# THE FIRST COLUMN OF AN INTERSPERSION

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## INTRODUCTION

In 1977, K B. Stolarsky [9] introduced an array of positive integers whose first row consists of the Fibonacci numbers $\{F_n : n \geq 2\}: 1\ 2\ 3\ 5\ 8\ 13 \ldots$ The subsequent rows are "generalized Fibonacci sequences." In fact, much more is true. The rows of the array are, in a sense, the set of *all* "positive Fibonacci sequences" of integers. This fact was proved by D. Morrison [7], who also introduced the Wythoff array and proved that it has many of the properties of the original Stolarsky array. In order to study from a general point of view the properties which the Stolarsky and Wythoff arrays have in common, the notion of *interspersion* was introduced in [4]. The name "interspersion" was chosen to match property I4 in the definition given below.

Much of the reason for interest in interspersions, especially those known as Stolarsky interspersions, lies with the first column of such an array: its high degree of regularity versus the possible unavailability of a nice formula for the $n^{\text{th}}$ term. In the case of the original Stolarsky and Wythoff arrays, however, such formulas are known (see Section 5). From Example 1.1(i) of [7, p. 307] and these formulas, we find that the first columns of the Stolarsky and Wythoff arrays are uniformly distributed mod $m$ for every positive integer $m$. In contrast to this, we construct in Section 4 a new Stolarsky interspersion for which every element of the first column, after the initial element 1, is *even*; we call it the *even first column array* (EFC).

## 1. WHAT IS AN INTERSPERSION?

Throughout this paper, the notation $A = A(i, j)$ denotes an array of distinct positive integers $a(i, j)$ with increasing first column. For such $A$, let $\hat{A} = \hat{A}(i, j)$ and $\check{A} = \check{A}(i, j)$ be the arrays of positive integers defined by

$$\hat{a}(i, j) = a(i, j+1) \text{ for } i \geq 1, j \geq 1,$$

and

$$\check{a}(i, j) = \text{the number of terms of } \hat{A} \text{ which are} \leq a(i, j+1),$$

respectively. Note that $\hat{A}$ is obtained from $A$ by simply removing the first column of $A$. If the terms of $\hat{A}$ are then ordered as an increasing sequence, then $\check{a}(i, j)$ is simply the rank of $\hat{a}(i, j)$ in this sequence. (The reader is urged to write out several terms of $A$ using the array in Table 1.) We call $\check{A}$ the *rank array of* $A$ and prove in Theorem 1.1 that an array $A$ is its own rank array iff $A$ is an *interspersion*, as defined in [4] by the following properties:

**I1**. the rows of $A$ comprise a partition of the positive integers;

**I2**. every row of $A$ is an increasing sequence;

**I3**. every column of $A$ is an increasing (possibly finite) sequence;

**I4**. if $\{u_j\}$ and $\{v_j\}$ are distinct rows of $A$, and $p$ and $q$ are indices for which $u_p < v_q < u_{p+1}$, then $u_{p+1} < v_{q+1} < u_{p+2}$.

Perhaps the simplest example of an interspersion is given by $a(i, j) = i + \frac{(i+j-1)(i+j-2)}{2}$.

***Theorem 1.1:*** An array $A$ is an interspersion iff $\check{A} = A$.

***Proof:*** First, suppose $A$ is an interspersion. Then, by Lemma 2 in [4],

$$a(i, j+1) = a(i, j) + C(a(i, j+1)),$$

where $C(m)$ denotes, for $m \geq 1$, the number of terms in the first column of $A$ that are $\leq m$. Thus, $a(i, j)$ is the number of terms of $A$ that are $\leq a(i, j+1)$ and are not in column 1. That is, $\check{a}(i, j) = a(i, j)$, as required.

For the converse, suppose $\check{a}(i, j) = a(i, j)$ for all $i$ and $j$. Then property I1 must hold, since $\check{a}(i, j)$ ranges through all the positive integers without repetition.

Now, since $a(i, j)$ is the number of terms of $\hat{A}$ that are $\leq a(i, j+1)$ for all $i$ and $j$, we have $a(i, j) \leq a(i, j+1)$, and this strengthens to $a(i, j) < a(i, j+1)$ since the terms of $A$ are distinct; thus, property I2 holds.

By hypothesis, column 1 of $A$ is increasing. Suppose for arbitrary $j \geq 1$ that column $j$ is increasing, and suppose $i \geq 1$. The number of terms of $\hat{A}$ that are $\leq a(i+1, j+1)$ is $a(i+1, j)$, and this by the induction hypothesis exceeds $a(i, j)$, which is the number of terms of $\hat{A}$ that are $\leq a(i, j+1)$. Therefore, $a(i+1, j+1) > a(i, j+1)$, and property I3 holds.

Arrange the numbers in $\hat{A}$ in increasing order, forming a sequence $s_n$ such that

$$\hat{a}(i, j) = s_{\hat{a}(i, j-1)} = s_{a(i, j)}.$$

If $u_p < v_q < u_{p+1}$, as in I4, then $s_{u_p} < s_{v_q} < s_{u_{p+1}}$, since $s_n$ is an increasing sequence. That is, property I4 holds. □

To summarize, Theorem 1.1 shows that an interspersion is an array $A$ whose characteristic property is that for any successive terms $u$ and $v$ in any row, $v$ is the $u^{\text{th}}$ term not in column 1.

## 2. STOLARSKY INTERSPERSIONS

Certain interspersions which have received much attention are the Stolarsky interspersions (e.g., [4], [5], [6], [8], [9]). These are shown in [6] to be in one-to-one correspondence with the set of all zero-one sequences $\{\delta_i\}$ that begin with 1. The correspondence is given as follows: for each row number $i$, the number $a(i, 2)$ in column 2 must be one of the two numbers $[\alpha a(i, 1) + \delta_i]$, where $\alpha = (1+\sqrt{5})/2$; thus, the numbers in column 2 depend on those in column 1 and, moreover, the numbers in columns numbered higher than 2 are determined by the recurrence

$$a(i, j) = a(i, j-1) + a(i, j-2), \quad j = 3, 4, 5, \ldots. \tag{1}$$

Accordingly, each row of a Stolarsky interspersion depends in a simple manner on whatever number occupies the first position in the row. This first number is always the least positive integer not appearing in any previous row. (See Tables 1-3.) We leave open the question of whether almost all Stolarsky interspersions have a uniformly distributed first column.

