FIBONACCI-LIKE DIFFERENTIAL EQUATIONS WITH A POLYNOMIAL NONHOMOGENEOUS PART

Peter R. J. Asveld
Department of Computer Science, Twente University of Technology
P.O. Box 217, 7500 AE Enschede, The Netherlands

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1. Introduction

In [1] and [2] we studied difference equations of the form

$$G_n = G_{n-1} + G_{n-2} + p(n) \tag{1}$$

where $G_0 = G_1 = 1$ and $p(n)$ is either a (ordinary or power) polynomial [1] or a factorial polynomial [2], i.e.,

$$p(n) = \sum_{i=0}^{k} \alpha_i n^i \quad \text{or} \quad p(n) = \sum_{i=0}^{k} \alpha_i n^{(i)}, \tag{2}$$

respectively, where

$$n^{(i)} = n(n-1)(n-2) \ldots (n-i+1) \text{ for } i \geq 1 \text{ and } n^{(0)} = 1.$$

The main results established in [1] and [2] provide expressions for the solution of (1) in terms of the coefficients $\alpha_1, \ldots, \alpha_k$ of (2) and in the Fibonacci numbers F_n, i.e., in the solution of the homogeneous difference equation

$$F_n = F_{n-1} + F_{n-2}, \tag{3}$$

where $F_0 = F_1 = 1$; cf. also [5].

In this note we derive similar expressions for the family of differential equations corresponding to (1) and (2), viz. we consider differential equations of the form

$$x''(t) + x'(t) - x(t) = p(t), \tag{4}$$

where $x(0) = c$, $x'(0) = d$,

$$p(t) = \sum_{i=0}^{k} \alpha_i t^i \quad \text{or} \quad p(t) = \sum_{i=0}^{k} \alpha_i t^{(i)},$$

and we express the solution of (4) in terms of the coefficients $\alpha_1, \ldots, \alpha_k$ and in the solution of the homogeneous differential equation corresponding to (3), i.e., the solution of

$$y''(t) + y'(t) - y(t) = 0 \tag{5}$$

where $y(0) = y'(0) = 1$.

Essential in our approach is the following proposition in which $p(t)$ now need not be a (factorial) polynomial at all; it may be an arbitrary function which, however, gives rise to a particular solution $x_p(t)$ of (4).

Let $F_{-1} = 0$ and $F_{-n} = (-1)^n F_{n-2}$ for each $n \geq 2$.

Proposition 1.1: Let $x_p(t)$ be a particular solution of (4). If $x(0) = c$ and $x'(0) = d$, then the solution of (4) can be expressed as

$$x(t) = (c - x_p(0))\left(\sum_{n=0}^{\infty} F_{-n}\frac{t^n}{n!}\right) + (d - x_p'(0))\left(\sum_{n=0}^{\infty} F_{-n-1}\frac{t^n}{n!}\right) + x_p(t). \qquad (6)$$

Proof: Using standard methods (cf. e.g., [3]), we first determine the solution $x_h(t)$ of the homogeneous equation corresponding to (4). To this end, we solve (5) with $y(0) = y'(0) = 1$:

$$y(t) = -(1 + \phi_2)(\sqrt{5})^{-1} \exp(-\phi_1 t) + (1 + \phi_1)(\sqrt{5})^{-1} \exp(-\phi_2 t),$$

where $\phi_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\phi_2 = \frac{1}{2}(1 - \sqrt{5})$. Then we obtain

$$y(t) = -(1 + \phi_2)(\sqrt{5})^{-1}\left(\sum_{n=0}^{\infty} \frac{(-\phi_1 t)^n}{n!}\right) + (1 + \phi_1)(\sqrt{5})^{-1}\left(\sum_{n=0}^{\infty} \frac{(-\phi_2 t)^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{\sqrt{5}}(\phi_1^{n-2} - \phi_2^{n-2})\frac{t^n}{n!} = \sum_{n=0}^{\infty} F_{-n+1}\frac{t^n}{n!},$$

since $(1 + \phi_2)\phi_1^2 = 1$ and $(1 + \phi_1)\phi_2^2 = 1$. Notice that

$$y'(t) = \sum_{n=0}^{\infty} F_{-n}\frac{t^n}{n!} \quad \text{and} \quad y''(t) = \sum_{n=0}^{\infty} F_{-n-1}\frac{t^n}{n!}. \qquad (7)$$

