Fibonacci and Lucas curves, may be thought of as members of a family of curves which we shall designate as *w-curves*.

The two Pell curves and the two Jacobsthal curves resemble the Fibonacci and Lucas curves, so we will not reproduce them here. Instead, the reader is invited to compare them in the mind's eye with the curves exhibited in [2].

2. GENERALITIES

Let a, b, p, and q be real numbers, usually integers.
Define the sequence $\{w_n\}$ by

$$w_{n+2} = pw_{n+1} - qw_n, \quad w_0 = 2b, \quad w_1 = a + b \quad (n \geqslant 0). \tag{2.1}$$

Extension to negative values of n may be made, but we do not require it here.

The explicit Binet form for w_n is

$$w_n = (A\alpha^n - B\beta^n)/(\alpha - \beta), \tag{2.2}$$

where α, β are the roots of the characteristic equation

$$\lambda^2 - p\lambda + q = 0, \tag{2.3}$$

so that
$$\begin{cases} \alpha = (p + \sqrt{p^2 - 4q})/2 \\ \beta = (p - \sqrt{p^2 - 4q})/2 \\ \alpha + \beta = p, \ \alpha\beta = q, \ \alpha - \beta = \sqrt{p^2 - 4q} \end{cases} \tag{2.4}$$

and
$$\begin{cases} A = a + b - 2b\beta, \\ B = a + b - 2b\alpha. \end{cases} \tag{2.5}$$

Special cases of $\{w_n\}$ are:

SEQUENCE	p	q	a	b	α	β	
$\{F_n\}$: Fibonacci	1	−1	1	0	$(1 + \sqrt{5})/2$	$(1 - \sqrt{5})/2$	(2.5)
$\{L_n\}$: Lucas	1	−1	0	1	$(1 + \sqrt{5})/2$	$(1 - \sqrt{5})/2$	(2.6)
$\{P_n\}$: Pell	2	−1	1	0	$1 + \sqrt{2}$	$1 - \sqrt{2}$	(2.7)
$\{Q_n\}$: Pell–Lucas	2	−1	1	1	$1 + \sqrt{2}$	$1 - \sqrt{2}$	(2.8)
$\{J_n\}$: Jacobsthal	1	−2	1	0	2	−1	(2.9)
$\{j_n\}$: Jacobsthal–Lucas	1	−2	0	1	2	−1	(2.10)

The Fibonacci and Lucas sequences are well known.

Some values for the other sequences are:

	$n = 0$	1	2	3	4	5	6	7	...	
$\{P_n\}$	0	1	2	5	12	29	70	169	...	(2.11)
$\{Q_n\}$	2	2	6	14	34	82	198	478	...	(2.12)
$\{J_n\}$	0	1	1	3	5	11	21	43	...	(2.13)
$\{j_n\}$	2	1	5	7	17	31	65	127	...	(2.14)

In what follows, $q < 0$.

Write

$$q = -1 \cdot r \quad (r > 0), \tag{2.15}$$

so, by (2.4)

$$\beta = -1 \cdot \frac{r}{\alpha}. \tag{2.16}$$

Only the cases

$$q = -1, \text{ i.e., } r = 1 \quad [\text{cf. } (2.5)-(2.8)] \tag{2.17}$$

and

$$\beta = -1, \text{ i.e., } r = \alpha \ (= 2) \quad [\text{cf. } (2.9)-(2.10)] \tag{2.18}$$

will concern us.

3. THE w-CURVES

For Cartesian coordintes x, y of a point in the plane, let

$$x = \left(A\alpha^{\theta} - B\left(\frac{\alpha}{r}\right)^{-\theta} \cos \theta\pi\right)\Big/(\alpha - \beta) \tag{3.1}$$

and

$$y = -B\alpha^{-\theta} \sin \theta\pi/(\alpha - \beta), \tag{3.2}$$

where θ is real.

Comparing (3.1) with (2.2), we may refer to (3.1) as the *generalized Binet form* of w_n. When $\theta = n$, integer, we have $x = w_n$ by (2.2) and (3.1).