**TABLE 1. The Original Stolarsky Array ([9], 1977)**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | ... |
| 4 | 6 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 | 466 | |
| 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | 521 | 843 | |
| 9 | 15 | 24 | 39 | 63 | 102 | 165 | 267 | 432 | 699 | 1131 | |
| 12 | 19 | 31 | 50 | 81 | 131 | 212 | 343 | 555 | 898 | 1453 | |
| 14 | 23 | 37 | 60 | 97 | 157 | 254 | 411 | 665 | 1076 | 1741 | |
| 17 | 28 | 45 | 73 | 118 | 191 | 309 | 500 | 809 | 1309 | 2118 | |
| 20 | 32 | 52 | 84 | 136 | 220 | 356 | 576 | 932 | 1508 | 2440 | |
| 22 | 36 | 58 | 94 | 152 | 246 | 398 | 644 | 1042 | 1686 | 2728 | |
| ⋮ | | | | | | | | | | | |

**TABLE 2. The Wythoff Array**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | ... |
| 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | 521 | |
| 6 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 | 466 | 754 | |
| 9 | 15 | 24 | 39 | 63 | 102 | 165 | 267 | 432 | 699 | 1131 | |
| 12 | 20 | 32 | 52 | 84 | 136 | 220 | 356 | 576 | 932 | 1508 | |
| 14 | 23 | 37 | 60 | 97 | 157 | 254 | 411 | 665 | 1076 | 1741 | |
| 17 | 28 | 45 | 73 | 118 | 191 | 309 | 500 | 809 | 1309 | 2118 | |
| 19 | 31 | 50 | 81 | 131 | 212 | 343 | 555 | 898 | 1453 | 2351 | |
| 22 | 36 | 58 | 94 | 152 | 246 | 398 | 644 | 1042 | 1686 | 2728 | |
| ⋮ | | | | | | | | | | | |

***Construction 2.1:*** Every Stolarsky interspersion can be constructed inductively using the rules described above: row 1 must be 1 2 3 5 8 13 21...; once $k$ rows have been constructed, there are two and only two possibilities for row $k+1$. The first term $u$ must be the least positive integer not already used in the first $k$ rows. The second term can be either $[\alpha u]$ or $[\alpha u+1]$, and the remaining terms are determined by the recurrence (1).

For any given zero-one sequence $\delta$ with initial term 1, the corresponding Stolarsky interspersion $A(\delta)$ would be easy to write out if only the first column were not, generally speaking, so mysterious. It turns out to be somewhat surprising how nearly determined these mysterious numbers are. This section is devoted to such determinations. We begin with a restatement of Lemma 1.5 of [6].

***Lemma 2.2:*** Suppose $\{r_j\}$ is a row of a Stolarsky interspersion. Then either

$$r_{2k}=[\alpha r_{2k-1}] \text{ and } r_{2k+1}=[\alpha r_{2k}+1] \text{ for all } k\geq 1, \text{ or else}$$
$$r_{2k}=[\alpha r_{2k-1}+1] \text{ and } r_{2k+1}=[\alpha r_{2k}] \text{ for all } k\geq 1.$$

***Lemma 2.3:*** Suppose $u$ and $v$ are adjacent terms in a row of a Stolarsky interspersion, and $u<v$. Then $u\in\left\{\left[\frac{v}{\alpha}\right],\left[\frac{v+1}{\alpha}\right]\right\}$.

***Proof:*** By Lemma 2.2, $v\in\{[\alpha u],[\alpha u+1]\}$, It is easy to confirm that if $v=[\alpha u]$, then $u=\left[\frac{v+1}{\alpha}\right]$, and if $v=[\alpha u+1]$, then $u=\left[\frac{v}{\alpha}\right]$.

***Theorem 2.4:*** Let $A$ be a Stolarsky interspersion. Let $\{s_n\}$ be the ordered complement of the first column of $A$. Then $s_n\in\{[n\alpha],[n\alpha+1]\}$ for $n=1,2,3,\ldots$.

***Proof:*** By Lemma 3 of [4], we have $s_{a(i,j)} = a(i, j+1)$ for all $i, j$. By Lemma 2.2,

$$a(i, j+1) \in \{[\alpha a(i, j)], [\alpha a(i, j)+1]\}.$$

Since $a(i, j)$ ranges through all the positive integers $n$, we therefore have

$$s_n \in \{[n\alpha], [n\alpha+1]\} \text{ for } n = 1, 2, 3, \ldots. \ \square$$

***Lemma 2.5:*** Suppose $\{c_i\}$ and $\{s_i\}$ are infinite complementary sequences of positive integers, $m \geq 0$, and $\{\sigma_i\}$ is a zero-one sequence in which the maximal string length of ones is $m$. Let $s_i^* = s_i + \sigma_i$ and suppose $s_{i+1}^* \neq s_i^*$ for all $i$. Let $\{c_j^*\}$ be the ordered complement of $\{s_i^*\}$. Then $0 \leq c_j - c_j^* \leq m$ for all $j$.

***Proof:*** The sequence of positive integers can be represented in increasing order as a sequence of strings of two types: $S$ strings consisting of consecutive $s_i$'s, and $C$ strings consisting of consecutive $c_j$'s. Each $S$ string is followed by a $C$ string, which is followed by an $S$ string. Either $s_1 = 1$ or else $c_1 = 1$; we assume the former, noting that the proof in case $c_1 = 1$ can easily be obtained from what follows and is, therefore, omitted. Write the initial string as $S_1 = s_1, s_2, \ldots, s_{m_1}$ $(= 1, 2, \ldots, m_1$, where $m_1 \geq 1)$, and the initial $C$ string as $C_1 = c_1, c_2, \ldots, c_{n_1}$ $(= m_1+1, \ldots, m_1+n_1$, where $n_1 \geq 1)$. Following $C_1$ is $S_2$, and so on, so that our representation of the positive integers is as a sequence of strings: $S_1C_1S_2C_2S_3\ldots$, where $S_i = s_{m_{i-1}+1}, \ldots, s_{m_i}$, $C_i = c_{n_{i-1}+1}, \ldots, c_{n_i}$, $m_0 = n_0 = 0$, $1 \leq m_1 < m_2 < \cdots$ and $1 \leq n_1 < n_2 < \cdots$.

Let $N$ denote the null string. Each string $S_i$ is a juxtaposition of two substrings, $L_i$ and $R_i$, which satisfy the following conditions:

(i) If $L_i = \mathcal{N}$, then $R_i \neq \mathcal{N}$;

(ii) If $L_i \neq \mathcal{N}$, then $L_i$ has the form $s_{m_{i-1}+1}, \ldots, s_{m_{i-1}+k_i}$ for some $k_i \geq 1$, and $s_\ell^* = s_\ell$ for $m_{i-1}+1 \leq \ell \leq m_{i-1}+k_i$;

(iii) If $R_i \neq \mathcal{N}$, then $R_i$ has the form $s_{m_{i-1}+k_i+1}, \ldots, s_{m_i}$, and $s_\ell^* = s_\ell + 1$ for $m_{i-1}+k_i+1 \leq \ell \leq m_i$.

Consider an arbitrary triple $L_iR_iC_i$. If $R_i = \mathcal{N}$, then clearly $s_\ell^* = s_\ell$ for $\ell$ as in (ii) and $c_\ell^* = c_\ell$ for $n_{i-1}+1 \leq \ell \leq n_i$. Otherwise, we have $s_\ell^* = s_\ell$ for the terms of $L_i$ and $s_\ell^* = s_\ell + 1$ for those of $R_i$, so that $c_{n_{i-1}+1}^* = s_{m_{i-1}+k_i+1}$, and $c_\ell^* = c_\ell$ for $\ell = n_{i-1}+2, \ldots, n_i$. Thus, $0 \leq c_\ell - c_\ell^* \leq m_i - k_i \leq m$ for $\ell = n_{i-1}+1, n_{i-1}+2, \ldots, n_i$. Now, putting the triples $L_iR_iC_i$ together in order, we conclude that $0 \leq c_\ell - c_\ell^* \leq m$ for all $\ell$. $\square$

***Lemma 2.6:*** Suppose $A$ is a Stolarsky interspersion. Let $\{s_n^*\}$ be the ordered sequence of terms of $A$ that are not in the first column of $A$. Let $s_n = [n\alpha]$. Let $\sigma_n = s_n^* - s_n$. Then $\{\sigma_n\}$ is a zero-one sequence, $s_{n+1}^* \neq s_n^*$ for all $n$, and the maximum string length of ones in $\{\sigma_n\}$ is 2.