Now it is straightforward to show that for the solution $x(t)$ of (4) we have

$$x(t) = x_h(t) + x_p(t) = (c - x_p(0))y'(t) + (d - x_p'(0))y''(t) + x_p(t),$$

which yields together with (7) the desired equality (6). □

From Proposition 1.1 it is clear that we now need a particular solution of (4). As in [1] and [2] we distinguish two cases, viz. $p(t)$ is a polynomial (Section 2) and $p(t)$ is a factorial polynomial (Section 3).

2. Polynomials

Throughout this section, we assume that $p(t)$ is an ordinary or power polynomial

$$p(t) = \sum_{i=0}^{k} \alpha_i t^i.$$

As a particular solution of (4) we try

$$x_p(t) = \sum_{i=0}^{k} A_i t^i.$$

For $p(t)$ and $x_p(t)$, we write

$$p(t) = \sum_{i=0}^{k} \beta_i \frac{t^i}{i!} \quad \text{and} \quad x_p(t) = \sum_{i=0}^{k} B_i \frac{t^i}{i!},$$

respectively, where $\beta_i = i!\alpha_i$ and $B_i = i!A_i$ for each i $(0 \le i \le k)$. Hence, (4) yields

$$\sum_{i=0}^{k-2} B_{i+2} \frac{t^i}{i!} + \sum_{i=0}^{k-1} B_{i+1} \frac{t^i}{i!} - \sum_{i=0}^{k} B_i \frac{t^i}{i!} = \sum_{i=0}^{k} \beta_i \frac{t^i}{i!}.$$

From a comparison of the coefficients of $t^i/i!$, it follows that

$$B_k = -\beta_k,$$
$$B_{k-1} = -\beta_{k-1} - \beta_k,$$
$$B_i = -\beta_i + B_{i+2} + B_{i+1}, \quad \text{for } 0 \le i \le k-2.$$

Thus, we can successively compute B_k, B_{k-1}, ..., B_0; B_i is a linear combination of β_i, ..., β_k. Therefore, we write

$$B_i = -\sum_{j=i}^{k} a_{ij}\beta_j$$

(cf. [1] and [2]), which gives

$$-\sum_{j=i}^{k} a_{ij}\beta_j = -\beta_i - \sum_{j=i+2}^{k} a_{i+2,j}\beta_j - \sum_{j=i+1}^{k} a_{i+1,j}\beta_j.$$

Comparing the coefficients of β_j yields the following difference equation for each j $(1 \le j \le k)$:

$$a_{ij} = a_{i+2,j} + a_{i+1,j}, \quad \text{for } j - i \ge 2,$$

where $a_{jj} = a_{j-1,j} = 1$. But this means that

$$a_{ij} = F_{j-i}, \quad \text{for } 0 \le i \le j,$$

and hence

$$x_p(t) = \sum_{i=0}^{k} B_i \frac{t^i}{i!} = -\sum_{i=0}^{k}\sum_{j=i}^{k} F_{j-i} j!\,\alpha_j \frac{t^i}{i!} = -\sum_{j=0}^{k}\alpha_j\left(\sum_{i=0}^{j} j^{(j-i)} F_{j-i} t^i\right)$$

which implies

$$x_p(0) = B_0 = -\sum_{j=0}^{k} j!F_j\alpha_j \quad \text{and} \quad x_p'(0) = B_1 = -\sum_{j=1}^{k} j!F_{j-1}\alpha_j.$$

These equalities together with Proposition 1.1 yield the following proposition.