As θ varies in (3.1) and (3.2), we obtain the *w-curves*.

Now, from (3.1) and (3.2), we have

$$\frac{dy}{d\theta} = B\alpha^{-\theta}(\log \alpha \sin \theta\pi - \pi \cos \theta\pi)/(\alpha - \beta) \tag{3.3}$$

and

$$\frac{dx}{d\theta} = \left[A\alpha^{\theta}\log \alpha + B\left(\frac{\alpha}{r}\right)^{-\theta}\left\{\log\left(\frac{\alpha}{r}\right)\cos \theta\pi + \pi \sin \theta\pi\right\}\right]\Big/(\alpha - \beta), \tag{3.4}$$

whence $\frac{dy}{dx} = 0$ if

$$\tan \theta\pi = \frac{\pi}{\log \alpha} \doteq \begin{cases} 6.528 & \text{for } (2.5), (2.6)\text{—see } [2] \\ 3.565 & \text{for } (2.7), (2.8) \\ 4.538 & \text{for } (2.9), (2.10) \end{cases} \tag{3.5}$$

yielding

$$\theta\pi \doteq 81°16', \ 74°21', \ 77°34',$$

respectively, i.e.,

$$\theta \doteq \begin{cases} 0.45 \\ 0.41 \\ 0.43, \end{cases} \tag{3.6}$$

respectively, for the three cases in (3.5).

Write (3.5) as

$$\begin{cases} \sin \theta\pi = \pm k\pi \\ \cos \theta\pi = \pm k \log \alpha, \end{cases} \tag{3.7}$$

i.e.,

$$k = [\pi^2 + (\log \alpha)^2]^{-1/2} \tag{3.8}$$

giving

$$k \doteq \begin{cases} 0.31 \\ 0.30 \\ 0.299, \end{cases} \tag{3.8)'}$$

respectively, for the three cases in (3.5).

Eliminate θ from (3.1) and (3.2) for the specific values of θ covered in (3.5) for stationary points, i.e., for

$$\tan(\theta - m)\pi = \frac{\pi}{\log \alpha} \qquad (m \text{ an integer}). \qquad (3.5)'$$

With the aid of (3.7), we find that the locus of the stationary points is generally given by

$$y^2 - \frac{\pi}{\log \alpha} xy = \pm \frac{ABk\pi^2}{\Delta^2 \log \alpha} \qquad (\Delta = \alpha - \beta), \qquad (3.9)$$

which represents the branches of two hyperbolas (a hyperbola and its conjugate hyperbola) in the first and fourth quadrants.

Common asymptotes of these hyperbolas have equations

$$y = 0 \quad \text{and} \quad y = \frac{\pi}{\log \alpha}x, \qquad (3.10)$$

the gradients of the oblique asymptote being given in (3.5).

Inflexions on these curves are established in the usual way. When x and y are replaced by the functional notation $x(\theta)$ and $y(\theta)$, it may be demonstrated as in [2] that (3.1) and (3.2) do reproduce the *w-type* recurrence relations, namely,

$$x(\theta) = px(\theta - 1) - qx(\theta - 2) \qquad (3.1)'$$
and
$$y(\theta) = py(\theta - 1) - qy(\theta - 2). \qquad (3.2)'$$

4. PELL CURVES

Consider the *generalized Pell sequence* $\{R_n\}$ defined by (2.1) in which $p = 2$ and $q = -1$, namely,

$$R_{n+2} = 2R_{n+1} + R_n, \quad R_0 = 2b, \quad R_1 = a + b \quad (n \geqslant 0). \qquad (4.1)$$

From (2.2), we have the Binet form

$$R_n = (A\alpha^n - B\beta^n)/2\sqrt{2}, \qquad (4.2)$$

where α, β are given in (2.7) [and (2.8)] and A and B in (2.5).

For the Pell numbers P_n given in (2.11), and for the Pell-Lucas numbers Q_n given in (2.12), we have

$$P_n : a = 1, \ b = 0, \ A = B = 1 \qquad (4.3)$$
and
$$Q_n : a = 1, \ b = 1, \ A = -B = 2\sqrt{2}. \qquad (4.4)$$

Binet forms for P_n and Q_n are then readily obtained from (4.2).