***Proof:*** By Theorem 2.4, $\{\sigma_n\}$ is a zero-one sequence. Also, $s_{n+1}^* \neq s_n^*$ for all $n$, since the terms of $A$ are distinct. Now suppose $n$ is a positive integer and write

$$n\alpha = [n\alpha] + \varepsilon_1, \ (n+2)\alpha = [n+2\alpha] + \varepsilon_2, \text{ where } 0 < \varepsilon_i < 1 \text{ for } i = 1, 2.$$

Then $[(n+2)\alpha] - [n\alpha] = 2\alpha - \varepsilon_2 + \varepsilon_1$, which, as an integer within 1 of $2\alpha$ $(= 1+\sqrt{5})$, must be 3 or 4. Therefore, the three integers $[n\alpha]+1$, $[(n+1)\alpha]+1$, $[(n+2)\alpha]+1$ cannot be consecutive integers. Consequently, there is no string of ones of length $\geq 3$ in $\{\sigma_n\}$.

***Theorem 2.7:*** Let $u_i = a(i,1)$, the $i^{\text{th}}$ term of column 1 of a Stolarsky interspersion $A$. Then $[i\alpha]+i-2 \le u_i \le [i\alpha]+i$ for every $i$.

***Proof:*** The ordered complement of $\{s_i\} = \{[i\alpha]\}$ is $\{c_i\} = \{[i\alpha]+i\}$, by the well-known Beatty theorem on complementary sequences (see Theorem XI in [1]). Lemmas 2.5 and 2.6 imply that $0 \le c_i - u_i \le 2$, from which the desired inequality immediately follows. □

***Corollary 2.8:*** Let $u_i = a(i,1)$, the $i^{\text{th}}$ term of column 1 of a Stolarsky interspersion $A$. Let $w_i$ be the $i^{\text{th}}$ term of the first column of the Wythoff array (see Table 2). Then $-1 \le w_i - u_i \le 1$ for every $i$.

***Proof:*** This follows immediately from Theorem 2.7 and the fact that $w_i = [i\alpha]+i-1$. □

***Lemma 2.9:*** If $u$ is a positive integer, then exactly one of the following statements is true:

***(i)*** $\exists n \ni u = [n\alpha]$ and $[(n+1)\alpha] = u+1$;
***(ii)*** $\exists n \ni u = [n\alpha]$ and $[(n+1)\alpha] = u+2$;
***(iii)*** $\exists n \ni u = [n\alpha+1]$ and $[(n+1)\alpha] = u+1$.

***Proof:*** If there exists $n$ satisfying $u = [n\alpha]$, then clearly $[(n+1)\alpha]$ must equal $u+1$ or $u+2$, since $0 < \alpha < 1$. If $u$ is not of the form $[m\alpha]$, then since $0 < \alpha < 2$, there must exist $n$ satisfying $u = [n\alpha+1]$. Since $u \ne [(n+1)\alpha]$, we have $1 \le n\alpha + \alpha - u$. Also, $n\alpha + 1 - u < 1$, so that $n\alpha + \alpha - u < 2$. That is, $u+1 < n\alpha + \alpha < u+2$, so that $[(n+1)\alpha] = u+1$. □

***Theorem 2.10:*** The first column of a Stolarsky interspersion does not contain two consecutive integers.

***Proof:*** If $u$ is a positive integer in column 1 of a Stolarsky array $A$, and $u$ is as in (i) or (ii) in Lemma 2.9, then the immediate successor of $n$ in a row of $A$ is, by Lemma 2.2, $u+1 = [n\alpha+1]$, so that $u+1$ is not in column 1.

By Lemma 2.9, the only remaining case is that $u = [n\alpha+1]$ and $u+1 = [(n+1)\alpha]$. Assume that both $u$ and $u+1$ lie in column 1 of $A$, and assume that $u$ is the least such positive integer. By Lemma 2.2, the immediate successor of $n$ in a row of $A$ must then be $u-1$, and the immediate follower of $n+1$ must be $u+2$. Since $n < u$, at least one of the numbers $n$ and $n+1$ does not lie in column 1. If $n$ is not in column 1, then by (1), $n$ is immediately preceded by $u-1-n$; and if $n+1$ is not in column 1, then $n+1$ is immediately preceded by $u+2-(n+1) = u+1-n$.

Now, $u = n\alpha + 1 - \varepsilon_1, 0 < \varepsilon_1 < 1$, so that $u\alpha - n\alpha^2 + \alpha - \alpha\varepsilon_1$. Since

$$\alpha^2 = \alpha + 1, \tag{2}$$

we have

$$u\alpha - n\alpha = n + \alpha(1-\varepsilon_1). \tag{3}$$

Also, $u+1 = n\alpha + \alpha - \varepsilon_2, 0 < \varepsilon_2 < 1$, so that $u\alpha + a = n\alpha^2 + \alpha^2 - \alpha\varepsilon_2$, which yields

$$u\alpha - n\alpha = n + 1 - \alpha\varepsilon_2. \tag{4}$$

Equations (3) and (4) show that $\alpha(1-\varepsilon_1) = 1 - \alpha\varepsilon_2 < 1$, so that (3) implies $n = [(u-n)\alpha]$. Now, by Lemma 2.2, in a row of $A$ the integer $u-n$ must immediately precede $n$ or $n+1$, whichever of these is not in column 1. However, it has already been proved that the immediate predecessor of $n$, if there is one, is $u-n-1$, and the immediate predecessor of $n+1$, if there is one, is $u-n+1$. This contradiction shows that $u$ and $u+1$ cannot both lie in column 1 of $A$. □

***Lemma 2.11:*** If $u$ is a positive integer, then exactly one of the following statements is true:

***(i)*** $\exists n \ni u=[n\alpha+1]=[(n+1)\alpha]$;
***(ii)*** $\exists n \ni u=[n\alpha+1]$ and $[(n+1)\alpha]=u+1$;
***(iii)*** $\exists n \ni u=[n\alpha]$ and $[(n-1)\alpha+1]=u-1$.

The proof of Lemma 2.9 can serve as a guide for proving Lemma 2.11. We omit a proof but do pause to note that each of these two lemmas partitions the set of positive integers into three subsets that can be expressed in terms of fractional parts. These are, in the order (i), (ii), (iii), as follows:

$$\{u:\{u\alpha\}>4-2\alpha\},\ \{u:2-\alpha<\{u\alpha\}<4-2\alpha\},\ \text{and}\ \{u:\{u\alpha\}<2-\alpha\}\ \text{for Lemma 2.9,}$$

$$\{u:2-\alpha<\{u\alpha\}<\alpha-1\},\ \{u:\{u\alpha\}<2-\alpha\},\ \{u:\{u\alpha\}>\alpha-1\}\ \text{for Lemma 2.11.}$$

***Theorem 2.12:*** Suppose successive terms of column 1 of a Stolarsky interspersion differ by 2: $a(i+1,1)-a(i,1)=2$. Then the integer $a(i,1)+1$ lies in a column numbered greater than 2.

***Proof:*** Let $u=a(i,1)+1$. By Theorem 2.10, $u$ does not lie in column 1; suppose $u$ lies in column 2. Let $n$ be the immediate predecessor of $u$ in a row of $A$. We shall see that $n$ must be related to $u$ as in one of the three cases in Lemma 2.11. The only possible exception would be if $u=[p\alpha]$ for some $p$ and also $u=[q\alpha+1]$ for some $q$. It is easy to check here that $q=p-1$. To see that $n=p-1$, suppose to the contrary that $n=p$. Then $[(n-1)\alpha+1]=u$ and $[(n-1)\alpha]=u-1$; now $u-1$ is in column 1, so that the immediate follower of $n-1$ in a row of $A$ must be $u$, by Lemma 2.2. However, this contradicts the hypothesis that $u$ follows $n$.