TABLE 1

j	$p_j(t)$
0	1
1	$t + 1$
2	$t^2 + 2t + 4$
3	$t^3 + 3t^2 + 12t + 18$
4	$t^4 + 4t^3 + 24t^2 + 72t + 120$
5	$t^5 + 5t^4 + 40t^3 + 180t^2 + 600t + 960$
6	$t^6 + 6t^5 + 60t^4 + 360t^3 + 1800t^2 + 5760t + 9360$
7	$t^7 + 7t^6 + 84t^5 + 630t^4 + 4200t^3 + 20160t^2 + 65520t + 105840$
8	$t^8 + 8t^7 + 112t^6 + 1008t^5 + 8400t^4 + 53760t^3 + 262080t^2 + 846720t + 1370880$
9	$t^9 + 9t^8 + 144t^7 + 1512t^6 + 15120t^5 + 120960t^4 + 786240t^3 + 3810240t^2 + 12337920t + 19958400$

Proposition 2.1: The solution of (4) with $x(0) = c$, $x'(0) = d$, and

$$p(t) = \sum_{i=0}^{k} \alpha_i t^i$$

can be expressed as

$$x(t) = (c + L_k)\left(\sum_{n=0}^{\infty} F_{-n} \frac{t^n}{n!}\right) + (d + \ell_k)\left(\sum_{n=0}^{\infty} F_{-n-1} \frac{t^n}{n!}\right) - \sum_{j=0}^{k} \alpha_j p_j(t),$$

where L_k and ℓ_k are linear combinations of $\alpha_0, \ldots, \alpha_k$, and for each j ($0 \le j \le k$), $p_j(t)$ is a polynomial of degree j:

$$L_k = \sum_{j=0}^{k} j! F_j \alpha_j; \quad \ell_k = \sum_{j=1}^{k} j! F_{j-1}\alpha_j; \quad p_j(t) = \sum_{i=0}^{j} j^{(j-i)} F_{j-i} t^i. \quad \square$$

The polynomials $p_j(t)$ are given in Table 1 above for $j = 0, 1, 2, \ldots, 9$.

The coefficients of α_j in L_k and ℓ_k are independent of k; cf. [1] and [2]. They give rise to two infinite sequences L and ℓ of natural numbers (not mentioned in [4]) as k tends to infinity. The first few elements of these new sequences are

L: 1,1,4,18,120,960,9360,105840,1370880,19958400,...,
ℓ: 0,1,2,12,72,600,5760,65520,846720,12337920,... .

3. Factorial Polynomials

This section is devoted to the case in which $p(t)$ is a factorial polynomial

$$p(t) = \sum_{i=0}^{k} \alpha_i t^{(i)}.$$

In order to try

$$x_p(t) = \sum_{i=0}^{k} A_i t^{(i)} \tag{8}$$

as a particular solution of (4), we first ought to determine the derivative of $t^{(n)}$.

Lemma 3.1: $\dfrac{dt^{(n)}}{dt} = \displaystyle\sum_{k=0}^{n-1} \binom{n}{k} t^{(k)} (-1)^{(n-k-1)}$.

Proof: The argument is by induction on n. The basis of which ($n = 1$) is trivial. Suppose the equality holds for $n - 1$:

$$\frac{dt^{(n-1)}}{dt} = \sum_{k=0}^{n-2} \binom{n-1}{k} t^{(k)} (-1)^{(n-k-2)}. \tag{9}$$

To perform the induction step, consider

$$dt^{(n)}/dt = d(t(t-1)^{(n-1)})/dt = (t-1)^{(n-1)} + td((t-1)^{(n-1)})/dt.$$

Now, by the Chain Rule, we have

$$d((t-1)^{(n-1)})/dt = d((t-1)^{(n-1)})/d(t-1).$$

Applying the Binomial Theorem from [2] to $(t-1)^{(n-1)}$ and the induction hypothesis (9) yields:

$$\begin{aligned}\frac{dt^{(n)}}{dt} &= \sum_{k=0}^{n-1}\binom{n-1}{k}t^{(k)}(-1)^{(n-k-1)} + t\sum_{k=0}^{n-2}\binom{n-1}{k}(t-1)^{(k)}(-1)^{(n-k-2)}\\ &= (-1)^{(n-1)} + \sum_{k=1}^{n-1}\binom{n-1}{k}t^{(k)}(-1)^{(n-k-1)} + \sum_{k=1}^{n-1}\binom{n-1}{k-1}t^{(k)}(-1)^{(n-k-1)}\\ &= (-1)^{(n-1)} + \sum_{k=1}^{n-1}\binom{n}{k}t^{(k)}(-1)^{(n-k-1)} = \sum_{k=0}^{n-1}\binom{n}{k}t^{(k)}(-1)^{(n-k-1)}\end{aligned}$$

which completes the induction. □

From Lemma 3.1, (4), and (8), we obtain

$$\sum_{i=2}^{k}A_i\left(\sum_{m=1}^{i-1}\binom{i}{m}\left(\sum_{\ell=0}^{m-1}\binom{m}{\ell}t^{(\ell)}(-1)^{(m-\ell-1)}\right)(-1)^{(i-m-1)}\right)$$
$$+\sum_{i=1}^{k}A_i\left(\sum_{m=0}^{i-1}\binom{i}{m}t^{(m)}(-1)^{(i-m-1)}\right) - \sum_{i=0}^{k}A_i t^{(i)} = \sum_{i=0}^{k}\alpha_i t^{(i)}.$$

Comparing the coefficients of $t^{(i)}$ yields

$$A_k = -\alpha_k,$$
$$A_{k-1} = -\alpha_{k-1} + k\alpha_k,$$
$$A_i = -\alpha_i + \sum_{n=i+1}^{k}A_n\binom{n}{i}(-1)^{(n-i-1)}$$
$$+\sum_{n=i+2}^{k}A_n\left(\sum_{m=i+1}^{n-1}\binom{n}{m}\binom{m}{i}(-1)^{(m-i-1)}(-1)^{(n-m-1)}\right)$$

for each i $(0 \le i \le k-2)$. As $(-x)^{(n)} = (-1)^n(x+n-1)^{(n)}$ and $n^{(n)} = n!$, this latter recurrence can be rewritten as

$$A_i = -\alpha_i + \sum_{n=i+1}^{k}A_n(-1)^{n-i-1}\frac{n^{(n-i)}}{n-i}$$
$$+\sum_{n=i+2}^{k}A_n\left(\sum_{m=i+1}^{n-1}(-1)^{n-i-2}\frac{n^{(n-i)}}{(n-m)(m-i)}\right)$$

or

$$A_i = -\alpha_i + (i+1)A_{i+1} + \sum_{n=i+2}^{k}\zeta_{in}A_n, \tag{10}$$

where

$$\zeta_{in} = (-1)^{n-i-1}n^{(n-i)}\left((n-i)^{-1} - \sum_{m=i+1}^{n-1}(n-m)^{-1}(m-i)^{-1}\right).$$

Now (10) enables us to compute $A_k, \ldots, A_0$: A_i is a linear combination of $\alpha_i, \ldots, \alpha_k$. Thus

$$A_i = -\sum_{j=i}^{k}b_{ij}\alpha_j$$

and (10) becomes

$$\sum_{j=i}^{k}b_{ij}\alpha_j = \alpha_i + (i+1)\sum_{j=i+1}^{k}b_{i+1,j}\alpha_j + \sum_{n=i+2}^{k}\zeta_{in}\sum_{j=n}^{k}b_{nj}\alpha_j.$$

From the coefficients of α_j, it follows that

$$b_{ii} = 1,$$

$$b_{i,i+1} = i + 1,$$

$$b_{ij} = (i+1)b_{i+1,j} + \sum_{n=i+2}^{j} \zeta_{in} b_{nj} \quad \text{for } j \geq i+2.$$