Substituting appropriately in (3.1) and (3.2), we derive

$$x = (\alpha^\theta - \alpha^{-\theta} \cos \theta\pi)/2\sqrt{2} \qquad (4.5)$$

and

$$y = -\alpha^{-\theta} \sin \theta\pi/2\sqrt{2} \qquad (4.6)$$

for the Pell case, and

$$x = \alpha^{\theta} + \alpha^{-\theta} \cos \theta\pi \qquad (4.7)$$

and

$$y = \alpha^{-\theta} \sin \theta\pi \qquad (4.8)$$

for the Pell-Lucas case.

Equations (4.5) and (4.7) are the *modified Binet forms* for P_n and Q_n, respectively. When $\theta = n$, integer, we get the usual Binet forms for P_n and Q_n, i.e., $x = P_n$ and $x = Q_n$, respectively.

The locus given by (4.5) and (4.6) is the *Pell curve*. Its stationary points lie on the two appropriate branches of the hyperbolas

$$y^2 - \frac{\pi}{\log \alpha} xy = \pm \frac{k\pi^2}{8 \log \alpha} \qquad (\alpha = 1 + \sqrt{2}). \qquad (4.9)$$

Equations (4.7) and (4.8) yield the *Pell-Lucas curve*. Stationary points of this curve lie on the hyperbolic curves

$$y^2 - \frac{\pi}{\log \alpha} xy = \pm \frac{k\pi^2}{\log \alpha} \qquad (\alpha = 1 + \sqrt{2}). \qquad (4.10)$$

In both (4.9) and (4.10), the value of k is given in (3.8)'. Asymptotes common to the curves in (4.9) and (4.10) are $y = 0$ and $y = (\pi/\log \alpha)x$, whose gradient is given in (3.5).

Suppose we put $a = 3$, $b = 1$ in (4.1) so that $A = 2\alpha$, $B = 2\beta$.

Then (4.1) or the Binet forms for P_n, Q_n, and R_n yield

$$R_n^* = 2P_n + Q_n \qquad (4.11)$$
$$= 2P_{n+1} \qquad \text{since } Q_n = P_{n+1} + P_{n-1}.$$

Thus, a composite curve for $2P_\theta + Q_\theta$ is equivalent to the Pell curve for $2P_{\theta+1}$. Furthermore, from (4.11) or the Binet forms, we deduce that

$$R_n = (a - b)P_n + bQ_n \qquad (4.12)$$
$$= (a - b)P_n + b(P_{n+1} + P_{n-1})$$
$$= (a + b)P_n + 2bP_{n-1}$$
$$= R_1 P_n + R_0 P_{n-1} \qquad \text{by (4.1)}$$
$$[= R_n^* = 2P_{n+1} \text{ when } a = 3, b = 1, \text{ as in (4.11)}].$$

5. JACOBSTHAL CURVES

Next, consider the *generalized Jacobsthal sequence* $\{\mathcal{J}_n\}$ given by

$$\mathcal{J}_{n+2} = \mathcal{J}_{n+1} + 2\mathcal{J}_n, \quad \mathcal{J}_0 = 2b, \quad \mathcal{J}_1 = a + b \quad (n \geq 0). \qquad (5.1)$$

From (2.2), we have the Binet form

$$\mathcal{J}_n = \frac{A\alpha^n - B\beta^n}{3} \tag{5.2}$$

in which α (= 2), β (= -1) are already given in (2.9) and (2.10), and A, B are given in (2.5).

Particular cases of (5.1) are the Jacobsthal numbers J_n given in (2.13) and the Jacobsthal-Lucas numbers j_n given in (2.14) for which

$$J_n : a = 1, \ b = 0, \ A = B = 1 \tag{5.3}$$

and

$$j_n : a = 0, \ b = 1, \ A = -B = 3. \tag{5.4}$$

Binet forms for J_n and j_n then readily follow from (5.2).