In case (1), $u=[n\alpha+1]=[(n+1)\alpha]$. In $A$, the integer $n+1$ must, by Lemma 2.2, be followed by $[(n+1)\alpha]$ or $[(n+1)\alpha+1]$. The former is $u$, which follows $n$, not $n+1$, and the latter is $u+1$, which lies in column 1. For $u$ as in (ii), a contradiction is similarly obtained.

In case (iii), $u<n\alpha$, so that $u\alpha<n\alpha^2=n\alpha+n$, and $u\alpha-n\alpha+1<n+1$. Also, $n\alpha-\alpha+1<u$, so that $n\alpha^2-\alpha^2+\alpha<u\alpha$, which yields $n<u\alpha-n\alpha+1$. Therefore, $[(u-n)\alpha+1]=n$. In a row of $A$, the term immediately following $u-n$ is not $[(u-n)\alpha+1]$, for this number, coming just before $u$, must lie in column 1 and, thus, has no immediate predecessor. Therefore, by Lemma 2.2, the follower must be $[(u-n)\alpha]$, which is $n-1$. By (1), the number $u-1=u-n+[(u-n)\alpha]$ must lie in column 3, contrary to the hypothesis that it lies in column 1. Therefore, if as in (iii), $u$ cannot lie in column 2.

Since $u$ does not lie in column 1 or column 2, it must, by property I1, lie elsewhere. □

## 3. TWO MORE THEOREMS ABOUT COLUMN 1

Following Construction 2.1, we indicated that it is a difficult problem to formulate the first column of a Stolarsky interspersion in terms of an arbitrary given classification sequence, but that, surprisingly, in view of this difficulty, these terms can be "almost formulated" without great difficulty. Theorem 2.7, especially, tells what we mean by "almost formulated," and in addition to it we present here two more theorems.

Let $S_i = \{k : \exists$ Stolarsky interspersion $A \ni k = a(i, 1)$ for some $i\}$. Thus, $S_i$ is the set of all possible values that can be taken by the $i^{\text{th}}$ element of column 1 in a Stolarsky interspersion; e.g., $S_1 = \{1\}$, $S_2 = \{4\}$, $S_3 = \{6, 7\}$, $S_4 = \{9, 10\}$, $S_5 = \{11, 12\}$, $S_6 = \{14, 15\}$, and $S_7 = \{16, 17, 18\}$.

***Theorem 3.1:*** The sets $\{S_i\}_{i=1}^{\infty}$ are pairwise disjoint.

***Proof:*** Suppose two of the sets $S_i$ and $S_j$, where $j > i$, have a common element. By Theorem 2.7, it is clear that $j$ must be $i + 1$ and that the only number that $S_i$ could possibly share with $S_{i+1}$ is

$$[(i+1)\alpha + i - 1] = [i\alpha + i]. \tag{5}$$

Assuming this possibility, let $B$ be a Stolarsky interspersion satisfying $b(i+1, 1) = [(i+1)\alpha + i - 1]$. Now, $b(i, 1) \in \{[i\alpha + i - 2], [i\alpha + i - 1], [i\alpha + i]\}$, by Theorem 2.7. Since $b(i, 1) \neq [i\alpha + i]$, by property I1, and $b(i, 1) \neq [i\alpha + i - 1]$, by Theorem 2.10, we have $b(i, 1) = [i\alpha + i - 2]$.

Let $\varepsilon = \{i\alpha\}$, the fractional part, $i\alpha - [i\alpha]$, of $i\alpha$. Then (5) can easily be proved equivalent to

$$0 < \varepsilon < 2 - \alpha. \tag{6}$$

Since $b(i+1, 1) = b(i, 1) + 2$, the position of the number $x = [i\alpha + i - 1]$ in $B$ is, by Theorem 2.12, in a column numbered $\geq 3$. Thus, the row of $B$ containing $x$ contains an immediate predecessor $w$ of $x$ and also an immediate predecessor $v$ of $w$. Now $x$ must be one of the numbers $[w\alpha]$ or $[w\alpha] + 1$, by Lemma 2.2. We consider these two cases separately.

**Case 1:** $x = [w\alpha]$. By Lemma 2.3, $w = \left[\frac{x}{\alpha} + 1\right] = [x(\alpha - 1) + 1] = [x\alpha] - x + 1$. Thus,

$$\begin{aligned} w &= \left[\alpha[i\alpha + i - 1]\right] - [i\alpha + i - 1] + 1 \\ &= \left[\left((\alpha - 1)[i\alpha] + i\right) + 2 - \alpha\right] \\ &= [(\alpha - 1)(i\alpha - \varepsilon + i) + 2 - \alpha] \\ &= [i\alpha^2 - \alpha\varepsilon + \varepsilon - i + 2 - \alpha] \\ &= [i\alpha - \alpha\varepsilon + \varepsilon + 2 - \alpha] \\ &= \left[[i\alpha] + (1 + \varepsilon)(2 - \alpha)\right] \\ &= [i\alpha], \end{aligned}$$

since $0 < (1+\varepsilon)(2-\alpha) < 1$. The equations $w = [i\alpha]$, $x = [i\alpha + i - 1]$, and $x = w + v$ imply $v = i - 1$. Then $[v\alpha + 2] = [i\alpha - \alpha - 2] = [[i\alpha] + \{i\alpha\} + 2 - \alpha]$, which by (6) equals $w$. Thus, neither $[\alpha v]$ nor $[\alpha v + 1]$ equals $w$. This contradiction to Lemma 2.2 completes the proof for Case 1.

**Case 2:** $x = [w\alpha + 1]$. By Lemma 2.3, $w = \left[\frac{x}{\alpha}\right] = [x(\alpha - 1)]$, so that

$$\begin{aligned} w &= [(i\alpha - \varepsilon)(\alpha - 1) + (\alpha - 1)(i - 1)] \\ &= [i\alpha + (1 - \alpha)(1 + \varepsilon)] \\ &= [i\alpha - 1], \text{ since } -1 < (1 - \alpha)(1 + \varepsilon) < 0. \end{aligned}$$

The equations $w = [i\alpha - 1]$, $x = [i\alpha + i - 1]$, and $x = w + v$ imply $v = i$. Then $w = [v\alpha - 1]$, contrary to Lemma 2.2. □

***Theorem 3.2:*** Let $S = \bigcup_{i=1}^{\infty} S_i$. Let $\mathbb{F} = \{2, 3, 5, 8, 13, \ldots\}$, the set of Fibonacci numbers $F_3 = 2$, $F_4 = 3, \ldots, F_n = F_{n-1} + F_{n-2}$. Then $S$ is the set of all positive integers not in $\mathbb{F}$.

***Proof:*** Each number in $\mathbb{F}$ necessarily lies in row 1 and not in column 1. We shall show that, for any positive integer $x$ other than these, there exists a Stolarsky interspersion containing $x$ in its first column.