Hence,

$$x_p(t) = -\sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} \alpha_j t^{(i)} = -\sum_{j=0}^{k} \alpha_j \left(\sum_{i=0}^{j} b_{ij} t^{(i)} \right)$$

and

$$x_p'(t) = -\sum_{j=1}^{k} \alpha_j \left(\sum_{i=1}^{j} b_{ij} \left(\sum_{\ell=0}^{i-1} \binom{i}{\ell} t^{(\ell)} (-1)^{(i-\ell-1)} \right) \right).$$

Since

$$x_p(0) = -\sum_{j=0}^{k} b_{0j} \alpha_j \quad \text{and} \quad x_p'(0) = -\sum_{j=1}^{k} \left(\sum_{i=1}^{j} (-1)^{(i-1)} b_{ij} \right) \alpha_j,$$

we have the following result.

Proposition 3.2: The solution of (4) with $x(0) = c$, $x'(0) = d$, and

$$p(t) = \sum_{i=0}^{k} \alpha_i t^{(i)}$$

can be expressed as

$$x(t) = (c + M_k) \left(\sum_{n=0}^{\infty} F_{-n} \frac{t^n}{n!} \right) + (d + m_k) \left(\sum_{n=0}^{\infty} F_{-n-1} \frac{t^n}{n!} \right) - \sum_{j=0}^{k} \alpha_j \pi_j(t),$$

where M_k and m_k are linear combinations of $\alpha_0, \ldots, \alpha_k$, and for each j $(0 \leq j \leq k)$, $\pi_j(t)$ is a factorial polynomial of degree j:

$$M_k = \sum_{j=0}^{k} b_{0j} \alpha_j; \quad m_k = \sum_{j=1}^{k} \left(\sum_{i=1}^{j} (-1)^{(i-1)} b_{ij} \right) \alpha_j; \quad \pi_j(t) = \sum_{i=0}^{j} b_{ij} t^{(i)}. \quad \square$$

For $j = 0, 1, \ldots, 9$, the factorial polynomials $\pi_j(t)$ are given in Table 2.

TABLE 2

j	$\pi_j(t)$
0	1
1	$t^{(1)} + 1$
2	$t^{(2)} + 2t^{(1)} + 3$
3	$t^{(3)} + 3t^{(2)} + 9t^{(1)} + 8$
4	$t^{(4)} + 4t^{(3)} + 18t^{(2)} + 32t^{(1)} + 50$
5	$t^{(5)} + 5t^{(4)} + 30t^{(3)} + 80t^{(2)} + 250t^{(1)} + 214$
6	$t^{(6)} + 6t^{(5)} + 45t^{(4)} + 160t^{(3)} + 750t^{(2)} + 1284t^{(1)} + 2086$
7	$t^{(7)} + 7t^{(6)} + 63t^{(5)} + 280t^{(4)} + 1750t^{(3)} + 4494t^{(2)} + 14602t^{(1)} + 11976$
8	$t^{(8)} + 8t^{(7)} + 84t^{(6)} + 448t^{(5)} + 3500t^{(4)} + 11984t^{(3)} + 58408t^{(2)} + {} + 95808t^{(1)} + 162816$
9	$t^{(9)} + 9t^{(8)} + 108t^{(7)} + 672t^{(6)} + 6300t^{(5)} + 26964t^{(4)} + 175224t^{(3)} + {} + 431136t^{(2)} + 1465344t^{(1)} + 1143576$

As in the previous section and [1] and [2], the coefficients of α_j in M_k and m_k are independent of k. The first few elements of the limit sequences (not mentioned in [4]) M and m (obtained from M_k and m_k for $k \to \infty$) are

M: 1,1,3,8,50,214,2086,11976,162816,1143576,...,
m: 0,1,1,8,16,224,608,13320,41760,1366152,... .

Finally, we remark that the coefficients b_{ij} (and hence the elements of the sequences M and m) can also be computed from a_{ij} by means of

$$b_{ij} = \sum_{m=i}^{j} S(i, m)\left(\sum_{\ell=m}^{j} a_{m\ell}s(\ell, j)\right) \qquad (i \le j),$$

where $s(\ell, j)$ and $S(i, m)$ are Stirling numbers of the first and of the second kind, respectively.

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