Appropriate substitution in (3.1) and (3.2) produces

$$x = (2^\theta - \cos \theta\pi)/3 \tag{5.5}$$

and

$$y = -2^{-\theta} \sin \theta\pi/3 \tag{5.6}$$

for J_n, and

$$x = 2^\theta + \cos \theta\pi \tag{5.7}$$

and

$$y = 2^{-\theta} \sin \theta\pi \tag{5.8}$$

for j_n. Note the effect of (2.18) on (3.1) in (5.5) and (5.7).

Equations (5.5) and (5.7) are the *modified Binet forms* for J_n and j_n, respectively. Setting $\theta = n$, integer, we have the ordinary Binet forms for J_n and j_n, i.e., $x = J_n$ and $x = j_n$, respectively.

The locus given by (5.5) and (5.6) is the *Jacobsthal curve*. Its stationary points lie on the appropriate branches of the rectangular hyperbolas

$$y\left(x \pm \frac{k \ \log \ 2}{3}\right) = \mp \frac{k\pi}{9} \ . \tag{5.9}$$

Equations (5.7) and (5.8) yield the *Jacobsthal-Lucas curve*. Its stationary points lie on the rectangular hyperbolic branches

$$y(x \pm k \ \log \ 2) = \mp k\pi. \tag{5.10}$$

In both (5.9) and (5.10), the value of k is given in (3.8)'.

Put $a = 1$, $b = 1$ in (5.1) so that $A = -B = 4$.

Hence, as for the Pell case,

$$\mathcal{J}_n^* = J_n + j_n \tag{5.11}$$

$$= 2J_{n+1}.$$

Thus, a composite curve for $J_\theta + j_\theta$ is equivalent to the Jacobsthal curve $2J_{\theta+1}$.

Finally,

$$\oint_n = aJ_n + bj_n \tag{5.12}$$

$$= aJ_n + b(2J_{n-1} + J_{n+1}) \qquad \text{since } j_n = J_{n+1} + 2J_{n-1}$$

$$= (a + b)J_n + 4bJ_{n-1}$$

$$= \oint_1 J_n + 2\oint_0 J_{n-1} \qquad \text{by (5.1)}$$

$$[= \oint_n^* = 2J_{n+1} \quad \text{when } a = 1, \ b = 1 \text{ as in (5.11)}].$$

Notice the formal similarity of the right-hand sides of (4.12) and (5.12).

In choosing the expression for y in (3.2), we could have opted to pick α/r instead of α, both of which seem to be permissible as extensions of Wilson's original idea [2] for Fibonacci curves. Choice of α/r, however, appears to be the less appropriate.

Some obvious limits might be noted, and compared with similar limits for $\{F_n\}$ and $\{L_n\}$. These are:

$$\lim_{n \to \infty}\left(\frac{Q_n}{P_n}\right) = 2\sqrt{2} \qquad \lim_{n \to \infty}\left(\frac{P_{n+1}}{P_n}\right) = \lim_{n \to \infty}\left(\frac{Q_{n+1}}{Q_n}\right) = 1 + \sqrt{2}; \tag{5.13}$$

$$\lim_{n \to \infty}\left(\frac{j_n}{J_n}\right) = 3 \qquad \lim_{n \to \infty}\left(\frac{J_{n+1}}{J_n}\right) = \lim_{n \to \infty}\left(\frac{j_{n+1}}{j_n}\right) = 2. \tag{5.14}$$

Our concluding comment is of a geometrical nature. If we consider sequences in which two terms have the same value, e.g., $J_1 = J_2 = 1$ and $F_1 = F_2 = 1$, we observe that, as θ passes through the set of values giving these coincident numbers, the curve will necessarily have a node (i.e., a loop at a double-point) there. Thus, in (5.5), the Jacobsthal curve has a node at $x = 1$, as θ lies in the range $2 \leqslant \theta \leqslant 3$. Similarly, the Fibonacci curve has a node occurring when $x = 1$ and $1 \leqslant \theta \leqslant 2$.

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