Let $E_k$ be the statement that, for all $m \le k$ such that $m \notin \mathbb{F}$, there exists a Stolarsky interspersion in which $m$ occurs in the first column. Clearly $E_k$ is true for $k$ = 1, 2, 3, 4. Assume for arbitrary $k \ge 4$ that $E_k$ is true. If $k+1 \in \mathbb{F}$, then clearly $E_{k+1}$ is true. Suppose $k+1 \notin \mathbb{F}$. Let $\delta = \left[\frac{k+2}{\alpha}\right] - \left[\frac{k+1}{\alpha}\right]$. Since $1 < \alpha < 2$, we have $\delta \in \{0, 1\}$. If $\delta = 1$, let $m = \left[\frac{k+2}{\alpha}\right]$ and obtain $k+1 = [m\alpha]$, but if $\delta = 0$, let $m = \left[\frac{k+1}{\alpha}\right]$ and obtain $k+1 = [m\alpha + 1]$.

**Case 1:** $m \notin \mathbb{F}$. Here, by the induction hypothesis, there exists a Stolarsky interspersion $B$ containing $m$ in its first column. Write $m = b(i_0, 1)$. We shall construct a new Stolarsky interspersion $A$ as follows: Define $a(i, j) = b(i, j)$ for all $i \le i_0 - 1, j \ge 1$. Define $a(i_0, 1) = m$. If $k+1 = [m\alpha]$, then define $a(i_0, 2) = [m\alpha + 1]$, but if $k+1 = [m\alpha + 1]$, then define $a(i_0, 2) = [m\alpha]$. Define the rest of row $i_0$ recursively: $a(i_0, j) = a(i_0, j-1) + a(i_0, j-2)$. Then finish defining $A$ as in Construction 2.1. By Theorem 5 of [6], $A$ contains $k+1$ in its first column.

**Case 2:** $m = F_p$ for some $p$. Here, $\delta = 1$, $k+2 = F_{p+1} = [m\alpha + 1]$, and $k = [(m-1)\alpha]$. Since $m-1 \notin \mathbb{F}$, there exists a Stolarsky interspersion $B$ having $m-1$ in its first column. Necessarily, $B$ contains $k+2$ in its first row, immediately following $m$. As in Case 1, we construct from $B$ a Stolarsky interspersion $A$ in which the immediate follower of $m-1$ is $k$. Now the only possible immediate predecessors of $k+1$ are $m$ and $m-1$. Since neither of these is followed by $k+1$ in $A$, we conclude that $k+1$ lies in the first column of $A$. □

## 4. A NEW STOLARSKY INTERSPERSION: THE EVEN FIRST COLUMN ARRAY

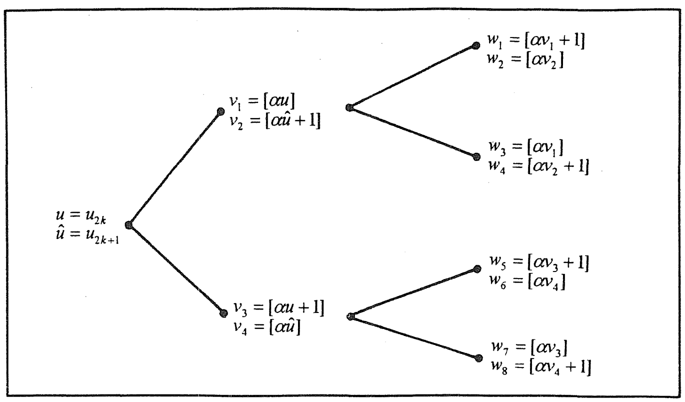
In addition to the two well-known Stolarsky interspersions of Tables 1 and 2 above, we introduce here a third, in which the only odd number in the first column is 1. Because of this property, we call this the *even first column array*, or *EFC array*. The array is defined by its classification sequence, namely, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, ... . In section 2, we encountered the sense in which a classification sequence defines a Stolarsky interspersion: if the sequence is $\{\delta_i\}$, then the number $a(i, 2)$ in column 2 must be $[\alpha a(i, 1) + \delta_i]$. [Recall that $a(i, 1)$ is always the least positive integer not in any previous row, and $a(i, j)$ for $j \ge 3$ is determined by (1).] The main objective in this section is to prove that the first column does indeed consist solely of even integers except for the first one.

Throughout this section, let $E = E(i, j)$, denote the EFC array with terms $e(i, j)$, and let $u_i = e(i, 1)$. Table 3 shows that the first few $u_i$ are 1, 4, 6, 10, 12, 14, 16. We shall deal with the $u_i$ in pairs: 4, 6; 10, 12; 14, 16; etc. Each such pair $u_{2k}, u_{2k+1}$ generates six terms $u_m$ where $m \ge 2k$. To describe these "higher $u_m$'s," we define the $u_{2k}$*-tree*, written as $T(u_{2k})$, as shown in Figure 1.

The classification sequence has $\delta_{2k} = 1$ and $\delta_{2k+1} = 0$, so that (7) shows that the numbers $v_3$ and $v_4$ must lie in column 2 of $E$, so that $v_1$ and $v_2$ must, by Lemma 2.2, be higher $u_m$'s. We shall show below that $v_1$ and $v_2$ are, respectively, of the forms $u_{2p+1}$ and $u_{2q}$. Assuming this for now, it follows by Lemma 2.2 that $w_3, w_4, w_7, w_8$ must lie in column 3 of $E$. Now $w_1, w_2, w_5, w_6$ must lie in $E$, but for each of these, its only possible immediate predecessor, as given by Lemma 2.3, is immediately followed by one of $w_3, w_4, w_7, w_8$. Therefore, $w_1, w_2, w_5, w_6$ are higher $u_m$'s.

**TABLE 3. The Even First Column Array**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | … |
| 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | 521 | |
| 6 | 9 | 15 | 24 | 39 | 63 | 102 | 165 | 267 | 432 | 699 | |
| 10 | 17 | 27 | 44 | 71 | 115 | 186 | 301 | 487 | 788 | 1275 | |
| 12 | 19 | 31 | 50 | 81 | 131 | 212 | 343 | 555 | 898 | 1453 | |
| 14 | 23 | 37 | 60 | 97 | 157 | 254 | 411 | 665 | 1076 | 1741 | |
| 16 | 25 | 41 | 66 | 107 | 173 | 280 | 453 | 733 | 1186 | 1919 | |
| 20 | 33 | 53 | 86 | 139 | 225 | 364 | 589 | 953 | 1542 | 2495 | |
| 22 | 35 | 57 | 92 | 149 | 241 | 390 | 631 | 1021 | 1652 | 2673 | |
| 26 | 43 | 69 | 112 | 181 | 293 | 474 | 767 | 1241 | 2008 | 3249 | |
| 28 | 45 | 73 | 118 | 191 | 309 | 500 | 809 | 1309 | 2118 | 3427 | |
| 30 | 49 | 79 | 128 | 207 | 335 | 542 | 877 | 1419 | 2296 | 3715 | |
| 32 | 51 | 83 | 134 | 217 | 351 | 568 | 919 | 1487 | 2406 | 3893 | |
| 36 | 59 | 95 | 154 | 249 | 403 | 652 | 1055 | 1707 | 2762 | 4469 | |
| 38 | 61 | 99 | 160 | 259 | 419 | 678 | 1097 | 1775 | 2872 | 4647 | |
| 40 | 65 | 105 | 170 | 275 | 445 | 720 | 1165 | 1885 | 3050 | 4935 | |
| 42 | 67 | 109 | 176 | 285 | 461 | 746 | 1207 | 1953 | 3160 | 5113 | |
| ⋮ | | | | | | | | | | | |



(7)

**FIGURE 1. The Tree $T(u_{2k})$**

***Lemma 4.1:*** Suppose $j$ and $k$ are nonzero integers. Let $\alpha = (1+\sqrt{5})/2$. Then

$$\{[j\alpha]\alpha\} - \{[k\alpha]\alpha\} = (1-\alpha)(\{j\alpha\} - \{k\alpha\}). \tag{8}$$

***Proof:*** For any nonzero integer $k$, we have

$$\begin{aligned} 1 &= \left\{-\{\alpha\}\{k\alpha\}\right\} + \{\alpha\}\{k\alpha\} = \left\{[k\alpha] - \{\alpha\}\{k\alpha\}\right\} + \{\alpha\}\{k\alpha\} \\ &= \left\{k\alpha + k - \{k\alpha\}\alpha\right\} + \{\alpha\}\{k\alpha\} = \left\{k\alpha^2 - \{k\alpha\}\alpha\right\} + \{k\alpha\}\{\alpha\}, \text{ by (2)} \\ &= \left\{(k\alpha - \{k\alpha\})\alpha\right\} + \{k\alpha\}\{\alpha\} \\ &= \{[k\alpha]\alpha\} + \{k\alpha\}\{\alpha\}. \end{aligned}$$

So, if $j$ and $k$ are nonzero integers, we have $\{[j\alpha]\alpha\}+\{j\alpha\}\{\alpha\}=\{[k\alpha]\alpha\}+\{k\alpha\}\{\alpha\}$, and (8) follows. □

***Lemma 4.2:*** Suppose $j$ and $k$ are nonzero integers. Let $\alpha=(1+\sqrt{5})/2$. Then

$$\{[j\alpha^2]\alpha\}-\{[k\alpha^2]\alpha\}=(2-\alpha)(\{j\alpha\}-\{k\alpha\}). \tag{9}$$

***Proof:*** For any nonzero integer $k$, we have $\{[k\alpha]\alpha\}+\{k\alpha\}\{\alpha\}=1$ from the proof of Lemma 4.1, so that $\{[k\alpha]\alpha\}+\{k\alpha\}>1$, a fact used below:

$$\begin{aligned}\{[k\alpha^2]\alpha\} &= \{[k\alpha+k]\alpha\}=\{[k\alpha]\alpha+k\alpha\}=\{[k\alpha]\alpha\}+\{k\alpha\}-1\\ &= \{(k\alpha-\{k\alpha\})\alpha\}+\{k\alpha\}-1=\left\{k\alpha^2-\{k\alpha\}\alpha\right\}+\{k\alpha\}-1\\ &= \left\{k\alpha-\{k\alpha\}\alpha\right\}+\{k\alpha\}-1=\left\{\{k\alpha\}-\{k\alpha\}\alpha\right\}+\{k\alpha\}-1\\ &= 1-(\alpha-1)\{k\alpha\}+\{k\alpha\}-1\\ &= (2-\alpha)\{k\alpha\}.\end{aligned}$$

So, if $j$ and $k$ are nonzero integers, then (9) holds.

***Lemma 4.3:*** An integer $u$ is of the form $[j\alpha]$ for some integer $j$ iff $\{u\alpha\}>2-\alpha$. Equivalently, an integer $u$ is of the form $[j\alpha]+j$ for some integer $j$ iff $\{u\alpha\}<2-\alpha$. (This inequality is stated without proof in [2].)

***Proof:*** Lemma 4.1 implies that, for any integers $j$ and $k$, we have $\{j\alpha\}>\{k\alpha\}$ iff $\{[j\alpha]\alpha\}<\{[k\alpha]\alpha\}$. The well-known fact that $\max\{\{j\alpha\}:1\le j\le F_{2n}\}=\{\alpha F_{2n}\}$ implies, therefore, that $\min\{\{[j\alpha]\alpha\}:1\le j\le F_{2n}\}=\{[\alpha F_{2n}]\alpha\}$. Since $\lim_{n\to\infty}\{[\alpha F_{2n}]\alpha\}=2-\alpha$, we have $\{[j\alpha]\alpha\}>2-\alpha$ for all positive integers $j$.

For the converse, Lemma 4.2 implies that, for any integers $j$ and $k$, we have $\{j\alpha\}>\{k\alpha\}$ iff $\{[j\alpha^2]\alpha\}>\{[k\alpha^2]\alpha\}$. The fact that $\max\{\{j\alpha\}:1\le j\le F_{2n}\}=\{\alpha F_{2n}\}$ implies, therefore, that $\max\{\{[j\alpha^2]\alpha\}:1\le j\le F_{2n}\}=\{[\alpha^2F_{2n}]\alpha\}$. Since $\lim_{n\to\infty}\{[\alpha^2F_{2n}]\alpha\}=2-\alpha$, we have, for all positive integers $j$, $\{[j\alpha^2]\alpha\}<2-\alpha$. But, by Beatty's theorem, as $j$ ranges through the positive integers, the numbers $[j\alpha^2]$ range through all the positive integers not of the form $[j\alpha]$. Since $[j\alpha^2]=[j\alpha]+j$, the proof is finished. □

***Lemma 4.4:*** Suppose $u$ has the form $2[n\alpha]+2n$ and $v=[u\alpha]$. Let $q=\left[\frac{u}{2\alpha}\right]$. Then

$$v=2[q\alpha]+2q+2 \quad\text{and}\quad u+v=[v\alpha+1].$$

***Proof:*** We have $\frac{u}{2\alpha}-1<q<\frac{u}{2\alpha}$, so that $\frac{u}{2}-\alpha<q\alpha<\frac{u}{2}$. Thus, $q\alpha$ is strictly less than the integer $\frac{u}{2}$. so that $[q\alpha]=\frac{u}{2}-1$. Also $q=\left[\frac{u}{2\alpha}\right]=\left[\frac{u}{2}(\alpha-1)\right]=\left[\frac{u\alpha}{2}\right]-\frac{u}{2}$. Accordingly,

$$2[q\alpha]+2q+2=u-2+2\left[\frac{u\alpha}{2}\right]-u+2=2\left[\frac{u\alpha}{2}\right]=2[[n\alpha]\alpha+n\alpha].$$

By Lemma 4.3, $\{[n\alpha]\alpha+n\alpha\}<1/2$, and this implies $2[[n\alpha]\alpha+n\alpha]=[2[n\alpha]\alpha+2n\alpha]$, which is $v$. Next,

$$\begin{aligned}[v\alpha+1]&=\big[\alpha[u\alpha]\big]+1=[\alpha(u\alpha-\varepsilon)]+1, \text{ where } \varepsilon=\{u\alpha\},\\ &=[u\alpha^2-\alpha\varepsilon]+1=[u\alpha+u-\alpha\varepsilon]+1=u\alpha-\alpha\varepsilon-\{u\alpha-\alpha\varepsilon\}+u+1\\ &=u\alpha-\alpha\varepsilon-\big\{u\alpha-\{u\alpha\}-\{\alpha\}\{u\alpha\}\big\}+u-1\\ &=u\alpha-\alpha\varepsilon-\big\{-\{\alpha\}\{u\alpha\}\big\}+u+1, \text{ since } u\alpha-\{u\alpha\} \text{ is an integer}\\ &=u\alpha-\alpha\{u\alpha\}+\{\alpha\}\{u\alpha\}+u=u\alpha-\{u\alpha\}+u\\ &=v+u\end{aligned}$$

***Lemma 4.5:*** Suppose $\hat{u}$ has the form $2[n\alpha]+2n+2$ and $\hat{v}=[\hat{u}\alpha+1]$. Let $q=\left[\frac{\hat{u}}{2\alpha}+1\right]$. Then $\hat{v}=2[q\alpha]+2q$ and $\hat{u}+\hat{v}=[\hat{v}\alpha]$.

***Proof:*** The proof is similar to that of Lemma 4.4 and is omitted. □

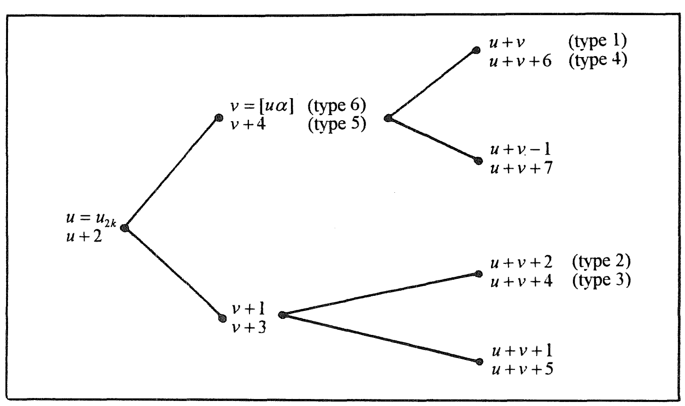
***Lemma 4.6:*** Suppose $u$ has the form $2[n\alpha]+2n$ and $\hat{u}=u+2$ in the $u_{2k}$-tree of Figure 1 (7). Then $v_2=v_1+4$.

***Proof:*** By Beatty's theorem, $[n/\alpha]$ is not of the form $[w\alpha]$, so that, by Lemma 4.3, $\{\alpha[n/\alpha]\}<2-\alpha$. Substituting $\alpha+1$ for $1/\alpha$ and multiplying by 2 gives $2\{\alpha([n\alpha]+n)\}<4-2\alpha$. Then $\{a([n\alpha]+n\}<1/2$ since $4-2\alpha<1$, so that $\{u\alpha\}=\{2\alpha([n\alpha]+n\}<4-2\alpha$. Since $\{2\alpha\}=2\alpha-3$, we have $\{u\alpha+2\alpha\}=\{u\alpha\}+\{2\alpha\}$, from which follows $[u\alpha+2\alpha]-[u\alpha]=3$. Equivalently, $v_2-v_1=[u\alpha+2\alpha+1]-[u\alpha]=4$. □

***Lemma 4.7:*** In the $u_{2k}$-tree (7), suppose $u$ has the form $2[n\alpha]+2n$ and $\hat{u}=u+2$. Let $v=v_1$. Then $v_2=v+4, v_3=v+1, v_4=v+3$. Moreover, $w_1=u+v, w_2=u+v+6, w_3=u+v+1$, and $w_4=u+v+7$; also, $w_5=u+v+2, w_6=u+v+2, w_7=u+v+1$, and $w_8=u+v+5$.

***Proof:*** Clearly $v_3=v+1$. By Lemma 4.6, $v_2=v+4$, so that $v_4=v+3$. By Lemma 4.4, $w_1=u+v$, so that $w_3=u+v-1$. Now $w_2=[\alpha v_2]$, which by Lemma 4.5 equals $\hat{u}+\hat{v}$, which is $u+v+6$, and then $w_4=u+v+7$. By recurrence (1), $w_7=u+v+1$ and $w_8=u+v+5$, and from these follow $w_5=u+v+2$ and $w_6=u+v+4$. □

Under the assumption that $u$ $(=u_{2k})$ is of the form $2[n\alpha]+2n$ and $\hat{u}=u+2$, we can summarize Lemma 4.7 by rewirting the tree $T(u_{2k})$ in (7) with new labels:




(10)

**FIGURE 2. The Tree $T(u_{2k})$, Relabeled**

***Lemma 4.8:*** Suppose $k$ is a positive integer, $u = 2[k\alpha]+2k$ and $\hat{u} = 2[k\alpha]+2k+2$. Then $\hat{u}-u = 4$ or $\hat{u}-u = 6$, according as $\{k\alpha\} < 2-\alpha$ or $\{k\alpha\} > 2-\alpha$.

***Proof:*** The proof is easy and is omitted. □

***Lemma 4.9:*** Let $E$ be the EFC array. The numbers $u_k$ in column 1 of $E$ are given by $u_1 = 1$ and

$$u_{2n} = 2[n\alpha]+2n, \tag{11}$$

$$u_{2n+1} = 2[n\alpha]+2n+2 \tag{12}$$

for $n = 1, 2, 3, \dots$ .

***Proof:*** It is easy to check that (11) and (12) hold for $1 \le n \le 3$ and that in the tree $T(u_2)$ we find $u_3 = 6$ of type 6, $u_4 = 10$ of type 5 (and also of type 1), $u_5 = 12$ of type 2, $u_6 = 14$ of type 3, $u_7 = 16$ of type 4. Suppose now that $m \ge 7$, and as an induction hypothesis, assume that for every $h$ satisfying $3 \le h \le m$ the following conditions hold:

(i) there exists $k$ such that $2k \le h-2$ and $u_h$ is a vertex of tree $T(u_{2k})$, and in that tree, $u_h$ is of one of the six types identified in (10);

(ii) in $T(u_{2k})$, if $u_h$ is of type 1, 3, or 5, then $h$ is even, and if $h = 2p$, then $u_h = 2[p\alpha]+2p$;

(iii) in $T(u_{2k})$, if $u_h$ is of type 2, 4, or 6, then $h$ is odd, and if $h = 2p+1$, then $u_h = 2[p\alpha]+2p+2$.

<u>**Case 1**</u>**:** $u_m$ is of type 1 (or type 3) in a tree $T(u_{2k})$. By (ii), $u_m = 2[p\alpha]+2p$, where $m = 2p$. Theorem 2.10 and Lemma 4.7 then imply $u_{m+1} = 2[p\alpha]+2p+2$, so that $u_{m+1}$ is of type 2 (or type 4) and satisfies (12).

<u>**Case 2**</u>**:** $u_m$ is of type 2 in a tree $T(u_{2k})$. By (iii), $u_m = 2[p\alpha]+2p+2$, where $m = 2p+1$. Theorem 2.10 and Lemma 4.7 then imply $u_{m+1} = 2[p\alpha]+2p+4 = 2[(p+1)\alpha]+2(p+1)$, so that $u_{m+1}$ is of type 3 and satisfies (12).

<u>**Case 3**</u>**:** $u_m$ is of type 4 in a tree $T(u_{2k})$. As in the proof of Lemma 4.6, we have $\{\alpha u_{2k}\} < 4-2\alpha$, so that

$$\frac{4\alpha-7}{\alpha-1} < \{\alpha u_{2k}\} < \frac{4\alpha-6}{\alpha-1} = 4-2\alpha,$$

which implies $0 < 4\alpha-6+(1-\alpha)\{\alpha u_{2k}\} < 1$, so that $6 = [(1-\alpha)\{\alpha u_{2k}\}+4\alpha]$ and $6 = [\alpha u_{2k} - [\alpha u_{2k}] - \alpha\{\alpha u_{2k}\}+4\alpha]$. Adding $u_{2k}+[\alpha u_{2k}]$ to both sides and applying Lemma 4.7 give

$$\begin{aligned} u_m &= [\alpha u_{2k}+u_{2k}-\alpha\{\alpha u_{2k}\}+4\alpha] \\ &= [\alpha(\alpha u_{2k}-\{\alpha u_{2k}\}+4)], \text{ by (2)} \\ &= [\alpha([\alpha u_{2k}]+4)], \end{aligned}$$

which is the number of type 6 in tree $T([\alpha u_{2k}]+4)$. By Lemma 4.6, the number $u_m+4$ is of type 5 in tree $T([\alpha u_{2k}]+4)$.

By Theorem 2.7, $u_{m+1} \le [(m+1)\alpha]+m+1$ and $[(m-3)\alpha]+m-5 \le u_{m-3}$, so that

$$u_{m+1}-u_{m-3} \ge [(m+1)\alpha]-[(m-3)\alpha]+6 \ge 11.472,$$

but, since $u_{m+1}-u_{m-3}$ is an integer, we have

$$u_{m+1}-u_{m-3}\geq 12. \tag{13}$$

Since $u_m$ is of type 4, the number $u_{m-i}$ must, by the induction hypothesis, be of type $4-i$, for $i=1,2,3$, so that $u_m=u_{m-1}+2$, $u_{m-1}=u_{m-2}+2$, and $u_{m-2}=u_{m-3}+2$; these imply

$$u_m-u_{m-3}=8. \tag{14}$$

By Theorem 2.7, $u_{m+1}\in\{u_m+2, u_m+3, u_m+4\}$, so that (13) and (14) force $u_{m+1}$ to be $u_m+4$.

By the induction hypothesis, $u_m=2[p\alpha]+2p+2$, where $m=2p+1$, $u_{m-1}=2[p\alpha]+2p$, and $u_{m-2}=2[(p-1)\alpha]+2(p-1)+2$. The equation $u_{m+1}-u_{m-2}=2$ therefore easily yields

$$[p\alpha]-[(p-1)\alpha]=1. \tag{15}$$

Now, if $[(p+1)\alpha]-[p\alpha]=1$, this and (15) imply $[(p+1)\alpha]-[(p-1)\alpha]=2$, which is easily seen to be impossible, since $1/2<\alpha<1$. Therefore, $[(p+1)\alpha]-[p\alpha]=2$, so that $u_{m+1}=u_m+4=2[p\alpha]+2p+6=2[(p+1)\alpha]+2(p+1)$, and (11) holds.

**Case 4:** $u_m$ is of type 5 in a tree $T(u_{2k})$. Before breaking this into two subcases, we note that

$$\{\alpha u_{2k}\}=\{\alpha(2[k\alpha]+2k)\}=\{4k\alpha-2\alpha\{k\alpha\}\}=(4-2\alpha)\{k\alpha\}. \tag{16}$$

**Case 4.1:** $\{k\alpha\}>2-\alpha$. In this case, (16) implies $\{\alpha u_{2k}\}>(4-2\alpha)(2-\alpha)=2(5-3\alpha)$. The inequality $\{\alpha u_{2k}\}>5-3\alpha$ implies

$$5-3\alpha=\frac{2\alpha-3}{\alpha}<\{\alpha u_{2k}\}<1<2\alpha-2,$$

which implies $[2\alpha-\alpha\{\alpha u_{2k}\}]=2$, so that

$$\begin{aligned}[\alpha u_{2k}+4]&=[\alpha u_{2k}]+2+[2\alpha-\alpha\{\alpha u_{2k}\}]\\ &=[\alpha u_{2k}-u_{2k}+2]+[\alpha^2u_{2k}-\alpha\{\alpha u_{2k}\}-\alpha u_{2k}+2\alpha]\\ &=[\alpha u_{2k}-u_{2k}+2]+[\alpha[\alpha u_{2k}-u_{2k}+2]].\end{aligned}$$

This shows that the number $u_m$ of type 5 in a tree $T(u_{2k})$, namely, $[\alpha u_{2k}+4]$, is the same as the number of type 1 in tree $T(\alpha u_{2k}-u_{2k}+2)$. It follows from Case 1 that $u_{m+1}$ is of type 2 in tree $T(\alpha u_{2k}-u_{2k}+2)$ and satisfies the required conditions.

**Case 4.2:** $\{k\alpha\}>2-\alpha$. Again (16) applies, giving $\{\alpha u_{2k}\}<2(5-3\alpha)<7-4\alpha=1-\{4\alpha\}$, so that $\{\alpha u_{2k}\}+\{4\alpha\}<1$. Consequently, $\{\alpha u_{2k}+4\alpha\}-\{\alpha u_{2k}\}=4\alpha-6$, so that

$$[(u_{2k}+4)\alpha]=[\alpha u_{2k}+4]+2. \tag{17}$$

Since $m\geq 7$, we have $2k\leq m-2$, by hypothesis (i), so that Lemma 4.8 gives $u_{2k+2}=u_{2k}+4$. Then (17) implies that $u_{m+1}$ is the number of type 6 in tree $T(u_{2k+2})$, and (12) holds.

**Case 5:** $u_m$ is of type 6 in a tree $T(u_{2k})$. We already know by Lemma 4.6 that the number $u_m+4$ is of type 5 in tree $T(u_{2k})$. If $u_{m+1}=u_m+2$, then we would have $u_{m+1}-u_{m-3}=10$ and a contradiction as in the proof for Case 3. Moreover, $u_{m+1}-u_m$ cannot be 1 or 3, by Theorem 2.10. Therefore, $u_{m+1}=u_m+4$, and as in the proof for Case 3, we find that (11) holds.

We have now shown that the conditions (i), (ii), and (iii) stated in the induction hypothesis all hold for $h = m+1$. Therefore, equations (11) and (12) hold for all positive integers $n$. □

## 5. CONCLUSION

It is clear from the induction method of the proof of Theorem 4.9 that the EFC array is the only Stolarsky interspersion having only even numbers in the first column, except for the initial 1.

We recount the connections between certain classification sequences $\{\delta_i\}$ and the first columns of the associated Stolarsky interspersions $\{u_i\}$.

**Wythoff Array (Table 2):** $\delta_i = 1$ for all $i$, and $u_i = [i\alpha]+i-1$ for all $i$. In fact, *all* the terms $a(i,j)$ of the Wythoff array are conveniently expressible: $a(i,j) = [i\alpha]F_{j+1} + (i-1)F_j$. Corollary 2.8 shows that the Wythoff array is "central" among Stolarsky interspersions.

**Dual of the Wythoff Array:** $\delta_1 = 1$ and $\delta_i = 0$ for all $i \geq 2$, and

$$u_i = \begin{cases} [i\alpha]+i & \text{if } i \text{ is of the form } [k\alpha]+k+1, \\ [i\alpha]+i-1 & \text{otherwise.} \end{cases}$$

**Stolarsky Array (Table 1):** $u_i = \left[\left(i-\frac{1}{2}\right)\alpha\right]+i$. No convenient formula for $\delta_i$ has been found; the sequence begins like this: 1 0 0 1 0 1 1 0 1 0 1 1 0 1 0 0 1 0 1.

**EFC Array (Table 3):** $\delta_i = 1$, $\delta_{2k} = 1$, $\delta_{2k+1} = 0$ for all $k \geq 1$, and

$$u_i = \begin{cases} 2\left[\frac{i\alpha}{2}\right]+i & \text{if } i \text{ is even,} \\ 2\left[\frac{i-1}{2}\alpha\right]+i+1 & \text{otherwise,} \end{cases}$$

by Theorem 4.9.

**ESC Array:** Introduced here by its classification sequence, $\{\delta_i\} = \{1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots\}$. We conjecture that the second column of this array consists solely of even integers, beginning with 2, 6, 12, 14, 18, 24, 28, 32, 36, 40. Can someone figure out a formula for $u_i$?

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